SIZE OF THE MEDIAL AXIS AND STABILITY OF FEDERER'S CURVATURE MEASURES

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Abstract. In this article, we study the $(d-1)$-volume and the covering numbers of the medial axis of a compact subset of $\mathbb{R}^d$. In general, this volume is infinite; however, the $(d-1)$-volume and covering numbers of a filtered medial axis (the μ-medial axis) that is at distance greater than ε from the compact set can be explicitely bounded. The behaviour of the bound we obtain with respect to μ, ε and the covering numbers of K is optimal.

From this result we deduce that the projection function on a compact subset K of $\mathbb{R}^d$ depends continuously on the compact set K, in the L^1 sense. This implies in particular that Federer's curvature measures of a compact subset of $\mathbb{R}^d$ with positive reach can be reliably estimated from a Hausdorff approximation of this subset, regardless of any regularity assumption on the approximating subset.

1. Introduction

We are interested in the following question: given a compact set K with positive reach, and a discrete approximation, is it possible to approximate Federer's curvature measures of K (see [10] or §2.2 for a definition) knowing the discrete approximation only ? A positive answer to this question has been given in [8] using convex analysis. In this article, we show that such a result can also be deduced from a careful study of the "size" — that is the covering numbers — of the medial axis.

The notion of medial axis, also known as ambiguous locus in Riemannian geometry, has many applications in computer science. In image analysis and shape recognition, the skeleton of a shape is often used as an idealized version of the shape [18], that is known to have the same homotopy type as the original shape [15]. In the reconstruction of curves and surfaces from point cloud approximations, the distance to the medial axis provides an estimation of the size of the local features that can be used to give sampling conditions for provably correct reconstruction [1]. The flow associated with the distance function d_K to a compact set K, that flows away from K toward local maxima of d_K (that lie in the medial axis of K) can be used for shape segmentation [9]. The reader that is interested by the computation and stability of the medial axis with some of these applications in mind can refer to the survey [2].

The main technical ingredient needed for bounding the covering numbers of the subsets of the medial axis that we consider is a Lipschitz regularity result for the so-called normal distance to the medial axis $\tau_K : \mathbb{R}^d \setminus K \to \mathbb{R}$. It is defined as follows: if x belongs to the medial axis of K, then $\tau_K(x) = 0$; otherwise, $\tau_K(x)$ is the infimum time t such that $x + t\nabla_x \mathrm{d}_K$ belongs to the medial axis of K. When K is a compact submanifold of class $\mathcal{C}^{2,1}$, this function is globally Lipschitz on any r-level set of the distance function to K, when the radius r is small enough [12, 14, 5]. When K is the analytic boundary of a bounded domain Ω of $\mathbb{R}^2$, the normal distance to the medial axis of $\partial\Omega$ is 2/3-Hölder on Ω [3].

However, without strong regularity assumption on the compact set K, it is hopeless to obtain a global Lipschitz regularity result for τ_K on a parallel set of K. Indeed, such a result would imply the finiteness of $(d-1)$-Hausdorff measure of the medial axis, which is known to be false — for instance, the medial axis of a generic compact set is dense.

We show however, that the normal distance to the medial axis is Lipschitz on a suitable subset of a parallel set. This enables us to prove the following theorem on the covering numbers of the μ-medial axis (see §3.1 for a definition):

Theorem 4.1. *For any compact set $K \subseteq \mathbb{R}^d$, a parameter ε smaller than the diameter of K, and η small enough,*

$$\mathcal{N}\left(\mathrm{Med}_\mu(K) \cap (\mathbb{R}^d \setminus K^\varepsilon), \eta\right) \leqslant \mathcal{N}(\partial K, \varepsilon/2)\, \mathrm{O}\left(\left[\frac{\mathrm{diam}(K)}{\eta\sqrt{1-\mu}}\right]^{d-1}\right)$$

From this theorem, we deduce a quantitative Hausdorff-stability results for projection function, which is the key to the stability of Federer's curvature measure (see Proposition 2.2):

Theorem 5.1. *Let E be a bounded open set of $\mathbb{R}^d$. The application that maps a compact subset of $\mathbb{R}^d$ to the projection function $\mathrm{p}_K \in \mathrm{L}^1(E)$ is locally h-Hölder, for any exponent h smaller than $1/(4d-2)$.*

Note that a similar result with a slightly better Hölder exponent has been obtained in [8]. However, the proofs in this article are very different and give a more geometric insight on the Hausdorff-stability of projection functions. Nonetheless, the main contribution of this article lies in Theorem 4.1.

2. Boundary measures and medial axes

2.1. Distance, projection, boundary measures.

Throughout this article, K will denote a compact set in the Euclidean d-space $\mathbb{R}^d$, with no additional regularity assumption unless specified otherwise. The *distance function* to K, denoted by $\mathrm{d}_K : X \to \mathbb{R}^+$, is defined by $d_K(x) = \min_{p\in K} \|p - x\|$. A point p of K that realizes the minimum in the definition of $\mathrm{d}_K(x)$ is called an *orthogonal projection* of x on K. The set of orthogonal projections of x on K is denoted by $\mathrm{proj}_K(x)$.

The locus of the points $x \in \mathbb{R}^d$ which have more than one projection on K is called the *medial axis* of K. Denote this set by $\mathrm{Med}(K)$. For every point x of $\mathbb{R}^d$ not lying in the medial axis of K, we let $\mathrm{p}_K(x)$ be the unique orthogonal projection of x on K. This defines a map $\mathrm{p}_K : \mathbb{R}^d \setminus \mathrm{Med}(K) \to K$, which we will refer to as the *projection function* on the compact set K.

Definition 2.1. Let K be a compact subset and E be a measurable subset of $\mathbb{R}^d$. We will call *boundary measure* of K with respect to E the pushforward of the restriction of the Lebesgue measure to E on K by the projection function p_K, or more concisely $\mu_{K,E} = \mathrm{p}_{K\#}\, \mathcal{H}^d\big|_E$.

