Strong c-concavity and stability in optimal transport

MAGA Days

Anatole Gallouet
Quentin Mérigot   Boris Thibert

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Laboratoire Jean Kuntzmann, Grenoble

Outline

1. Introduction
2. Stability of transport maps under strong c-concavity
3. Sufficient condition for strong c-concavity
4. Reflector cost on the sphere

Introduction

Motivation

Numerical optimal transport naturally raises questions on the stability of the optimal transport maps with respect to the measures, namely if we approximate $T : \mu \to \nu$ by $\tilde{T} : \tilde{\mu} \to \tilde{\nu}$ we want

$$d(T, \tilde{T}) \leq d((\mu, \nu), (\tilde{\mu}, \tilde{\nu}))$$

Some stability results in optimal transport:

- (Ambrosio Gigli '09) Local stability near Lipschitz transport map
- (Berman '18) Global stability
- (Merigot Delalande Chazal '19) Global stability, independant of dimension
- (Li Nochetto '20) Local stability with discretization of both source and target

Local stability with respect to target measure

Theorem (Ambrosio-Gigli '09)

Let $\mathcal{X}$ and $\mathcal{Y}$ be compact domains of $\mathbb{R}^d$. Let $i \in \{0,1\}$ and $T_i : \mathcal{X} \to \mathcal{Y}$ optimal transports maps between measures μ to ν_i for the cost $c(x,y) = \|x-y\|^2$. Assuming that μ and ν_0 are absolutely continuous and T_0 is K-Lipschitz, then

$$\|T_1 - T_0\|^2_{L^2(\mu)} \leq 4 M_{\mathcal{X}} K W_1(\nu_0, \nu_1)$$

- The map T_0 is K-Lipschitz if and only if its associated potential $\psi_0 : \mathcal{Y} \to \mathbb{R}$ is $1/K$ strongly convex. It typically implies that $\mathrm{spt}(\nu_0)$ is connected.
- (Li-Nochetto '20) have a similar result with a discretization of both measures.

Global stability

Theorem (Berman 18')

Assume μ is the Lebesgue measure on $\mathcal{X}$ convex and compact, and $\nu_0, \nu_1 \in \mathcal{P}(\mathcal{Y})$ with $\mathcal{Y}$ compact. Then

$$\| T_1 - T_0 \|^2_{L^2(\mu)} \leq C W_1(\nu_0, \nu_1)^{\alpha} \quad \textit{with} \quad \alpha = \frac{1}{2^{d-1}(d+1)}$$

- (Merigot Delalande Chazal '19) Have the same result with $\alpha = 1/6$
- Open problem: Can $\alpha = 1/6$ be upgraded ? The theoretical bound is $1/2$.

Framework

Here we work on manifolds instead of domains, and cost functions that are not necessarly the squared distance:

- $M, N \subset \mathbb{R}^n$ be two d dimensionnal manifolds and $c : M \times N \to \mathbb{R}$.
- $\mu \in \mathcal{P}(M)$, $\nu \in \mathcal{P}(N)$ are two absolutely continuous measures
- $\mathcal{X} = \mathrm{spt}(\mu)$ and $\mathcal{Y} = \mathrm{spt}(\nu)$ are two compact set
- $T : \mathcal{X} \to \mathcal{Y}$ is a transport map if $T_{\#}\mu = \nu$
- $c \in \mathcal{C}^4(D)$ with $D \subset M \times N$ compact and $\mathcal{X} \subset \mathrm{proj}_M(D)$, $\mathcal{Y} \subset \mathrm{proj}_N(D)$

c-transforms, c-superdifferential and c-concavity

Definition (c-transform)

Let $\varphi : \mathcal{X} \to \mathbb{R}$ and $\psi : \mathcal{Y} \to \mathbb{R}$

$$\varphi^c(y) = \inf_{x \in \mathcal{X}} c(x,y) - \varphi(x) \qquad \psi^c(x) = \inf_{y \in \mathcal{Y}} c(x,y) - \psi(y)$$

Definition (c-superdifferential)

$$\partial^c \psi(y) = \{x \in \mathcal{X} | \psi^c(x) + \psi(y) = c(x,y)\}$$

Definition (c-concavity)

We say that $\psi : \mathcal{Y} \to \mathbb{R}$ is c-concave if there exists $\phi : \mathcal{X} \to \mathbb{R}$ such that $\psi = \phi^c$, which means that for any $y \in \mathcal{Y}$, $\psi(y) = \inf_{x \in \mathcal{X}} c(x,y) - \phi(x)$.

