Monotone discretization of the second boundary value problem for the Monge-Ampère equation

Discrétisation monotone du second problème aux limites pour l'équation de Monge-Ampère

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Setting

Optimal transport problem*:

$$\inf_{T\colon X \to Y,\, T_{\#}\mu=\nu} \int_X c(x, T(x))\, d\mu(x).$$

We assume that:

- The nonempty open sets X and Y are convex and bounded.
- The measure μ has a bounded density $f\colon \overline{X} \to \mathbb{R}_+$.
  - (allows measures with non-convex / non-connected support)
- The measure ν has a positive, Lipschitz continuous density $g\colon \overline{Y} \to \mathbb{R}_+^*$.
  - (implies convex support)
- (For now) c is the quadratic cost function $(x, y) \mapsto |x - y|^2$.

*The constraint $T_{\#}\mu = \nu$ means that $\mu(T^{-1}(E)) = \nu(E)$, for any Borel set $E \subset Y$. Measures μ and ν must have the same mass.

Monge-Ampère equation of optimal transport

The optimal transport map is of the form $T = \nabla u$ where $u\colon X \to \mathbb{R}$ is solution to the *second boundary value problem* for the Monge-Ampère equation:

$$\begin{cases} \det \nabla^2 u(x) = f(x)/g(\nabla u(x)) & \text{in } X, \\ \nabla^2 u(x) \succeq 0 & \text{in } X, \\ \nabla u(X) \subset \overline{Y}. \end{cases}$$

- $\nabla^2 u(x) \succeq 0$ ($\nabla^2 u(x)$ is positive semidefinite) is a convexity constraint.
- $\nabla u(X) \subset \overline{Y}$ is a boundary condition (by convexity, equivalent to $\nabla u(\partial X) \subset \overline{Y}$).

How to discretize the Monge-Ampère problem in order to solve it numerically?

Outline

1. Discretization of the Monge-Ampère equation.
2. Handling of the optimal transport boundary condition.
3. Numerical results.

Discretization of the Monge-Ampère equation

For simplicity, we consider the simple Monge-Ampère equation:

$$\det \nabla^2 u(x) = f(x) \quad \text{in } X.$$

- The left-hand side is monotone with respect to $\nabla^2 u(x)$, for the Loewner order, provided that $\nabla^2 u(x) \succeq 0$.
- Thus the Monge-Ampère equation belongs to the class of *degenerate elliptic* equations.
- *Monotone schemes* is a category of numerical schemes that is well-suited for the discretization of degenerate elliptic equations.

How to discretize $\det \nabla^2 u(x)$ in a monotone way?

Reformulation

- Denote by $\mathcal{S}_d^+$ (resp. $\mathcal{S}_d^{++}$) the set of symmetric positive semidefinite (resp. definite) matrices of size d.
- Then for $M \in \mathcal{S}_d^+$,

$$d(\det M)^{1/d} = \inf_{\mathcal{D} \in \mathcal{S}_d^{++},\, \det \mathcal{D} = 1} \operatorname{Tr}(\mathcal{D} M)$$

  - (Justified using the inequality of arithmetic and geometric means on eigenvalues of $\mathcal{D}M$.)
- Successive reformulations of the Monge-Ampère equation:

$$\begin{aligned}
df(x)^{1/d} - d(\det \nabla^2 u(x))^{1/d} &= 0 && \text{in } X,\\
\sup_{\mathcal{D} \in \mathcal{S}_d^{++},\, \det \mathcal{D} = 1} \Big(df(x)^{1/d} - \operatorname{Tr}(\mathcal{D} \nabla^2 u(x)) \Big) &= 0 && \text{in } X,\\
\max_{\mathcal{D} \in \mathcal{S}_d^{+},\, \operatorname{Tr}(\mathcal{D}) = 1} \Big(d(f(x) \det \mathcal{D})^{1/d} - \operatorname{Tr}(\mathcal{D} \nabla^2 u(x)) \Big) &= 0 && \text{in } X.
\end{aligned}$$

Discussion of the reformulation

$$\max_{\mathcal{D}\in\mathcal{S}_d^+,\,\mathrm{Tr}(\mathcal{D})=1} \Big(d(f(x)\det\mathcal{D})^{1/d} - \mathrm{Tr}(\mathcal{D}\nabla^2 u(x)) \Big) = 0 \quad \text{in } X.$$