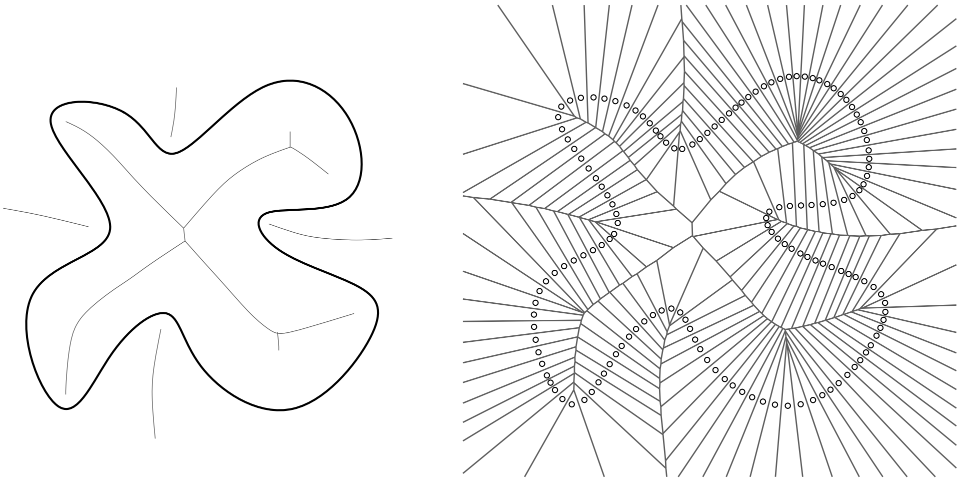

FIGURE 1. Medial axis of a curve C in the plane, and Voronoi diagram of a point cloud P sampled on the curve.

We will be especially interested in the case where E is of the form K^r, where K^r denotes the r-tubular neighborhood of K, i.e. $K^r = \mathrm{d}_K^{-1}([0, r])$.

Example 2.1 (Steiner-Minkowski)**.** If P is a convex solid polyhedron of $\mathbb{R}^3$, F its set of faces, E its set of edges and V its set of vertices, then the following formula holds:

$$\mu_{P,P^r} = \mathcal{H}^3\big|_P + r \sum_{f \in F} \mathcal{H}^2\big|_f + \frac{1}{2} r^2 \sum_{e \in E} K(e)\ \mathcal{H}^1\big|_e + \frac{1}{3} r^3 \sum_{v \in V} K(v) \delta_v$$

where $K(e)$ is the angle between the normals of the faces adjacent to the edge e, and $K(v)$ the solid angle formed by the normals of the faces adjacent to the vertex v.

For a general convex polyhedra the measure μ_{K,K^r} can similarly be written as a sum of weighted Hausdorff measures supported on the i-skeleton of K, whose local density is the local external dihedral angle.

Example 2.2 (Weyl)**.** Let M be a compact smooth hypersurface of $\mathbb{R}^d$, and denote by $\sigma_i(p)$ the ith elementary symmetric polynomial of the $(d-1)$ principal curvatures of M at a point p in M. Then, for any Borel subset B of $\mathbb{R}^d$, and r small enough, the μ_{K,K^r}-measure of B can be written as

$$\mu_{K,K^r}(B) = \sum_{i=0}^{d-1} \mathrm{const}(i, d) \int_{B \cap M} \sigma_i(p) \mathrm{d}M(p).$$

This formula can be generalized to submanifolds of any codimension [19].

2.2. **Federer curvature measures and reach.** Following Federer [10], we will call *reach* of a compact subset K of $\mathbb{R}^d$ the smallest distance between K and its medial axis, i.e. $\mathrm{reach}(K) = \min_{x \in \mathrm{Med}(K)} \mathrm{d}_K(x)$.

Generalizing Steiner-Minkowski and Weyl tubes formula, Federer proved that as long as r is smaller than the *reach* of K, the dependence in r of the boundary measure μ_{K,K^r} is a polynomial in r, of degree bounded by the ambient dimension d:

Theorem 2.1 (Federer). *For any compact set $K \subseteq \mathbb{R}^d$ with reach greater than R, there exists $(d+1)$ uniquely defined (signed) measures $\Phi_K^0, \dots, \Phi_K^d$ supported on K such that for any $r \leqslant R$,*

$$\mu_{K,K^r} = \sum_{i=0}^{d} \omega_{d-i} \Phi_K^i r^i$$

where ω_k is the volume of the k-dimensional unit sphere.

These measures are uniquely defined and Federer calls them *curvature measures* of the compact set K.

2.3. **Stability of boundary and curvature measures.** The question of the stability of boundary measures is a particular case of the more general question of geometric inference. Given a (discrete) approximation of a compact subset K of $\mathbb{R}^d$, what amount of geometry and topology of K is it possible to recover ? In our case, the question is to bound the Wasserstein distance between the boundary measures of two compact subsets as a function of their Hausdorff distance.

Recall that the *Hausdorff distance* between two compact subsets K and K' is defined by $\mathrm{d_H}(K, K') = \|\mathrm{d}_K - \mathrm{d}_{K'}\|_\infty$. The *Wasserstein distance* (of exponent one) between two measures μ and ν with finite first moment on $\mathbb{R}^d$ is defined by $\mathrm{W}_1(\mu, \nu) = \min_{X,Y} \mathbb{E}[\|X - Y\|]$ where the minimum is taken over all the couples of random variables X, Y whose law are μ and ν respectively.

Proposition 2.2. *Let E be an open subset and K, K' be two compact subsets of $\mathbb{R}^d$. Then,*

$$\mathrm{W}_1\left(\frac{\mu_{K,E}}{\mathcal{H}^d(E)}, \frac{\mu_{K',E}}{\mathcal{H}^d(E)}\right) \leqslant \frac{1}{\mathcal{H}^d(E)} \|\mathrm{p}_K - \mathrm{p}_{K'}\|_{\mathrm{L}^1(E)}$$

Proof. See [8, Proposition 3.1]. □

Hence, in order to obtain a Hausdorff stability result for boundary measures, one only needs to obtain a bound of the type $\|\mathrm{p}_K - \mathrm{p}_{K'}\|_{\mathrm{L}^1(E)} = o(\mathrm{d_H}(K, K'))$. The possibility to estimate Federer's curvature measures from a discrete approximation can also be deduced from a L^1 stability result for projection functions (see [8, §4]).

3. A first non-quantitative stability result

Intuitively, one expects that the projections $p_K(x)$ and $p_{K'}(x)$ of a point x on two Hausdorff-close compact subsets can differ dramatically only if x lies close to the medial axis of one of the compact sets. This makes it reasonable to expect a L^1 convergence property of the projections. However, since the medial axis of a compact subset of $\mathbb{R}^d$ is generically dense (see [20] or [16, Proposition I.2]), translating the above intuition into a proof isn't completely straightforward.

3.1. **Semi-concavity of d_K and μ-medial axis.** The semi-concavity of the distance function to a compact set has been remarked and used in different contexts [11, 17, 4, 15]. More precisely, we will use the fact that for any compact subset $K \subseteq \mathbb{R}^d$, the squared distance function to K is 1-concave. This is equivalent to the function $v_K : \mathbb{R}^d \to \mathbb{R}, x \mapsto \|x\|^2 - \mathrm{d}_K^2(x)$ being convex. Thanks to its semiconcavity one is able to define a notion of generalized gradient for the distance function d_K, that is defined even at points where d_K isn't differentiable.