Strong c-concavity

Note that ψ c-concave implies $\forall y : \partial^c \psi(y) \neq \emptyset$

Definition (strong c-concavity)

We say that a c-concave function $\psi : \mathcal{Y} \to \mathbb{R}$ is strongly c-concave with modulus ω if for all $y, z \in \mathcal{Y}$, and $x \in \partial^c \psi(y)$

$$\psi(z) \leq \psi(y) + c(x, z) - c(x, y) - \omega(y, z)$$

where $\partial^c \psi(y) = \{x \in \mathcal{X} | \psi^c(x) + \psi(y) = c(x, y)\}$

For $c(x, y) = -\langle x | y \rangle$ and $\omega(y, z) = C \|y - z\|^2$, we have $\partial^c \psi(y) = \nabla \psi(y)$ which gives

$$\psi(z) \leq \psi(y) + \langle \nabla \psi(y) | y - z \rangle - C \|y - z\|^2$$

Stability of transport maps under strong c-concavity

Local stability with respect to target

Theorem (Generalizes Ambrosio-Gigli '09)

$T_i : \mathcal{X} \to \mathcal{Y}$ optimal transports maps from μ to ν_i, associated with c-concave potentials $\psi_i : \mathcal{Y} \to \mathbb{R}$ with Lipshitz constant L. ψ_1 strongly c-concave with modulus ω. Then we have

$$\int_{\mathcal{X}} \omega(T_0(x), T_1(x)) d\mu(x) \leq 2LW_1(\nu_0, \nu_1)$$

In particular if $\omega(y, z) = C \|y - z\|^2$ for some $C > 0$, then the results writes

$$\|T_1 - T_0\|^2_{L^2(\mu)} \leq \frac{2L}{C} W_1(\nu_0, \nu_1)$$

Proof

Let $A = \int_{\mathcal{Y}} \psi_1 d(\nu_1 - \nu_0)$ and $B = \int_{\mathcal{Y}} \psi_0 d(\nu_0 - \nu_1)$.
Since $T_{i\#}\mu = \nu_i$ and $x \in \partial^c \psi_i(T_i(x))$ we have by c-concavity

$$\begin{aligned} A &= \int_{\mathcal{X}} \psi_1(T_1(x)) d\mu(x) - \int_{\mathcal{X}} \psi_1(T_0(x)) d\mu(x) \\ &\geq \int_{\mathcal{X}} c(x, T_1(x)) - c(x, T_0(x)) + \omega(T_0(x), T_1(x)) d\mu \end{aligned}$$

$$B \geq \int_{\mathcal{X}} -c(x, T_1(x)) + c(x, T_0(x)) d\mu$$

$$\begin{aligned} \int_{\mathcal{X}} \omega(T_0(x), T_1(x)) d\mu(x) &\leq \int_{\mathcal{Y}} \psi_1 - \psi_0 d(\nu_1 - \nu_0) \leq \max_{\mathrm{Lip}(f) \leq 2L} \int_{\mathcal{Y}} f d(\nu_1 - \nu_0) \\ &\leq 2L \max_{\mathrm{Lip}(f) \leq 1} \int_{\mathcal{Y}} f d(\nu_1 - \nu_0) \leq 2L W_1(\nu_0, \nu_1) \end{aligned}$$

Stability of transport plans

Theorem (Li-Nochetto '20)

Let $\mathcal{X}$ and $\mathcal{Y}$ be domains of $\mathbb{R}^d$. Let $\varphi : \mathcal{X} \to \mathbb{R}$ such that $T := \nabla\varphi$ is an optimal transport map between absolutely continuous measures μ and ν, and assume that φ^* is $1/\lambda$ strongly convex. Then for any $\gamma \in \Gamma(\mu,\nu)$:

$$\int_{\mathcal{X}\times\mathcal{Y}} \|y - T(x)\|^2 \, d\gamma(x,y) \leq \lambda \left(\int_{\mathcal{X}\times\mathcal{Y}} \|x-y\|^2 \, d\gamma(x,y) - \int_{\mathcal{X}} \|T(x) - x\|^2 \, d\mu(x) \right)$$

