From Euler's Hydrodynamics to Einstein's Gravitation through Monge's Optimal Transport

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General Relativity GR and Optimal Transport OT

Recent works linking GR and OT:
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Lott-Sturm-Villani OT definition of Ricci curvature.
Our approach is different and related to the
hydrodynamical formulation of OT (Benamou-B. 2000).

ref: Y.B. CRAS 2021 (https://hal.archives-ouvertes.fr/hal-03311171).

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Euler 1757 : the 1st multiD evolution PDEs ever writen...

XXI. Nous n'avons donc qu'à égaler ces forces accélératrices avec les accélerations actuelles que nous venons de trouver, & nous obtiendrons les trois équations suivantes :

$$P - \frac{1}{q}\left(\frac{dp}{dx}\right) = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right)$$

$$Q - \frac{1}{q}\left(\frac{dp}{dy}\right) = \left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right)$$

$$R - \frac{1}{q}\left(\frac{dp}{dz}\right) = \left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right)$$

Si nous ajoutons à ces trois équations premièrement celle, que nous a fournie la consideration de la continuité du fluide :

$$\left(\frac{dq}{dt}\right)+\left(\frac{d.qu}{dx}\right)+\left(\frac{d.qv}{dy}\right)+\left(\frac{d.qw}{dz}\right)=0.$$

Si le fluide n'étoit pas compreſſible, la denſité q ſeroit la même en Z, & en Z′, & pour ce cas on auroit cette équation :

$$\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=0.$$

qui eſt auſſi celle ſur laquelle j'ai établi mon Mémoire latin allégué ci-deſſus.

tombent dans la ſurface même. Or nous voyons par là ſuffiſamment, combien nous ſommes encore éloignés de la connoiſſance complette du mouvement des fluides, & que ce que je viens d'expliquer, n'en contient qu'un foible commencement. Cependant tout ce que la Théorie des fluides renferme, eſt contenu dans les deux équations rapportées cy-deſſus (§. XXXIV.), de ſorte que ce ne ſont pas les principes de Méchanique qui nous manquent dans la pourſuite de ces recherches, mais uniquement l'Analyſe, qui n'eſt pas encore aſſés cultivée, pour ce deſſein: & partant on voit clairement, quelles découvertes nous reſtent encore à faire dans cette Science, avant que nous puiſſions arriver à une Théorie plus parfaite du mouvement des fluides.

Euler's conclusion still correct after 264 years

The least action principle for the Euler equations

It amounts to looking for fields
$(t,x) \in \mathbb{R} \times \mathbb{R}^d \to (\rho, v)(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, critical points

$$\text{of} \quad \int \left(\frac{\rho |v|^2}{2} - \Phi(\rho)\right) dx dt \quad (\text{where} \;\; r\Phi''(r) = p'(r))$$

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Example: $p(r) = c^2 r$, $\Phi(r) = c^2 r(logr - 1)$.

The pressureless case and the Monge problem

In the limit case $p = 0$, the action principle reads

$$\mathrm{crit}\ \left(\int \frac{\rho |v|^2}{2} dxdt \right), \quad \text{s.t.} \ \ \partial_t \rho + \nabla \cdot (\rho v) = 0.$$