- Reformulation first used numerically in Feng, Jensen, 2017.
- We could have stopped earlier in the reformulation process (maximizing over $\mathcal{D} \in \mathcal{S}_d^{++}$ satisfying $\det\mathcal{D} = 1$ instead of $\mathrm{Tr}(\mathcal{D}) = 1$).
    - This would have yielded (some variant of) the MA-LBR scheme, see Benamou, Collino, Mirebeau, 2016.
- Benefits of the Feng and Jensen reformulation:
    - Maximum of a concave function over a compact set.
    - This reformulation enforces the convexity of its solutions.
        - No need to discretize the convexity constraint separately.
        - No need of damping when solving the resulting scheme with the Newton method, as opposed to the MA-LBR scheme.

Finite difference discretization in dimension two

For any $\mathcal{D} \in \mathcal{S}_2^{++}$, we have to discretize the second-order term $\mathrm{Tr}(\mathcal{D}\nabla^2 u(x))$ in a monotone way.
Selling's decomposition (a tool from low-dimensional lattice geometry) is of the form

$$\mathcal{D} = \sum_{i=1}^{3} \lambda_i e_i e_i^\top,$$

with weights $\lambda_i \geq 0$ and offsets $e_i \in \mathbb{Z}^d$ (*not the eigendecomposition*).
Then, with consistency at the order two,

$$\mathrm{Tr}(\mathcal{D}\nabla^2 u(x)) \approx \Delta_h^{\mathcal{D}} u(x),$$

where

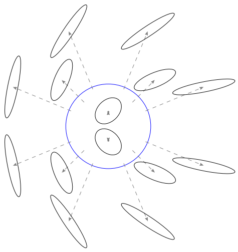
$$\Delta_h^{\mathcal{D}} u(x) := \sum_{i=1}^{3} \lambda_i \frac{u(x+he_i) + u(x-he_i) - 2u(x)}{h^2}.$$

Selling's decomposition — illustration

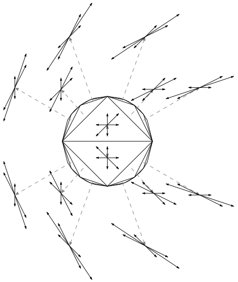
The set $\{\mathcal{D} \in \mathcal{S}_2^+ \mid \mathrm{Tr}(\mathcal{D}) = 1\}$ is a disk:

$$\{\mathcal{D} \in \mathcal{S}_2^+ \mid \mathrm{Tr}(\mathcal{D}) = 1\} = \left\{ \frac{1}{2} \begin{pmatrix} 1+\rho_1 & \rho_2 \\ \rho_2 & 1-\rho_1 \end{pmatrix} \;\Big|\; |\rho| \leq 1 \right\}.$$

Offsets of Selling's decomposition are constant on each cell of some infinite triangulation of this disk (but weights vary on those cells).



Matrices



Offsets

Discretization of the Feng and Jensen operator

On a Cartesian grid $\mathcal{G}_h \subset X \cap h\mathbb{Z}^2$, $\mathcal{G}_h \approx X \cap h\mathbb{Z}^2$, we let

$$(F_h u)[x] :\approx \max_{\mathcal{D}\in\mathcal{S}_2^+,\, \mathrm{Tr}(\mathcal{D}=1)} \left(2(f(x)\det\mathcal{D})^{1/2} - \Delta_h^{\mathcal{D}} u(x)\right).$$

Not an exact definition since we need to explain how to compute or approximate the maximum. Our strategy:

- Keep only a finite number of cells in the triangulation of the parameter set.
- Use a closed-form formula for the maximum on each of those cells (B., Mirebeau, 2021: this closed-form formula exists and is numerically exploitable).

Numerically more efficient than the alternative (discretizing the parameter set).

More general Monge-Ampère equations

General form of the Monge-Ampère equation:

$$\det\left(\nabla^2 u(x) - A(x, \nabla u(x))\right) = B(x, \nabla u(x)) \quad \text{in } X,$$

with admissibility constraint

$$\nabla^2 u(x) \succeq A(x, \nabla u(x)) \quad \text{in } X.$$

Feng and Jensen reformulation:

$$\max_{\mathcal{D}\in\mathcal{S}_2^+,\,\mathrm{Tr}(\mathcal{D})=1} \Big(d(B(x, \nabla u(x)) \det \mathcal{D})^{1/d} + \mathrm{Tr}(\mathcal{D}A(x, \nabla u(x))) \\ - \mathrm{Tr}(\mathcal{D}\nabla^2 u(x)) \Big) = 0 \quad \text{in } X.$$

In this setting, we use a Lax-Friedrichs approximation of $\nabla u(x)$.