Given a compact set $K \subseteq \mathbb{R}^d$, the subdifferential of the distance function to K at a point $x \in \mathbb{R}^d$ is by definition the set of vectors $v \in \mathbb{R}^d$ such that

$$\mathrm{d}_K^2(x+h) \leqslant \mathrm{d}_K^2(x) + \langle h|v\rangle - \|h\|^2 \tag{3.1}$$

for all $h \in \mathbb{R}^d$. The subdifferential of d_K at a point x is denoted by $\partial_x \mathrm{d}_K$, it is the convex hull of the set $\{(p-x)/\|p-x\| \;;\; p \in \mathrm{proj}_K(x)\}$.

The gradient $\nabla_x \mathrm{d}_K$ of the distance function d_K at a point $x \in \mathbb{R}^d$ is defined as the vector of $\partial_x \mathrm{d}_K$ whose Euclidean norm is the smallest, or equivalently as the projection of the origin on $\partial_x \mathrm{d}_K$ (see [17] or [15]). Given a point $x \in \mathbb{R}^d$, denote by $\gamma_K(x)$ the center and $\mathrm{r}_K(x)$ the radius of the smallest ball enclosing the set of orthogonal projections of x on K. Then,

$$\begin{aligned} \nabla_x \mathrm{d}_K &= \frac{x - \gamma_K(x)}{\mathrm{d}_K(x)} \\ \|\nabla_x \mathrm{d}_K\| &= \left(1 - \frac{\mathrm{r}_K^2(x)}{\mathrm{d}_K^2(x)}\right)^{1/2} = \cos(\theta) \end{aligned} \tag{3.2}$$

where θ is the (half) angle of the cone joining x to $\mathrm{B}(\gamma_K(x), r_K(x))$

3.2. **μ-Medial axis of a compact set.** The notion of μ-medial axes and μ-critical point of the distance function to a compact subset K of $\mathbb{R}^d$ were introduced by Chazal, Cohen-Steiner and Lieutier in [6]. We recall the definitions and properties we will need later.

A point x of $\mathbb{R}^d$ will be called a μ-critical point for the distance function to K(with $\mu \geqslant 0$), or simply a μ-critical point of K if for every $h \in \mathbb{R}^d$,

$$\mathrm{d}_K^2(x+h) \leqslant \mathrm{d}_K^2(x) + \mu \|h\| \, \mathrm{d}_K(x) + \|h\|^2 .$$

By the definition of the subdifferential of d_K (Eq. (3.1)), the point x is μ-critical iff the norm of the gradient $\|\nabla_x \mathrm{d}_K\|$ is at most μ. The *μ-medial axis* $\mathrm{Med}_\mu(K)$ of a compact set $K \subseteq \mathbb{R}^d$ is the set of μ-critical points of the distance function. Is is easily seen that the medial axis is the union of all μ-medial axes, with $0 \leqslant \mu < 1$:

$$\mathrm{Med}(K) = \bigcup_{0 \leqslant \mu < 1} \mathrm{Med}_\mu(K).$$

Moreover, from the lower semicontinuity of the map $x \mapsto \|\nabla_x \mathrm{d}_K\|$, one obtains that for every $\mu < 1$, the μ-medial axis $\mathrm{Med}_\mu(K)$ of K is a compact subset of $\mathbb{R}^d$. The main result of [6] that we will use is the following quantitative critical point stability theorem.

THEOREM 3.1 (Critical point stability theorem). *Let K, K' be two compact sets with $\mathrm{d}_{\mathrm{H}}(K, K') \leqslant \varepsilon$. For any point x in the μ-medial axis of K, there* *exists a point y in the μ'-medial axis of K' with $\mu' = \mu + 2\sqrt{\varepsilon/\mathrm{d}_K(x)}$ and $\|x-y\| \leqslant 2\sqrt{\varepsilon \mathrm{d}_K(x)}$.*

3.3. **A first non-quantitative stability result.** The goal of this paragraph is to prove the following non-quantitative L^1 convergence result for projections:

PROPOSITION 3.2. *If (K_n) Hausdorff converges to a compact $K \subseteq \mathbb{R}^d$, then for any bounded open set E, $\lim_{n\to+\infty} \|\mathrm{p}_{K_n} - \mathrm{p}_K\|_{\mathrm{L}^1(E)} = 0$.*

In order to do so, for any $L > 0$, and two compact sets K and K', we will denote $\Delta_L(K,K')$ the set of points x of $\mathbb{R}^d \setminus (K \cup K')$ whose projections on K and K' are at least at distance L, i.e. $\|\mathrm{p}_K(x) - \mathrm{p}_{K'}(x)\| \geqslant L$. For technical reasons, we remove all points of the medial axes of K and K' from $\Delta_L(K,K')$. Since the Lebesgue measure both medial axes vanishes, this does not affect the measure of $\Delta_L(K,K')$.

A consequence of the critical point stability theorem is that $\Delta_L(K,K')$ lie close to the μ-medial axis of K for a certain value of μ (this Lemma is similar to [7, Theorem 3.1]):

LEMMA 3.3. *Let $L > 0$ and K, K' be two compact sets and $\delta \leqslant L/2$ denote their Hausdorff distance. Then for any positive radius R, one has*

$$\Delta_L(K,K') \cap K^R \subseteq \mathrm{Med}_\mu(K)^{2\sqrt{R\delta}}$$

with

$$\mu = \left(1 + \left[\frac{L-\delta}{4R}\right]^2\right)^{-1/2} + 4\sqrt{\frac{\delta}{L}}$$

Proof. Let x be a point in $\Delta_L(K,K')$ with $\mathrm{d}_K(x) \leqslant R$, and denote by p and p' its projections on K and K' respectively. By assumption, $\|p - p'\|$ is at least L. We let q be the projection of p' on the sphere $\mathcal{S}(x, \mathrm{d}_K(x))$, and let K_0 be the union of K and q. By hypothesis on the Hausdorff distance between K and K', there exists a point p'' in K such that $\|p'' - p'\| \leqslant \delta$. By definition of the distance to K, $\|x - p''\| \geqslant \mathrm{d}_K(x)$: this means that $\|x - p'\| \geqslant \mathrm{d}_K(x) - \delta$. Thus, because q is the projection of p' on the sphere $\mathcal{S}(x, \mathrm{d}_K(x))$, the distance between p' and q is at most δ. Hence, $\mathrm{d_H}(K, K_0)$ is at most 2δ.