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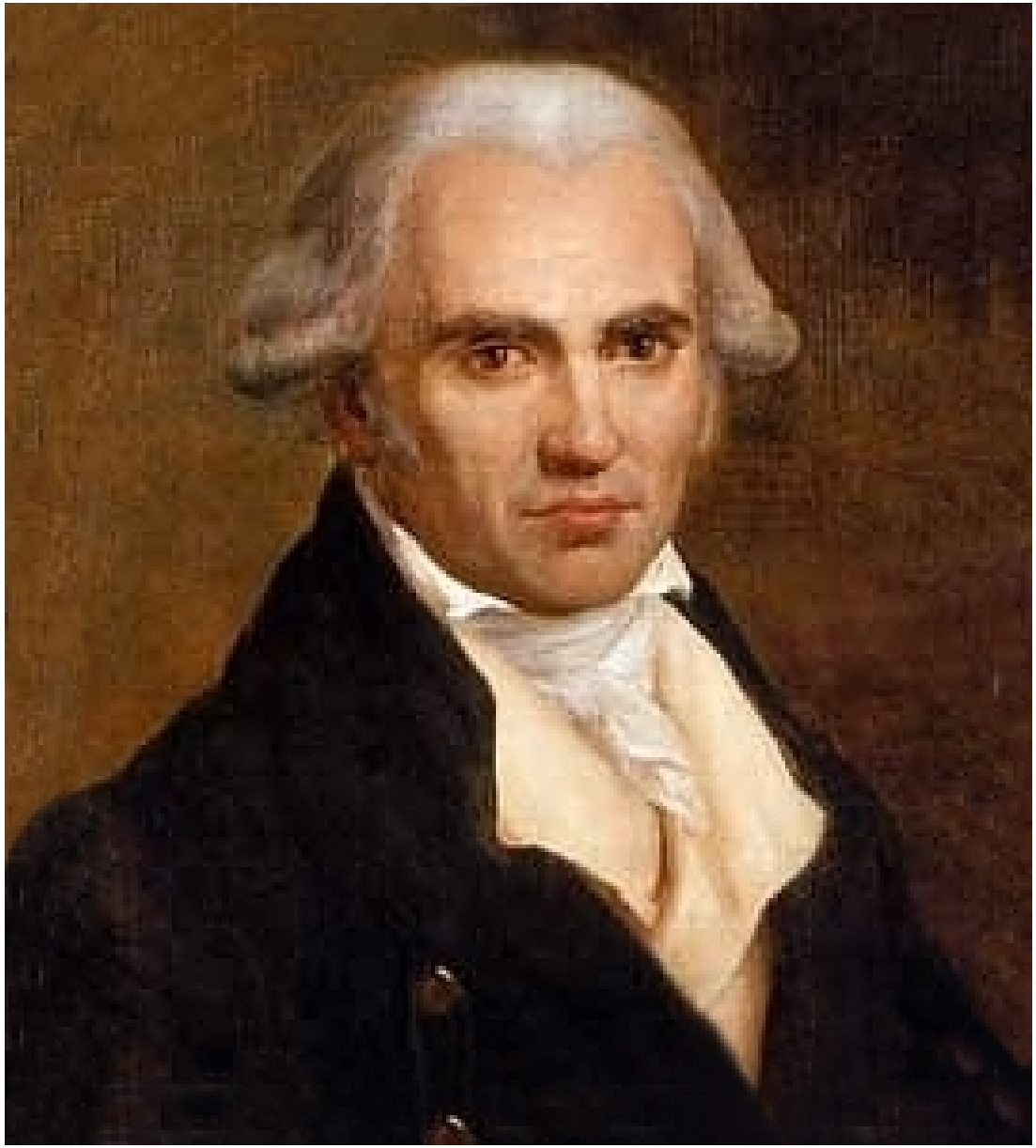
This is very different from the initial value problem and rather fits to the optimal transport problem of Monge

(problème des déblais et des remblais, 1780, cf. books by Villani and Santambrogio).



The (quadratic) Monge OT problem

$$Monge_2(\rho_0, \rho_1)^2 = \inf \int_{\mathbb{R}^d} |T(x) - x|^2 \rho_0(x) dx,$$

for all Borel maps T for which $\rho_1(y)dy$ is the image by $y = T(x)$ of $\rho_0(x)dx$.

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Then, one can show:

$$Monge_2(\rho_0, \rho_1)^2 = \inf \int_0^1 dt \int_{\mathbb{R}^d} \rho(t,x)|v(t,x)|^2 dx,$$

where (ρ, v) is subject to $\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \rho(0,\cdot) = \rho_0 \quad \rho(1,\cdot) = \rho_1$.

(Benamou-B. 2000, see also Otto 2001, Ambrosio-Gigli-Savaré 2005.)

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N.B. Optimality equations read: $v = \nabla \phi, \quad \partial_t v + \nabla(|v|^2/2) = 0$.

Proposal: a matrix-valued generalization of OT

We look for 4×4 real matrix-valued fields $(C, V)(x, \xi)$ over the phase-space $(x, \xi) \in \mathbb{R}^8$, critical points of

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and the linear symmetry constraints $\partial_{\xi^i} V^k_j = \partial_{\xi^j} V^k_i$,

$$\partial_{\xi^m} C^\gamma_\gamma \, \delta^j_k - 3\partial_{\xi^m} C^j_k = \partial_{\xi^k} C^\gamma_\gamma \, \delta^j_m - 3\partial_{\xi^k} C^j_m.$$

Theorem: Let (g, Γ) be a smooth solution to the Einstein equations in vacuum. Let us define

$$V^j_k(x,\xi) = -\Gamma^j_{k\gamma}(x)\xi^\gamma,$$

$$C^j_k(x,\xi) = \partial_{\xi^k} A^j(x,\xi) - \partial_{\xi^q} A^q(x,\xi)\ \delta^j_k,$$

$$A^j(x,\xi) = \xi^j\ \mathrm{det} g(x)\ \cos\left(\frac{g_{\alpha\beta}(x)\xi^\alpha\xi^\beta}{2}\right).$$

Then (C, V) solves the matrix-valued OT problem.

ref: to appear in CRAS, https://hal.archives-ouvertes.fr/hal-03311171


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Comment: Einstein vs Monge

$$(x,\xi) \in \mathbb{R}^{4+4} \;\Leftarrow\; (t,x) \in \mathbb{R}^{1+d} \;\;;\; C^j_k(x,\xi) \;\;\Leftarrow\;\; \rho(t,x)$$

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SO, OUR MATRIX-VALUED OT FORMULATION OF GRV SOUNDS LIKE CONTINUUM MECHANICS (A LA EULER, D'ALEMBERT, LAGRANGE, CAUCHY) MUCH MORE THAN LORENTZIAN GEOMETRY!