Numerical efficiency

In the smooth case, we assume that the Monge-Ampère problem has a solution of class C^2 with a uniformly admissible Hessian.

	General case	Smooth case	Smooth case, Lax-Friedrichs
Consistency error	$O(h^{2/3})$	$O(h^2)$	$O(h)$
Numerical cost	$O(h^{-8/3}\log(1+h^{-1}))$	$O(h^{-2})$	$O(h^{-2})$
Numerical cost (discretized maximum)	$O(h^{-10/3})$	$O(h^{-6})$	$O(h^{-4})$

The numerical cost with the discretized maximum is to retain the same order of consistency.

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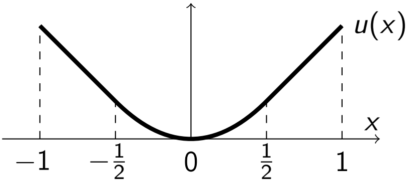
One-dimensional problem

For simplicity, let us consider the one-dimensional Monge-Ampère problem, with $X = Y = (-1, 1)$ and $g \equiv 1$:

$$\begin{cases} u''(x) = f(x) & \text{in } (-1, 1), \\ u'(x) \in [-1, 1], & \forall x \in (-1, 1). \end{cases}$$

Example: solution with $f = 2\chi_{(-\frac{1}{2}, \frac{1}{2})}$:



Reformulation suitable for discretization (1/2)

- The optimal transport boundary condition

$$u'(x) \in [-1, 1], \quad \forall x \in (-1, 1),$$

  may be reformulated in the inequality form

$$|u'(x)| - 1 \leq 0 \quad \text{in } (-1, 1).$$

  - (Generalizes to higher dimensions using the signed distance function to ∂Y.)
- We have both an equality and an inequality on the whole domain $(-1, 1) \implies$ How to turn them into a single equation?

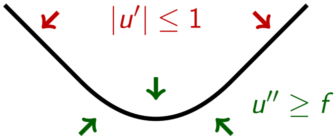
Reformulation suitable for discretization (2/2)

- Following Froese, 2019, we consider the maximum between the Monge-Ampère operator and the optimal transport boundary condition operator.
- Need to add a condition on $\partial(-1,1)$.
  - Appropriate choice: Dirichlet boundary condition $u(-1) = u(1) = +\infty$, in the *weak sense* of viscosity solutions.
  - (Induces no boundary layer.)
- Resulting system:

$$\begin{cases} \max\left\{f - u'', |u'| - 1\right\} = 0 & \text{in } (-1,1), \\ u(-1) = u(1) = +\infty. \end{cases}$$

Justification of the reformulation

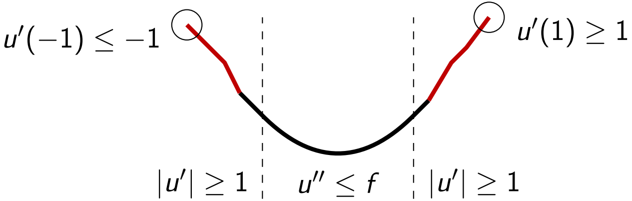
- Froese, 2019: all subsolutions to the reformulated problem are solutions to the original Monge-Ampère problem.
- Particularly strong result (concerns subsolutions, not only solutions).
- Justification:
  - If $\max\{f - u'', |u'| - 1\} \leq 0$, then both $u'' \geq f$ and $|u'| \leq 1$.
  - By a competition argument between both inequalities, deduce that actually $u'' = f$ in $(-1, 1)$.



- What about supersolutions?

Case of supersolutions

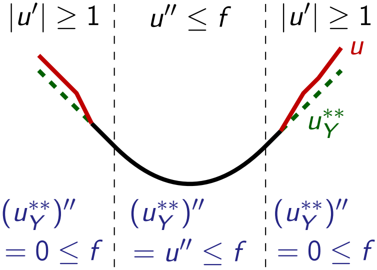
- If $\max\{f - u'', |u'| - 1\} \geq 0$, then either $u'' \leq f$ or $|u'| \geq 1$.
- Thanks to the appropriate choice of the boundary condition $u(-1) = u(1) = +\infty$ in the viscosity sense, one can show that supersolutions also satisfy $u'(-1) \leq -1$ and $u'(1) \geq 1$.
- Pathological example of a supersolution that is not a solution:



- Hope: supersolutions are *not too dissimilar from solutions*.