By construction, the point x has two projections on K_0, and must belong to the μ_0-medial axis of K_0 for some value of μ. Letting m be the midpoint of the segment $[p,q]$, we are able to upper bound the value of μ_0:

$$\mu_0^2 \leqslant \|\nabla_x \mathrm{d}_{K_0}\|^2 \leqslant \cos\left(\frac{1}{2}\angle(p-x, q-x)\right)^2 = \|x-m\|^2 / \|x-p\|^2$$

Since p, q belong to the sphere $\mathrm{B}(x, \mathrm{d}_K(x))$, one has $(p-q) \perp (m-x)$ and $\|x-p\|^2 = \|x-m\|^2 + \frac{1}{4}\|p-q\|^2$. This gives

$$\mu_0 \leqslant \left(1 + \frac{1}{4}\frac{\|p-q\|^2}{\|x-m\|^2}\right)^{-1/2} \leqslant \left[1 + \left(\frac{L-\delta}{2R}\right)^2\right]^{-1/2}$$

To get the second inequality we used $\|x-m\| \leqslant R$ and $\|p-q\| \geqslant L - \delta$.

In order to conclude, one only need to apply the critical point stability theorem (Theorem 3.1) to the compact sets K and K_0 with $\mathrm{d}_H(K, K_0) \leqslant 2\delta$. Since x is in the μ_0-medial axis of K_0, there should exist a point y in $\mathrm{Med}_\mu(K)$ with $\|x - y\| \leqslant 2\sqrt{R\delta}$ and $\mu = \mu_0 + 4\sqrt{\delta/L}$. □

Proof of Proposition 3.2. Fix $L > 0$, and suppose K and K' are given. One can decompose the set E between the set of points where the projections differ by at least L (i.e. $\Delta_L(K, K') \cap E$) and the remaining points. This gives the bound:

$$\|\mathrm{p}_{K'} - \mathrm{p}_K\|_{\mathrm{L}^1(E)} \leqslant L\mathcal{H}^d(E) + \mathcal{H}^d(\Delta_L(K, K') \cap E)\,\mathrm{diam}(K \cup K')$$

Now, take $R = \sup_E \|\mathrm{d}_K\|$, so that E is contained in the tubular neighborhood K^R, and fix $L = \varepsilon/\mathcal{H}^d(E)$. Then, for $\delta = \mathrm{d_H}(K, K')$ small enough (e.g. less than some δ_0), the value of μ given in Lemma 3.3 is smaller than one. Denote by μ_0 the value given by the lemma for δ_0. Then

$$\|\mathrm{p}_{K'} - \mathrm{p}_K\|_{\mathrm{L}^1(E)} \leqslant \varepsilon + \mathcal{H}^d(\mathrm{Med}_{\mu_0}(K)^{2\sqrt{R\delta}})\,\mathrm{diam}(K \cup K') \qquad (3.3)$$

Being compact, $\mathrm{Med}_{\mu_0}(K)$ is the intersection of its tubular neighborhoods. Combining this with the outer-regularity of the Lebesgue measure gives:

$$\lim_{\delta \to 0} \mathcal{H}^d(\mathrm{Med}_{\mu_0}(K)^{2\sqrt{R\delta}}) = \mathcal{H}^d(\mathrm{Med}_{\mu_0}(K)) = 0.$$

Putting this limit in equation (3.3) concludes the proof. □

4. Size and volume of the μ-medial axis

From the proof of Proposition 3.2, one can see that a way to get a quantitative stability of the projection functions is to control the volume of tubular neighborhoods of some part of the μ-medial axis. Recall that the *ε-covering number* of a subset $X \subseteq \mathbb{R}^d$ is the minimum number N of points $x_1, \dots, x_N$ such that X is contained in the union of balls $\cup_{i=1}^N \overline{\mathrm{B}}(x_i, \varepsilon)$. The following inequality is then straightforward:

$$\mathcal{H}^d(X^\varepsilon) \leqslant \mathcal{H}(\mathrm{B}(0, \varepsilon))\mathcal{N}(X, \varepsilon) \qquad (4.4)$$

Our goal in this section is to obtain a bound on the covering numbers of the considered part of the μ-medial axis (see Theorem 4.1) that will allow to control the growth of the volume of its tubular neighborhoods.

Because of its compactness, one could expect that the μ-medial axis of a well-behaved compact set will have finite $\mathcal{H}^{d-1}$-measure. This is not the case in general: if one considers a “comb”, i.e. an infinite union of parallel segments of fixed length in $\mathbb{R}^2$, such as $\mathcal{C} = \cup_{i \in \mathbb{N}^*}[0, 1] \times \{2^{-i}\} \subseteq \mathbb{R}^2$ (see Figure 2), the set of critical points of the distance fonction to $\mathcal{C}$ contains an imbricate comb. Hence $\mathcal{H}^{d-1}(\mathrm{Med}_\mu(\mathcal{C}))$ is infinite for any $\mu > 0$.

However, for any positive ε, the set of points of the μ-medial axis of $\mathcal{C}$ that are ε-away from $\mathcal{C}$ (that is $\mathrm{Med}_\mu(\mathcal{C}) \cap \mathbb{R}^d \setminus \mathcal{C}^\varepsilon$) only contains a finite union of segments, and has finite $\mathcal{H}^{d-1}$-measure. The goal of this section is to prove (quantitatively) that this remains true for any compact set. Precisely, we have:

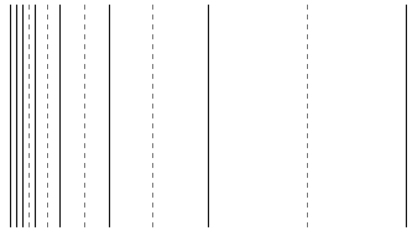

FIGURE 2. The "comb" and a part of its medial axis (dotted)

THEOREM 4.1. *For any compact set $K \subseteq \mathbb{R}^d$, $\varepsilon \leqslant \operatorname{diam}(K)$, and η small enough,*

$$\mathcal{N}\left(\operatorname{Med}_\mu(K) \cap (\mathbb{R}^d \setminus K^\varepsilon), \eta\right) \leqslant \mathcal{N}(\partial K, \varepsilon/2)\, \mathrm{O}\left(\left[\frac{\operatorname{diam}(K)}{\eta\sqrt{1-\mu}}\right]^{d-1}\right)$$

In particular, one can bound the $(d-1)$-volume of the μ-medial axis

$$\mathcal{H}^{d-1}\left(\operatorname{Med}_\mu(K) \cap (\mathbb{R}^d \setminus K^\varepsilon)\right) \leqslant \mathcal{N}(\partial K, \varepsilon/2)\, \mathrm{O}\left(\left[\frac{\operatorname{diam}(K)}{\sqrt{1-\mu}}\right]^{d-1}\right)$$

Remark (Sharpness of the bound)**.** Let x, y be two points at distance D in $\mathbb{R}^d$ and $K = \{x, y\}$. Then, $\operatorname{Med}(K)$ is simply the medial hyperplane between x and y. A point m in $\operatorname{Med}(K)$ belongs to $\operatorname{Med}_\mu(K)$ iff the cosine of the angle $\theta = \frac{1}{2}\angle(x-m, y-m)$ is at most μ.