Remark: as a by-product, we also found a strikingly similar OT form of the (free) Schrödinger equation (based on the 'Madelung transform'):

We look for complex fields $(C, V_1, \cdots, V_d)(t, x)$ over space-time $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, critical points of

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subject to the complex 'dispersive' continuity equation

$$\partial_t C + \sum_{k=1}^{d} \partial_{x^k}(C V_k) + i \Delta C = 0.$$

Back to General Relativity (in vacuum)

'Free fall' follows geodesic curves $s \in \mathbb{R} \to x(s) \in \mathbb{R}^4$

$$\text{i.e. critical points of } \int g_{ij}(x(s)) \frac{dx^i(s)}{ds} \frac{dx^j(s)}{ds} ds$$

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$$\frac{dx^i(s)}{ds} = \xi^i(s), \quad \frac{d\xi^i(s)}{ds} = -\Gamma^i_{jk}(x(s))\xi^j(s)\xi^k(s),$$

$$2g_{mi}\Gamma^i_{jk} + \partial_m g_{jk} - \partial_j g_{mk} - \partial_k g_{mj} = 0.$$

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Einstein equations in vacuum
just means that g has zero Ricci curvature.

OT form of GRV: some ingredients of the proof

Key idea: view Γ as a collection of 4 vector fields over the phase space $(x, \xi) \in \mathbb{R}^8$ which are linear in ξ:
$V_k^j(x, \xi) = -\Gamma^j_{k\gamma}(x)\xi^\gamma,$

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$$R^n_{jkm}(x)\xi^m = \left((\partial_{x^k} + V^\gamma_k \partial_{\xi^\gamma}) V^n_j - (\partial_{x^j} + V^\gamma_j \partial_{\xi^\gamma}) V^n_k \right)(x,\xi)$$

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The zero-Ricci curvature 'phase-space' equation

$$\partial_{x^k} V^j_j + \partial_{\xi^j}(V^\gamma_k V^j_\gamma) - \partial_{x^j} V^j_k - \partial_{\xi^k}(V^\gamma_j V^j_\gamma) = 0$$

is going to play for GRV the role taken by the multiD Burgers equation $\partial_t v + \nabla(|v|^2/2) = 0$ for the quadratic OT problem in its hydrodynamical form.

A toy model : the multiD Burgers equation (1/4)

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Ignoring BC, let us look for critical points (A, V) of

$$\int \left(-\partial_t A \cdot V - \frac{(\nabla \cdot A)|V|^2}{2} \right) dxdt, \quad A = A(t,x) \in \mathbb{R}^d.$$

A toy model : the multiD Burgers equation (2/4)

Critical points (A, V) of

$$\mathcal{I}(A, V) = \int \left(-\partial_t A \cdot V - \frac{(\nabla \cdot A)|V|^2}{2} \right) dxdt.$$

$$\partial_A \mathcal{I}(A, V) = 0 \;\Rightarrow\; (1) \quad \partial_t V + \nabla\left(\frac{|V|^2}{2}\right) = 0$$

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$$\partial_V \mathcal{I}(A, V) = 0 \;\Rightarrow\; (2) \quad \partial_t A + V(\nabla \cdot A) = 0$$

(additional information that we are now going to use).

A toy model : the multiD Burgers equation (3/4)

We use (2) $\partial_t A + V(\nabla \cdot A) = 0$ to rewrite $\mathcal{I}(A, V)$ as:

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Assuming that (A, V) is critical for $\mathcal{I}(A, V)$, we have

$\partial_t A + V(\nabla \cdot A) = 0$ and $\partial_t V + \nabla(|V|^2/2) = 0$.

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$\partial_A \mathcal{L}(A, V, B) = 0 \;\; \Rightarrow \;\; -\nabla(|V|^2/2) + \partial_t B + \nabla(B \cdot V) = 0$.

Assuming that (A, V) is critical for $\mathcal{I}(A, V)$, we have

$\partial_t A + V(\nabla \cdot A) = 0$ and $\partial_t V + \nabla(|V|^2/2) = 0$. Setting $B = V$, we are just in business!

A toy model : the multiD Burgers equation (4/4)

Let us now write everything in terms of $(\rho = \nabla \cdot A, V)$:

$$(2) \quad \partial_t A + V(\nabla \cdot A) = 0 \;\Rightarrow\; \partial_t \rho + \nabla \cdot (\rho V) = 0,$$

$$\mathcal{I}_2(A, V) = \int \frac{(\nabla \cdot A)|V|^2}{2} dxdt \;\Rightarrow\; \int \frac{\rho |V|^2}{2} dxdt.$$

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