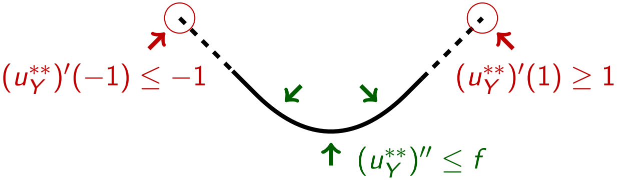
Slope-limited convex envelope

- We define a *slope-limited convex envelope* u_Y^{**} of the supersolution u.
  - Supremum of supporting hyperplanes *whose slope belong to the target set* $Y = [-1, 1]$.
- u_Y^{**} satisfies $(u_Y^{**})'' \le f$ on the whole domain $(-1, 1)$.



Competition argument

- Remember that $u'(-1) \leq -1$ and $u'(1) \geq 1$.
- One can deduce that $(u_Y^{**})'(-1) \leq -1$ and $(u_Y^{**})'(1) \geq 1$ (actually with equalities).
- By a competition argument with the inequality $(u_Y^{**})'' \leq f$ on $(-1, 1)$, one can show that $(u_Y^{**})'' = f$ on $(-1, 1)$.



- Conclusion: if u is a supersolution to the reformulated Monge-Ampère problem, then u_Y^{**} is a solution to the original problem.

Summary

- Reformulated system for the Monge-Ampère problem:

$$\begin{cases} \max\{f - u'', |u'| - 1\} = 0 & \text{in } (-1, 1), \\ u(-1) = u(1) = +\infty. \end{cases}$$

- Subsolutions are solutions to the original system.
- Supersolutions may be turned into solutions to the original system.
- B., Mirebeau, 2021: proof for systems associated to optimal transport problems *with quadratic cost*, in arbitrary dimension and with potentially nonconstant target density g.
  - Need to use the appropriate notions of weak solutions:
    - Viscosity solutions for the reformulated system.
    - Aleksandrov (equivalently Brenier) solutions for the original system.
- The numerical scheme is a discretization of the reformulated system.

Mass balance condition

- In order for the systems to be well-posed, one has to assume the *mass balance condition*

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} g(y)\,dy$$

  (or $\int_{-1}^{1} f(x)\,dx = 2$ in the particular case $g \equiv 1$).
- Usually, no discrete counterpart to the mass balance condition holds at the discrete level.
- Therefore, a scheme that is a direct discretization of the reformulated system often does not admit solutions.
- How to modify the reformulated system in order to weaken the need for the mass balance condition?

Weakening the need for the mass balance condition

- Approach 1: replace the (weak) Dirichlet boundary condition $u = +\infty$ on ∂X by $u = 0$ on ∂X.
  - Approach used in Froese, 2019.
  - Theoretical guarantees of existence and convergence of solutions to numerical schemes (for quadratic transport costs).
  - Schemes have to be *underestimating*, numerical artifacts may appear near the boundary.
- Approach 2: Add an unknown $\alpha \in \mathbb{R}$ and solve the modified system

$$\begin{cases} \max\{f - u'' + \alpha, |u'| - 1\} = 0 & \text{in } (-1,1), \\ u(-1) = u(1) = +\infty. \end{cases}$$

  - Approach used as a numerical trick in Benamou, Duval, 2019.
  - Our contribution: theoretical guarantees for this approach (existence and convergence of solutions to numerical schemes, for quadratic transport costs).

Study of the augmented system

System with the additional unknown:

$$\begin{cases} \max\{f - u'' + \alpha, |u'| - 1\} = 0 & \text{in } (-1, 1), \\ u(-1) = u(1) = +\infty. \end{cases}$$

Properties depending on the sign of α:

$\alpha < 0$	$\alpha = 0$	$\alpha > 0$
no supersolutions (many subsolutions)	existence of a solution	no subsolutions (many supersolutions)

Proof of no sub- / supersolutions: refinement of the competition arguments described previously.