$$\cos^2(\theta) = 1 - \frac{\|x-y\|^2}{4\mathrm{d}_K^2(m)} = 1 - \frac{\operatorname{diam}(K)^2}{4\mathrm{d}_K^2(m)}$$

Hence, $\cos(\theta) \geqslant \mu$ iff $\mathrm{d}_K(m) \leqslant \frac{1}{2}\operatorname{diam}(K)/\sqrt{1-\mu^2}$. Let z denote the midpoint between x and y; then $\mathrm{d}_K(m)^2 = \|z-m\|^2 + \operatorname{diam}(K)^2/4$. Then, $\operatorname{Med}_\mu(K)$ is simply the intersection of the ball centered at z and of radius $\frac{1}{2}\operatorname{diam}(K)\sqrt{\mu^2/(1-\mu^2)}$ with the medial hyperplane. Hence,

$$\mathcal{H}^{d-1}(\operatorname{Med}_\mu(K)) = \Omega\left(\left[\frac{\operatorname{diam}(K)\mu^2}{\sqrt{1-\mu}}\right]^{d-1}\right)$$

This shows that the behaviour in $\operatorname{diam}(K)$ and μ of the theorem is sharp as μ converges to one.

4.1. **Outline of the proof.** In order to obtain the bound on the covering numbers of the 2ε-away μ-medial axis $\operatorname{Med}_\mu(K) \cap (\mathbb{R}^d \setminus K^{2\varepsilon})$ given in Theorem 4.1, we prove that this set can be written as the image of a part of the level set ∂K^ε under the so-called *normal projection on the medial axis* $\ell : \mathbb{R}^d \setminus K \to \overline{\operatorname{Med}}(K)$.

The main difficulty is to obtain a Lipschitz regularity statement for the restriction of the map ℓ to a suitable subset of ∂K^ε. There is no such statement for the whole surface ∂K^ε in general. However, we are able to introduce a subset $S^\varepsilon_\mu \subseteq \partial K^\varepsilon$ whose image under ℓ cover the ε-away μ-medial axis, and such that the restriction of ℓ to S^ε_μ is Lipschitz. This is enough to conclude.

4.2. **Covering numbers of the μ-medial axis.** We now proceed to the proof of Theorem 4.1.

Definition 4.1. For any point $x \in \mathbb{R}^d$, we define the *normal distance of* x *to the medial axis* as $\tau_K(x) := \inf\{t \geqslant 0\,;\, x + t\nabla_x \mathrm{d}_K \in \mathrm{Med}(K)\}$. We will set $\tau_K(x)$ to zero at any point in K or in the medial axis $\mathrm{Med}(K)$.

For any time t smaller than $\tau(x)$, we denote by $\Psi_K^t(x)$ the point $\Psi_K^t(x) = x + t\nabla_x \mathrm{d}_K$. Finally, for any $x \notin K$, we let $\ell_K(x)$ be the first intersection of the half-ray starting at x with direction $\nabla_x \mathrm{d}_K$ with the medial axis. More precisely, we define $\ell_K(x) = \Psi_K^{\tau(x)}(x) \in \overline{\mathrm{Med}}(K)$.

Lemma 4.2. *Let m be a point of the medial axis* $\mathrm{Med}(K)$ *with* $\mathrm{d}(x, K) > \varepsilon$*, and x be a projection of m on ∂K^ε. Then $\ell(x) = m$.*

Proof. By definition of K^ε, $d(m, K) = d(m, K^\varepsilon) + \varepsilon$, so that the projection p of x on K must also be a projection of m on K. Hence, m, x and p must be aligned. Since the open ball $B(m, d(m, p))$ does not intersect K, for any point $y \in]p, m[$ the ball $B(y, d(y, p))$ intersects K only at p. In particular, by definition of the gradient, $\nabla_x \mathrm{d}_K$ must be the unit vector directing $]p, m[$, i.e. $\nabla_x \mathrm{d}_K = (m - x)/d(m, x)$. Moreover, since $[x, p[$ is contained in the complement of the medial axis, $\tau(x)$ must be equal to $\mathrm{d}(x, m)$. Finally one gets $\Psi^{\tau(x)}(x) = x + \mathrm{d}(x, m)\nabla_x \mathrm{d}_K = m$. □

This statement means in particular that ε-away medial axis, that is the intersection $\mathrm{Med}(K) \cap (\mathbb{R}^d \setminus K^\varepsilon)$, is contained in the image of the piece of hypersurface $\{x \in \partial K^\varepsilon\,;\, \tau_K(x) \geqslant \varepsilon\}$ by the map ℓ.

Recall that the radius of a set $K \subseteq \mathbb{R}^d$ is the radius of the smallest ball enclosing K, while the diameter of K is the maximum distance between two points in K. The following inequality between the radius and the diameter is known as Jung's theorem [13]: $\mathrm{radius}(K)\sqrt{2(1 + 1/d)} \leqslant \mathrm{diam}(K)$.

Lemma 4.3. *For any point m in the μ-medial axis* $\mathrm{Med}_\mu(K)$*, there exist two projections $x, y \in$* $\mathrm{proj}_K(m)$ *of m on K such that the cosine of the angle $\frac{1}{2}\angle(x - m, y - m)$ is smaller than $\left(\frac{1+\mu^2}{2}\right)^{1/2}$.*

Proof. We use the characterization of the gradient of the distance function given in equation (3.2). If $\mathrm{B}(\gamma_K(m), \mathrm{r}_K(m))$ denotes the smallest ball enclosing $\mathrm{proj}_K(m)$, then $\mu^2 \geqslant 1 - \mathrm{r}_K^2(m)/\mathrm{d}_K^2(m)$. Using Jung's theorem and the definition of the diameter, there must exists two points x, y in $\mathrm{proj}_K(m)$ whose distance r' is larger than $\sqrt{2}\mathrm{r}_K(m)$. The following bound on the cosine of the angle $\theta = \frac{1}{2}\angle(x - m, y - m)$ concludes the proof:

$$\cos^2(\theta) = 1 - \frac{(r'/2)^2}{d_K^2(m)} \leqslant 1 - \frac{1}{2}\frac{\mathrm{r}_K^2(m)}{d_K^2(m)} \leqslant (1 + \mu^2)/2 \tag{4.5}$$

□

Lemma 4.4. *The maximum distance from a point in* $\mathrm{Med}_\mu(K)$ *to K is bounded by $\frac{1}{\sqrt{2}}\,\mathrm{diam}(K)/\left(1 - \mu^2\right)^{1/2}$*

Proof. Let x, y be two orthogonal projections of $m \in \mathrm{Med}_\mu(K)$ on K as given by the previous lemma. Then, using equation (4.5), one obtains

$$1 - \frac{\|x-y\|^2/4}{\mathrm{d}_K^2(m)} \leqslant (1+\mu^2)/2.$$

Hence, $\mathrm{d}_K^2(m) \leqslant \frac{1}{2}(1-\mu^2)^{-1}\|x-y\|^2$, which proves the result. □