- Similar to (Ambrosio Gigli '09), but for transport plans
- Left hand size is a distance between T and γ

Stability of transport plans

Theorem (Generalizes Li-Nochetto 20)

Let $T : \mathcal{X} \to \mathcal{Y}$ optimal transport map from μ to ν, with associated potential $\psi : \mathcal{Y} \to \mathbb{R}$ strongly c-concave with modulus ω. Then for any $\gamma \in \Gamma(\mu, \nu)$:

$$\int_{\mathcal{X}\times\mathcal{Y}} \omega(T(x), y) d\gamma(x, y) \leq \int_{\mathcal{X}\times\mathcal{Y}} c(x, y) d\gamma(x, y) - \int_{\mathcal{X}} c(x, T(x)) d\mu(x)$$

Corollary

For any transport map $\widetilde{T} : \mathcal{X} \to \mathcal{Y}$ between μ and ν, if $\omega(y, z) = C \|y - z\|^2$

$$C \left\| T - \widetilde{T} \right\|_{L^2(\mu)}^2 \leq \int_{\mathcal{X}} c(x, \widetilde{T}(x)) d\mu(x) - \int_{X} c(x, T(x)) d\mu(x)$$

Proof

$$\begin{aligned}
0 &= \int_{\mathcal{Y}} \psi(y) d\nu(y) - \int_{\mathcal{X}} \psi(T(x)) d\mu(x) \\
&= \int_{\mathcal{X}\times\mathcal{Y}} \psi(y) - \psi(T(x)) d\gamma(x,y) \\
&\leq \int_{\mathcal{X}\times\mathcal{Y}} c(x,y) - c(x,T(x)) - \omega(T(x),y) d\gamma(x,y) \\
&= \int_{\mathcal{X}\times\mathcal{Y}} c(x,y) d\gamma(x,y) - \int_{\mathcal{X}} c(x,T(x)) d\mu(x) - \int_{\mathcal{X}\times\mathcal{Y}} \omega(T(x),y) d\gamma(x,y)
\end{aligned}$$

Sufficient condition for strong c-concavity

Some important notions

Definition (c-exponential)

For $x \in M$, $\text{c-exp}_x = (-\nabla_x c(x, \cdot))^{-1}$

Definition (c-segment)

$$y_t = \text{c-exp}_x((1-t)p_0 + tp_1)$$

$p_0 = (-\nabla_x c)^{-1}(x, y_0) \in T_x M$ and $p_1 = (-\nabla_x c)^{-1}(x, y_1) \in T_x M$

Definition (symmetrically c-convex set)

Let $A \subset \mathcal{Y}$. $D \subset M \times N$ is symmetrically c-convex if for any $(x_0, y_0) \in D$ and $(x_1, y_1) \in D$:

$$[x_0, x_1]_{y_0} \subset D \quad \text{and} \quad [y_0, y_1]_{x_0} \subset D$$

MTW Tensor

Assume $c \in \mathcal{C}^4(D)$, The Ma-Trudinger-Wang tensor is defined for $(x_0, y_0) \in D$ and $(\zeta, \eta) \in T_x M \times T_y N$ by

$$\mathfrak{S}_c(x_0, y_0)(\zeta, \eta) = -\frac{3}{2} \frac{\partial^2}{\partial p_{\hat{\eta}}^2} \frac{\partial^2}{\partial x_{\zeta}^2} \left(c(x, \text{c-exp}_{x_0}(p))\right)\Big|_{x=x_0, p=-\nabla_x c(x_0, y_0)}$$

with $\hat{\eta} = -\nabla^2_{xy} c(x_0, y_0)\eta$.

Definition (Weak MTW hypothesis)

$\exists C > 0$ such that $\mathfrak{S}_c(x, y)(\eta, \xi) \geq -C|\langle \eta | \xi \rangle| \, \|\eta\| \, \|\xi\|$.

