Inégalités optimales pour l'échantillonnement d'intégrales en termes de distances de Wasserstein et normes du gradient

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Wasserstein distances and Wasserstein spaces – 1

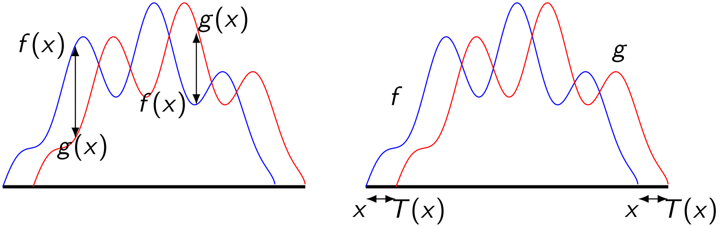
Let's start from Kantorovich optimal transport' :

$$\min\Big\{\int_{\Omega\times\Omega} c(x,y)d\gamma(x,y)\ :\ \gamma\in\Pi(\mu,\nu)\Big\},\qquad \textbf{(KP)}$$

Starting from the values **(KP)** we can define a set of distances over $\mathcal{P}(X)$, in the following way : for any $p\in[1,+\infty[$ set

$$W_p(\mu,\nu)=\big(\min\ \ \textbf{(KP)}\text{ with } c(x,y)=|x-y|^p\big)^{1/p}.$$

Compared to L^p distances between densities we can say that they are "horizontal" instead of "vertical".



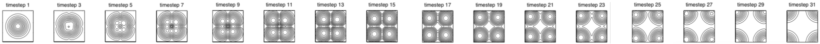
L. Kantorovich, On the transfer of masses, 1942.

Wasserstein distances and Wasserstein spaces – 2

There is also a dynamical formulation whenever $\Omega \subset \mathbb{R}^d$ is convex :

$$W_p^p(\mu,\nu) = \inf\left\{ \int_0^1 \int_\Omega |v_t|^p \, \rho_t \, dt \ : \ \partial_t \rho + \nabla \cdot (\rho v) = 0, \ \rho_0 = \mu, \rho_1 = \nu \right\}$$

This kinetic energy minimization is the so-called *Benamou-Brenier* formulation, which is also a valuable tool for numerical computations.



The measures ρ_t which minimize the Benamou-Brenier formulation are indeed constant-speed geodesics (they satisfy $\int |v_t|^p d\rho_t = W_p^p(\mu,\nu)$ for every t) connecting μ to ν and they have an explicit expression

$$\rho_t = ((1-t)id + tT)_\# \mu,$$

where T is the optimal transport map from μ to ν.

J.-D. Benamou, Y. Brenier A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, *Numer. Math.*, 2000.

Wasserstein distances and negative Sobolev norms

In the case $p = 1$, i.e. $c(x, y) = |x - y|$ the dual problem

$$\sup\Big\{\int_\Omega \varphi d\mu + \int_\Omega \psi d\nu \ : \ \varphi(x) + \psi(y) \leq c(x, y)\Big\}. \qquad \textbf{(DP)}$$

becomes easier since we can restrict the maximization to $\phi \in \mathrm{Lip}_1$ and $\psi = -\phi$, so that we get

$$W_1(\mu, \nu) = \sup\Big\{\int_\Omega \phi d(\mu - \nu) \ : \ \phi \in \mathrm{Lip}_1(\Omega)\Big\}.$$

We then have

$$||\mu - \nu||_{(\mathrm{Lip})'} \leq W_1(\mu, \nu) \quad \text{i.e.} \quad \left|\int_\Omega \phi d(\mu - \nu)\right| \leq ||\nabla\phi||_{L^\infty} W_1(\mu, \nu).$$

This suggests a form of duality between $W^{1,q}$ and W_p. Can this be generalized ?

The general answer should be no : for $q \leq d$ Sobolev functions are not continuous and cannot be integrated against arbitrary measures. But some results can hold under additional assumptions on μ and/or ν.

Geodesic convexity and estimate for L^∞ measures.

Given $r > 1$, set $F_r(\rho) := \int_\Omega \rho(x)^r dx$. For every $p > 1$ and every $r > 1$, F_r is *geodesically convex*, i.e.

$$t \mapsto F_r(\rho_t) \quad \text{is convex}$$

when ρ_t is the constant-speed geodesic between two measures μ and ν. Sending $r \to \infty$, we also have

$$||\rho_t||_{L^\infty} \leq \max\{||\mu||_{L^\infty}, ||\nu||_{L^\infty}\}.$$

We can then deduce

$$\int_\Omega \phi d(\nu - \mu) = \int_0^1 \frac{d}{dt} \int_\Omega \phi d\rho_t = \int_0^1 \int_\Omega \nabla\phi \cdot v_t d\rho_t,$$

hence, with $p' = p/(p-1)$, if we suppose $\mu, \nu \leq C$, we have

$$\left|\int_\Omega \phi d(\mu - \nu)\right| \leq W_p(\mu,\nu) \int_0^1 \left(\int_\Omega |\nabla\phi|^{p'} d\rho_t\right)^{1/p'} \leq C^{1/p'} W_p(\mu,\nu)||\nabla\phi||_{L^{p'}}.$$

Generalizations also exist using L^r norms : $\left|\int \phi d(\mu-\nu)\right| \leq C^{1/p'} W_p(\mu,\nu)||\nabla\phi||_{L^q}$ if $||\mu||_{L^r}, ||\nu||_{L^r} \leq C$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{pr}$.