Convergence result

- B., Mirebeau, 2021: under suitable assumptions, solutions (u_h, α_h) to the finite difference scheme converge to $(u, 0)$ where u solves the Monge-Ampère problem.
- Main assumptions: X is strongly convex, Y is convex, $f \geq 0$ is bounded and almost everywhere continuous, g is positive and Lipschitz continuous.
- Sketch of proof:
  1. Arzelà-Ascoli: (u_h, α_h) converge, up to extraction, to some (u, α).
  2. Barles, Souganidis, 1991: u solution to the reformulated problem with additional parameter α.
  3. Solutions only exist for $\alpha = 0$, so $\alpha = 0$.
  4. u solution with $\alpha = 0 \implies u$ subsolution with $\alpha = 0 \implies u$ solution to the original Monge-Ampère problem.
  5. Conclude using uniqueness for the original problem.

Existence of solutions

- B., Mirebeau, 2021: under suitable assumptions, there exists a solution (u_h, α_h) to the scheme.
- Proved in a general setting which allows general optimal transport costs (or even non-Monge-Ampère equations).
- Existence of solutions for monotone schemes is usually proved using Perron's method.
- Main difficulty here: the scheme is monotone with respect to u for fixed α, but not monotone with respect to the pair of unknowns (u, α).
- We add to adapt Perron's method to this setting, handling the unknown α separately in the proof.

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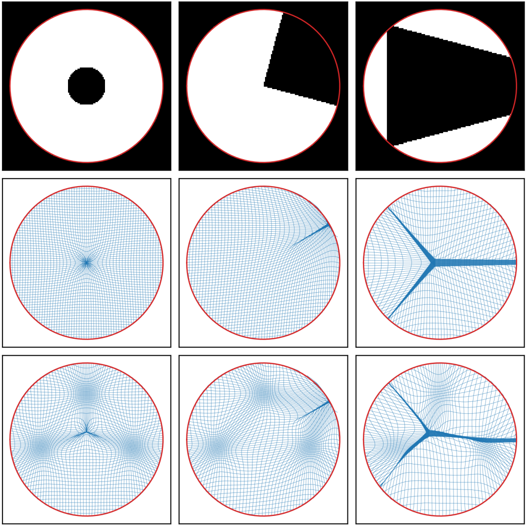
Application to quadratic optimal transport problems

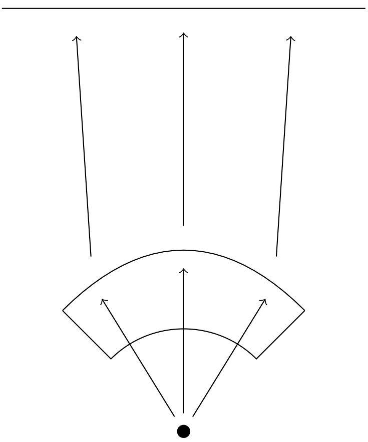


- Top: source density.
- Middle: results with constant target density.
- Bottom: results with nonconstant target density.

Application to nonimaging optics

We solve the far field refractor problem in nonimaging optics: given a uniform point light source, what should be the shape of the lens so that a given target image is reconstructed on the screen?

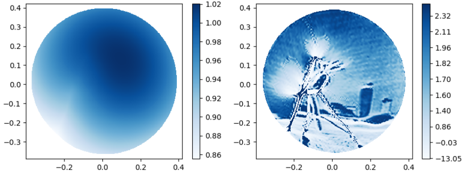
- The screen is assumed to be at infinite distance from the light source.
- This problem reduces to solving a Monge-Ampère equation (in the general form).



Application to nonimaging optics

Target image

Simulation using the appleseed® rendering engine



Shape and curvature of the lens (numerical solution)

Conclusion and perspectives

Conclusion:

- Monotone finite difference scheme for the Monge-Ampère equation of optimal transport.
- In dimension two, closed-form formula for the maximum at the discrete level, which improves the efficiency of the scheme.
- Existence of solutions, and convergence in the setting of quadratic optimal transport.

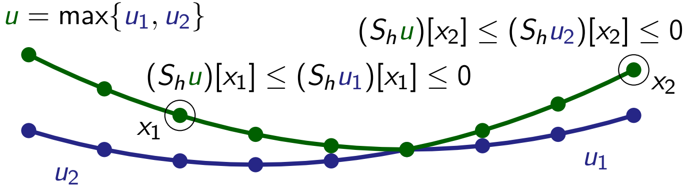
Perspectives:

- Adaptation of the closed-form formula to other equations (see Bonnans, B., Mirebeau, 2021 for the Pucci equation).
- Convergence for Monge-Ampère problems with non absolutely continuous source measures or with general transport costs.
- Analysis for yet more general equations, for instance $\det\left(\nabla^2 u(x) - A(x, u(x), \nabla u(x))\right) = B(x, u(x), \nabla u(x))$.