Definition (STwist)

The cost satisfies the strong Twist condition (STwist) if c is $\mathcal{C}^2$, $\nabla_x c$ is one-to-one and $D^2_{xy} c$ is non singular.

Differential criterion for c-convexity

Theorem (Villani 12.46)

Let $c : M \times N \to \mathbb{R}$ such that c and $\check{c}$ satisfies (STwist) and c is $\mathcal{C}^4$ on a set closed D which is symmetrically c-convex. We assume that weak MTW is satisfied on D. Let $\psi \in \mathcal{C}^2(\mathcal{Y}, \mathbb{R})$ with $\mathcal{Y} \subset \mathrm{proj}_N(D)$. If for any $y \in \mathcal{Y}$, there exists x such that $(x, y) \in D$ and

$$\begin{cases} \nabla\psi(x) + \nabla_y c(x, y) = 0 \\ D^2\psi(x) + D^2_{yy} c(x, y) \geq 0 \end{cases}$$

Then ψ is c-convex on $\mathcal{Y}$.

- Local criterion for c-convexity
- Requires strong and global hypothesis

Theorem (Strong c-concavity)

Let $D \subset M \times N$ be a symmetrically c-convex compact set. We assume that $c \in \mathcal{C}^4(D, \mathbb{R})$, that c and $\check{c}$ satisfy (STwist) and that weak MTW is satisfied on D. Let $\psi \in \mathcal{C}^3(\mathcal{Y}, \mathbb{R})$ such that the map $T : \mathcal{X} \to \mathcal{Y}$ defined by $T(x) = \mathrm{argmin}_y\, c(x, y) - \psi(y)$ is a diffeomorphism and satisfies for any $x \in \mathcal{X}$, $(x, T(x)) \in D$.

Then ψ is strongly c-concave with modulus $\omega(y, z) = C \|y - z\|^2$, i.e.

$$\forall y, z \in \mathcal{Y}, x \in \partial^c \psi(y) : c(x, y) - \psi(y) \leq c(x, z) - \psi(z) - C \|y - z\|^2$$

Remark

This theorem is a natural development of Villani 12.46

Sketch of proof

- Let $h(t) = c(\bar{x}, y_t) - \psi(y_t)$ with $\bar{x} \in \partial^c \psi(\bar{y})$, $y \in \mathcal{Y}$ and $y_t = [\bar{y}, y]_{\bar{x}}$
- We want $h(1) \geq h(0) + C \left\|y - \bar{y}\right\|^2 \iff \int_0^1 \dot{h}(t)dt \geq C \left\|y - \bar{y}\right\|^2$
- $\dot{h}(t) = \langle \nabla_{xy}^2 c^{-1}.\eta | \zeta \rangle$ and $\ddot{h}(t) = (D^2\psi(y_t) + D_{yy}^2 c(x_t, y_t))(\eta, \eta) + \int_0^1 \mathfrak{S}_c(\cdots)$
- T diffeomorphism + MTW weak $\implies \ddot{h}(t) \geq \lambda \left\|\eta\right\|^2 - C\dot{h}(t)$.
- By compactness arguments, $\left\|\eta\right\| \geq C \left\|y - \bar{y}\right\|$
- Conclude by Grönwall's lemma.

Reflector cost on the sphere

Lemma

Let $M = N = \mathbb{S}^{d-1}$ *and* $c = -\ln(1 - \langle x|y\rangle)$. *Let* μ *and* ν *be asolutely continuous measures and* $\beta > 0$ *such that* $M(\beta) \leq 1/8$ *where*

$$M(\beta) = \sup_{x \in \mathbb{S}^{d-1}} (\mu(B(x,\beta)), \nu(B(x,\beta)))$$

Then the stability theorems applies.

Proof

- c is $\mathcal{C}^4$ on $M^2 \setminus \Delta$ with $\Delta = \{(x,x), x \in \mathcal{S}^{d-1}\}$ and satisfies weak MTW.
- Δ is repulsive i.e. $c(x,x) = +\infty$.
- $D_\varepsilon = \{(x,y) \in M |\, \|x-y\| \geq \varepsilon\}$ is a symmetrically c-convex compact set.
- Let $\gamma \in \Gamma(\mu,\nu)$ optimal, then $\mathrm{spt}(\gamma) \subset D_\varepsilon$.