G. Loeper Uniqueness of the solution to the Vlasov-Poisson system with bounded density, *JMPA*, 2006.
F. Santambrogio Optimal Transport for Applied Mathematicians, 2015.

One-sided estimates (looking at grids)

What if μ is nice and ν is not, as it is the case when estimating the difference $\int_\Omega \phi dx - \frac{1}{N}\sum \phi(x_i)$, i.e. when discretizing integrals ?
The points x_i are often taken on a regular grid of step $N^{-1/d}$ and all Wasserstein distances $W_p(dx, \frac{1}{N}\sum_i \delta_{x_i})$ are of order $N^{-1/d}$. We then look at the case where p is very large, trying to obtain estimates with the smallest possible norm on $\nabla\phi$. Let's take $p = \infty$:

$$\begin{aligned} W_\infty(\mu,\nu) &:= \min\{||x-y||_{L^\infty(\gamma)} \,:\, \gamma \in \Pi(\mu,\nu)\} \\ &= \min\{||v||_{L^\infty} \,:\, \partial_t\rho + \nabla\cdot(\rho_t v_t) = 0\}. \end{aligned}$$

We then have

$$\left|\int_\Omega \phi d(\mu-\nu)\right| \leq W_\infty(\mu,\nu)\int_0^1\int_\Omega |\nabla\phi| d\rho_t = W_\infty(\mu,\nu)\int_\Omega |\nabla\phi| dm,$$

with $m = \int_0^1 \rho_t dt$. What can we say about the integrability of m ?
A simple scaling argument shows that, even in the worst case ($\nu = \delta_0$), we have $||\rho_t||_{L^r} \leq (1-t)^{-d/r'}||\mu||_{L^r}$. Hence, $m \in L^r$ for every $r < d'$.
F. Santambrogio Absolute continuity and summability of transport densities : simpler proofs and new estimates, 2009.

Sharp and Lorentz estimates – 1

From the previous estimate we easily obtain, for $p > d$

$$\left|\int_\Omega \phi d(\mu - \nu)\right| \leq W_\infty(\mu, \nu)||\nabla\phi||_{L^p}\frac{p}{p-d}||\mu||_{L^{p'}}.$$

Yet, for $\mu = dx$ and $\nu = \delta_0$, the measure m exploses at 0 as $|x|^{1-d}$. This is the limit behavior for not being in $L^{d'}$. But we can obtain better in the class of Lorentz spaces $L^{p,q}$ if we find the Lorentz space to which m belongs.

Lorentz summability of m was already considered by Dweik for transport densities, who proved $\mu \in L^{p,q} \Rightarrow m \in, L^{p,q}$ for $r < d'$ but did not consider the limit case $p = d'$.

S. Dweik $L^{p,q}$ estimates on the transport density, *Comm. Pures Appl. An.*, 2019.

Sharp and Lorentz estimates – 2

Let us recall the definition of Lorentz spaces and norms :

$$||u||_{L^{p,q}} \approx ||s \mapsto s|\{|u| > s\}|^{1/p}||_{L^q(\mathbb{R}_+, \frac{ds}{s})}.$$

In particular,

$$||u||_{L^{p,\infty}} \approx \sup_s s|\{|u| > s\}|^{1/p} \quad ; \quad ||u||_{L^{p,1}} \approx \int_0^\infty |\{|u| > s\}|^{1/p} ds.$$

We can prove $(L^{p,q})' = L^{p',q'}$. Since $|x|^{1-d}$ belongs to $L^{d',\infty}$ around the origin, the conjecture was

$$\left|\int_\Omega \phi dx - \int_\Omega \phi d\nu\right| \leq C(d,\Omega) W_\infty(\mu,\nu) ||\nabla\phi||_{L^{d,1}}.$$

S. Steinerberger On a Kantorovich-Rubinstein inequality, *JMAA*, 2021.

The final result

We can prove

$$\mu \in L^{d',1} \Rightarrow m \in L^{d',\infty}$$

(with a small loss in the q-exponent, differently from the case $r < d'$). This is sharp in the sense that for $\mu = |x|^{1-d}dx$ on B_1 we have $\mu \in L^{d',\infty}$ but $m \approx |x|^{1-d}(1 - \log(|x|)) \notin L^{d',\infty}$. The result also applies to the transport density, but sharpness is not clear.

Anyway, we obtain

$$\left| \int_\Omega \phi d(\mu - \nu) \right| \leq C(d) ||\mu||_{L^{d',1}} W_\infty(\mu, \nu) ||\nabla \phi||_{L^{d,1}},$$

thus proving Steinerberger's conjecture.

Don't think I'm done, let me give some proofs

Thanks for your attention

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