Let us denote by S_μ^ε the set of points x of the hypersurface ∂K^ε that satisfies the three conditions below:

(i) the normal distance to the medial axis is bounded below: $\tau(x) \geqslant \varepsilon$;
(ii) the image of x by ℓ is in the μ-medial axis of K: $\ell(x) \in \mathrm{Med}_\mu(K)$;
(iii) there exists another projection y of $m = \ell(x)$ on ∂K^ε with

$$\cos\left(\frac{1}{2}\angle(p-m, q-m)\right) \leqslant \sqrt{\frac{1+\mu^2}{2}}$$

A reformulation of Lemmas 4.3 and 4.2 is the following corollary:

Corollary 4.5. *The image of S_μ^ε by the map ℓ covers the whole 2ε-away μ-medial axis: $\ell(S_\mu^\varepsilon) = \mathrm{Med}_\mu(K) \cap (\mathbb{R}^d \setminus K^{2\varepsilon})$*

4.3. **Lipschitz estimations for the map ℓ.** In this paragraph, we bound the Lipschitz constants of the restriction of the maps $\nabla \mathrm{d}_K$, τ and (finally) ℓ to the subset $S_\mu^\varepsilon \subseteq \partial K^\varepsilon$.

First, let $\partial K^{\varepsilon,t}$ be the set of points x in ∂K^ε where the distance function is differentiable, and such that $\tau(x)$ is bounded from below by t. In particular, notice that S_μ^ε is contained in $\partial K^{\varepsilon,\varepsilon}$. The following Lemma proves that the functions Ψ^t and $\nabla_x \mathrm{d}_K$ are Lipschitz on $\partial K^{\varepsilon,t}$:

Lemma 4.6. *(i) The restriction of Ψ^t to $\partial K^{\varepsilon,t}$ is $(1+t/\varepsilon)$-Lipschitz.*
(ii) The gradient of the distance function, $x \mapsto \nabla_x \mathrm{d}_K$, is $3/\varepsilon$-Lipschitz on $\partial K^{\varepsilon,\varepsilon}$.

Proof. (i) Let x and x' be two points of ∂K^ε with $\tau(x), \tau(x') > t$, p and p' their projections on K and y and y' their image by Ψ^t. We let $u = 1 + t/\varepsilon$ be the scale factor between $x - p$ and $y - p$, i.e.:

$$(*)\ y' - y = u(x'-x) + (1-u)(p'-p)$$

Using the fact that y projects to p, and the definition of u, we have:

$$\|y-p\|^2 \leqslant \|y-p'\|^2 = \|y-p\|^2 + \|p-p'\|^2 + 2\langle y-p|p-p'\rangle$$
$$\text{i.e.} 0 \leqslant \|p-p'\|^2 + 2u\langle x-p|p-p'\rangle$$
$$\text{i.e.} \langle p-x|p-p'\rangle \leqslant \frac{1}{2}u^{-1}\|p-p'\|^2$$

Summing this last inequality, the same inequality with primes and the equality $\langle p'-p|p-p'\rangle = -\|p'-p\|^2$ gives

$$(**)\ \langle x'-x|p'-p\rangle \geqslant \left(1-u^{-1}\right)\|p'-p\|^2$$

Using $(*)$ and $(**)$ we get the desired Lipschitz inequality

$$\begin{aligned}\|y-y'\|^2 &= u^2\|x-x'\|^2 + (1-u)^2\|p'-p\|^2 + 2u(1-u)\langle x'-x|p'-p\rangle \\ &\leqslant u^2\|x-x'\|^2 - (1-u)^2\|p'-p\|^2 \leqslant (1+t/\varepsilon)^2\|x-x'\|^2\end{aligned}$$

(ii) If x belongs to $\partial K^{\varepsilon,\varepsilon}$, then $\nabla_x \mathrm{d}_K = \frac{1}{\varepsilon}(\Psi^\varepsilon(x) - x)$. The result follows from the Lipschitz estimation of (i). □

The second step is to prove that the restriction of τ to the set S_μ^ε is also Lipschitz. The technical core of the proof is contained in the following geometric lemma:

Lemma 4.7. *Let $t(x,v)$ denote the intersection time of the ray $x + tv$ with the medial hyperplane $H_{x,y}$ between x and another point y, and $t(x',v')$ the intersection time between the ray $x' + tv'$ and $H_{x'y}$. Then, assuming:*

$$\alpha \|x - y\| \leqslant \langle v | x - y\rangle, \tag{4.6}$$

$$\|x' - y\| \leqslant D, \tag{4.7}$$

$$\|v' - v\| \leqslant \lambda \|x' - x\|, \tag{4.8}$$

$$\varepsilon \leqslant t(x,v) \tag{4.9}$$

one obtains the following bound:

$$t(x',v') \leqslant t(x,v) + \frac{6}{\alpha^2}(1+\lambda D)\|x' - x\|$$

as soon as $\|x' - x\|$ is small enough (namely, smaller than $\varepsilon\alpha^2(1+3\lambda D)^{-1}$).

Proof. We search the time t such that $\|x' + tv' - x'\|^2 = \|x' + tv' - y\|^2$, i.e.

$$t^2\|v'\|^2 = \|x' - y\|^2 + 2t\langle x' - y | v'\rangle + t^2\|v'\|^2$$

Hence, the intersection time is $t(x',v') = \|x' - y\|^2 / 2\langle y - x' | v'\rangle$. The lower bound on $t(x,y)$ translates as

$$\varepsilon \leqslant \frac{1}{2}\frac{\|x-y\|^2}{\langle x - y | v\rangle} \leqslant \frac{1}{2\alpha}\|x - y\|$$

If $\nabla_{x'} t$ and $\nabla_{v'} t$ denote the gradients of this function in the direction of v' and x', one has:

$$\nabla_{v'} t(x',v') = \frac{1}{2}\frac{\|x'-y\|^2 (x'-y)}{\langle y - x' | v'\rangle^2}$$

$$\nabla_{x'} t(x',v') = \frac{1}{2}\frac{\|x'-y\|^2 v' + 2\langle y - x' | v'\rangle (x'-y)}{\langle y - x' | v'\rangle^2}$$

Now, we bound the denominator of this expression:

$$\begin{aligned}\langle x' - y | v'\rangle &= \langle x' - y | v' - v\rangle + \langle x' - x | v\rangle + \langle x - y | v\rangle \\ &\geqslant \alpha\|x - y\| - (1 + \lambda\|x' - y\|)\|x' - x\| \\ &\geqslant \alpha\|x' - y\| - (2 + \lambda D)\|x' - x\|\end{aligned}$$