Thank you for your attention.

Monotone schemes

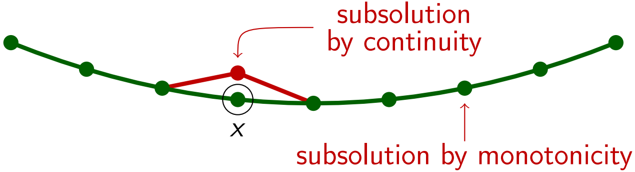
- On a grid $\mathcal{G}_h$, a scheme $S_h \colon \mathbb{R}^{\mathcal{G}_h} \to \mathbb{R}^{\mathcal{G}_h}$ is monotone if its residue $(S_h u)[x]$ at point $x \in \mathcal{G}_h$ is nonincreasing with respect to the values $\{u(y) \mid y \in \mathcal{G}_h,\, y \neq x\}$.
- If the scheme is monotone, then the maximum $u = \max\{u_1, u_2\}$ of two subsolutions u_1 and u_2 remains a subsolution.



- By a compactness argument, a finite-valued supremum of subsolutions is still a subsolution (if the scheme operator S_h is continuous).

Perron's method

- Perron's method: If the scheme operator S_h is monotone and continuous, and if the pointwise supremum u of all subsolutions is finite-valued, then u is a solution to the scheme.
- Sketch of proof:
  1. We already know that u is a subsolution.
  2. If it is not a solution, then there exists $x \in \mathcal{G}_h$ such that $(S_h u)[x] < 0$.
  3. Then by perturbation we can build a subsolution $\hat{u}$ such that $\hat{u}(x) > u(x)$.
  4. Impossible since u is the pointwise supremum of all subsolutions.



Extension to our setting

- Scheme in our setting: $(S_h^\alpha u)[x] = 0$ in $\mathcal{G}_h$, where:
  - Unknowns are $u \in \mathbb{R}^{\mathcal{G}_h}$ and $\alpha \in \mathbb{R}$.
  - For fixed $\alpha \in \mathbb{R}$, the operator $S_h^\alpha \colon \mathbb{R}^{\mathcal{G}_h} \to \mathbb{R}^{\mathcal{G}_h}$ is monotone.
- Perron's method is not directly applicable since:
  - The scheme is not monotone with respect to the pair (u, α).
  - (Moreover for fixed α, the pointwise supremum of all subsolutions u is everywhere $+\infty$.)
- Definition: a subsolution (u, α) to the scheme is a solution to $(S_h^\alpha u)[x] \leq 0$ in $\mathcal{G}_h$.
- Stability property: there is $\alpha_* \in \mathbb{R}$ such that $\alpha \leq \alpha_*$ for all subsolutions to the scheme.
  - (Remark: similarly, one has $\alpha \leq 0$ for all subsolutions to the continuous problem.)
- Stability + compactness $\implies$ there exists a subsolution $(\overline{u}, \alpha)$ which maximizes α among all subsolutions.

Proof of existence of solutions

1. There exists a nonempty set $\mathcal{G}_h^*$ on which $(S_h^\alpha \overline{u})[x] = 0$, since otherwise α could be increased.
2. Let $\tilde{u}$ be the pointwise supremum of u such that (u, α) is subsolution and $u = \overline{u}$ on $\mathcal{G}_h^*$ (1).
3. By Perron's argument, one has $(S_h^\alpha \tilde{u})[x] = 0$ on $\mathcal{G}_h \setminus \mathcal{G}_h^*$.
4. One of the following holds:
   - $(S_h^\alpha \tilde{u})[x] = 0$ on $\mathcal{G}_h^*$. Then $(\tilde{u}, \alpha)$ is a solution.
   - There exists $x \in \mathcal{G}_h^*$ such that $(S_h^\alpha \tilde{u})[x] < 0$. Then one can build a perturbation $\hat{u}$ of $\tilde{u}$ (2) for which the cardinal of $\mathcal{G}_h^*$ is reduced upon taking $\overline{u} \leftarrow \hat{u}$ (3). Repeat with $\overline{u} \leftarrow \hat{u}$.