The scalar product $\langle x' - y | v'\rangle$ will be larger than (say) $\frac{\alpha}{2}\|x' - y\|$ provided that

$$(2 + \lambda D)\|x' - x\| \leqslant \frac{\alpha}{2}\|x' - y\|$$

or, bounding from below $\|x' - y\|$ by $\|x - y\| - \|x - x'\| \geqslant 2\alpha\varepsilon - \|x - x'\|$, provided that:

$$(3 + \lambda D)\|x' - x\| \leqslant \alpha^2\varepsilon$$

This is the case in particular if $\|x' - x\| \leqslant \alpha^2 \varepsilon (3 + \lambda D)^{-1}$. Under that assumption, we have the following bound on the norm of the gradient, from which the Lipschitz inequality follows:

$$\left\|\nabla_{x'} t(x', v')\right\| \leqslant 6/\alpha^2 \text{ and } \left\|\nabla_{v'} t(x', v')\right\| \leqslant 4D/\alpha^2$$

□

Using this Lemma, we are able to show that the function ℓ is locally Lipschitz on the subset $S_\mu^\varepsilon \subseteq \partial K^\varepsilon$:

PROPOSITION 4.8. *The restriction of* τ *to* S_μ^ε *is locally L-Lipschitz, in the sense that if* $(x, y) \in S_\mu^\varepsilon$ *are such that* $\|x - y\| \leqslant \delta_0$*, then* $\|\ell(x) - \ell(y)\| \leqslant L \|x - y\|$ *with*

$$L = \mathrm{O}\left(\frac{1 + \mathrm{diam}(K)/\varepsilon}{(1-\mu)^{1/2}}\right) \text{ and } \delta_0 = \mathrm{O}(\varepsilon/L)$$

In order to simplify the proof of this Proposition, we will make use of the following notation, where f is any function from $X \subseteq \mathbb{R}^d$ to $\mathbb{R}$ or $\mathbb{R}^d$:

$$\mathrm{Lip}_\delta\, f|_X := \sup\{\|f(x) - f(y)\| \,/\, \|x - y\| \,;\, (x, y) \in X^2 \text{ and } \|x - y\| \leqslant \delta\}.$$

Proof. We start the proof by evaluating the Lipschitz constant of the restriction of τ to S_μ^ε, using Lemma 4.7 (Step 1), and then deduce the Lipschitz estimate for the function ℓ (Step 2).

STEP 1. Thanks to Lemma 4.3, for any x in S_μ^ε, there exists another projection y of $m = \ell(x)$ on ∂K^ε such that the cosine of the angle $\theta = \frac{1}{2}\angle(x - m, y - m)$ is at most $\sqrt{(1 + \mu^2)/2}$. Let us denote by $v = \nabla_x \mathrm{d}_K$ the unit vector from x to m. The angle between $\overrightarrow{xy}$ and v is $\pi/2 - \theta$. Then,

$$\cos(\pi/2 - \theta) = \sin(\theta) = \sqrt{1 - \cos^2(\theta)} \geqslant \alpha := \left(\frac{1 - \mu^2}{2}\right)^{1/2}$$

As a consequence, with the α introduced above, one has $\alpha \|x - y\| \leqslant |\langle v | x - y\rangle|$. Moreover, $\|x - y\|$ is smaller than $D = \mathrm{diam}(K^\varepsilon) \leqslant \mathrm{diam}(K) + \varepsilon$. For any other point x' in S_μ^ε, and $v' = \nabla_{x'} \mathrm{d}_K$, one has $\|v - v'\| \leqslant \lambda \|x - x'\|$ with $\lambda = 3/\varepsilon$ (thanks to Lemma 4.6).

These remarks allow us to apply Lemma 4.7. Using the notations of this lemma, one sees that $t(x, v)$ is simply $\tau(x)$ while $t(x', v')$ is an upper bound for $\tau(x')$. This gives us:

$$\begin{aligned}
\tau(x') &\leqslant \tau(x) + \frac{6}{\alpha^2}(1 + \lambda D) \left\|x - x'\right\| \\
&\leqslant \tau(x) + M \left\|x - x'\right\| \\
\text{where } M &= \mathrm{O}\left(\frac{1 + \mathrm{diam}(K)/\varepsilon}{\sqrt{1 - \mu^2}}\right)
\end{aligned}$$

as soon as x' is close enough to x. From the statement of Lemma 4.7, one sees that $\|x - x'\| \leqslant \delta_0$ with $\delta_0 = \mathrm{O}(\varepsilon/M)$ is enough. Exchanging the role of x and x', one proves that $|\tau(x) - \tau(x')| \leqslant M \|x - x'\|$, provided that $\|x - x'\| \leqslant \delta_0$. As a conclusion,

$$\mathrm{Lip}_{\delta_0}\left[\tau|_{S_\mu^\varepsilon}\right] = \mathrm{O}\left(\frac{1 + \mathrm{diam}(K)/\varepsilon}{\sqrt{1 - \mu^2}}\right) \tag{4.10}$$

STEP 2. We can use the following decomposition of the difference $\ell(x)-\ell(x')$:

$$\ell(x) - \ell(x') = (x - x') + (\tau(x) - \tau(x'))\nabla_x \mathrm{d}_K + \tau(x')(\nabla_x \mathrm{d}_K - \nabla_{x'} \mathrm{d}_K) \quad (4.11)$$

in order to bound the (local) Lipschitz constant of the restriction of ℓ to S^ε_μ from those computed earlier. One deduces from this equation that

$$\mathrm{Lip}_{\delta_0}\left[\ell|_{S^\varepsilon_\mu}\right] \leqslant 1 + \mathrm{Lip}_{\delta_0}\left[\tau|_{S^\varepsilon_\mu}\right] + \|\tau\|_\infty \mathrm{Lip}_{\delta_0}\left[\nabla \mathrm{d}_K|_{S^\varepsilon_\mu}\right] \quad (4.12)$$

Thanks to Lemma 4.4, one has $|\tau(x)| = \mathrm{O}(\mathrm{diam}(K)/(1-\mu)^{1/2})$; combining this with the estimate from Lemma 4.6 that $\mathrm{Lip}\ \nabla \mathrm{d}_K|_{S^\varepsilon_\mu} \leqslant 3/\varepsilon$, this gives

$$\|\tau\|_\infty \mathrm{Lip}_{\delta_0}\left[\nabla \mathrm{d}_K|_{S^\varepsilon_\mu}\right] = \mathrm{O}(\mathrm{diam}(K)/[\varepsilon(1-\mu)^{1/2}]) \quad (4.13)$$

Putting the estimates (4.10) and (4.13) into (4.12) concludes the proof. □

In order to be able to deduce Theorem 4.1 from Proposition 4.8 we need the following bound on the covering numbers of a levelset ∂K^r, where K is any compact set in $\mathbb{R}^d$ (see [8, Proposition 4.2]):

$$\mathcal{N}(\partial K^r, \varepsilon) \leqslant \mathcal{N}(\partial K, r)\mathcal{N}(\mathcal{S}^{d-1}, \varepsilon/2r) \quad (4.14)$$

Proof of Theorem 4.1. Applying Proposition 4.8, we get the existence of

$$L = \mathrm{Lip}_{\delta_0}\left[\ell|_{S^{\varepsilon/2}_\mu}\right] = \mathrm{O}(\mathrm{diam}(K)/(\varepsilon\sqrt{1-\mu})) \text{ and } \delta_0 = \mathrm{O}(\varepsilon/L)$$

such that ℓ is locally L-Lipschitz. In particular, for any η smaller than δ_0,

$$\begin{aligned}\mathcal{N}\left(\mathrm{Med}_\mu(K) \cap (\mathbb{R}^d \setminus K^\varepsilon), \eta\right) &= \mathcal{N}\left(\ell(S^{\varepsilon/2}_\mu), \eta\right)\\ &\leqslant \mathcal{N}\left(S^{\varepsilon/2}_\mu, \eta/L\right) \\ &\leqslant \mathcal{N}(\partial K^{\varepsilon/2}, \eta/L).\end{aligned} \quad (4.15)$$

The bound on the covering number of the boundary of tubular neighborhoods (equation (4.14)) gives:

$$\mathcal{N}(\partial K^{\varepsilon/2}, \eta/L) \leqslant \mathcal{N}(\partial K, \varepsilon/2)\mathcal{N}\left(\mathcal{S}^{d-1}, \frac{\eta}{L\varepsilon}\right). \quad (4.16)$$

Equations (4.15) and (4.16), and the estimation $\mathcal{N}(\mathcal{S}^{d-1}, \rho) \sim \omega_{d-1}\rho^{d-1}$ yield

$$\mathcal{N}\left(\mathrm{Med}_\mu(K) \cap (\mathbb{R}^d \setminus K^\varepsilon), \eta\right) = \mathcal{N}(\partial K, \varepsilon/2)\,\mathrm{O}\left(\left[\frac{\eta}{L\varepsilon}\right]^{d-1}\right).$$

It suffices to replace L by its value from Proposition 4.8 to finish the proof. □

5. A quantitative stability result for boundary measures

In this paragraph, we show how to use the bound on the covering numbers of the ε-away μ-medial axis given in Theorem 4.1 in order to get a quantitative version of the L^1 convergence results for projections. Notice that the meaning of *locally* in the next statement could also be made quantitative using the same proof.

Theorem 5.1. *The map $K \mapsto \mathrm{p}_K \in \mathrm{L}^1(E)$ is locally h-Hölder for any exponent $h < \frac{1}{2(2d-1)}$.*

Proof. As in the previous proof, we will let $R = \|\mathrm{d}_K\|_{E,\infty}$, so that E is contained in the tubular neighborhood K^R. Remark first that if a point x is such that $\mathrm{d}_K(x) \leqslant \frac{1}{2}L - \mathrm{d_H}(K,K')$, then by definition of the Hausdorff distance, $\mathrm{d}_{K'}(x) \leqslant \frac{1}{2}L$. In particular, the orthogonal projections of x on K and K' are at distance at most L. Said otherwise, the set $\Delta_L(K,K')$ is contained in the complementary of the $\frac{L}{2} - \delta$ tubular neighborhood of K, with $\delta := \mathrm{d_H}(K,K')$. Using this fact and the result of Lemma 3.3, we have:

$$\Delta_L(K,K') \cap K^R \subseteq \mathrm{Med}_\mu(K)^{2\sqrt{R\delta}} \cap (\mathbb{R}^d \setminus K^{\frac{L}{2}-\delta}) \tag{5.17}$$

$$\subseteq \left(\mathrm{Med}_\mu(K) \cap \left(\mathbb{R}^d \setminus K^{\frac{L}{2}-\delta-2\sqrt{R\delta}}\right)\right)^{2\sqrt{R\delta}} \tag{5.18}$$

$$\text{where } \mu = \left(1 + \left[\frac{L-\delta}{4R}\right]^2\right)^{-1/2} + 4\sqrt{\frac{\delta}{L}} \tag{5.19}$$

We now choose L to be δ^h, where $h > 0$, and see for which values of h we are able to get a converging bound. For $h < 1/2$, the radius $\frac{1}{2}(L-\delta) - 2\sqrt{R\delta}$ will be greater than $L/3$ as soon as as soon as δ is small enough. For these values,

$$\Delta_L(K,K') \cap K^R \subseteq \left(\mathrm{Med}_\mu(K) \cap (\mathbb{R}^d \setminus K^{L/3})\right)^{2\sqrt{R\delta}} \tag{5.20}$$

The μ above, given by Lemma 3.3 can then be bounded as follows. Note that the constants in the "big O" will always be positive in the remaining of the proof. From Eq. (5.19), one deduces:

$$\mu = 1 + \mathrm{O}(-\delta^{2h} + \delta^{1/2-h/2})$$

This term will be asymptotically smaller than 1 provided that $2h < 1/2 - h/2$ i.e. $h < 1/5$, in which case $\mu = 1 - \mathrm{O}(\delta^{2h})$. By definition of the covering number, one has:

$$\begin{aligned}\mathcal{H}^d(\Delta_L(K,K') \cap K^R) &\leqslant \mathcal{H}^d\left[\left(\mathrm{Med}_\mu(K) \cap \left(\mathbb{R}^d \setminus K^{L/3}\right)\right)^{2\sqrt{R\delta}}\right] \\ &\leqslant \mathcal{N}\left(\mathrm{Med}_\mu(K) \cap \left(\mathbb{R}^d \setminus K^{L/3}\right), 2\sqrt{R\delta}\right) \times \mathrm{O}(\delta^{d/2})\end{aligned} \tag{5.21}$$

The covering numbers of the intersection $\mathrm{Med}_\mu(K) \cap \left(\mathbb{R}^d \setminus K^{L/3}\right)$ can be bounded using Theorem 4.1:

$$\begin{aligned}&\mathcal{N}\left(\mathrm{Med}_\mu(K) \cap \left(\mathbb{R}^d \setminus K^{L/2}\right), 2\sqrt{R\delta}\right) \\ &\qquad = \mathcal{N}(\partial K, L/4)\,\mathrm{O}\left(\left[\frac{\mathrm{diam}(K)/\sqrt{R\delta}}{\sqrt{1-\mu^2}}\right]^{d-1}\right) \\ &\qquad = \mathcal{N}(\partial K, L/4)\,\mathrm{O}\left(\delta^{-(h+\frac{1}{2})(d-1)}\right)\end{aligned} \tag{5.22}$$

Combining equations (5.21) and (5.22), and using the (crude) estimation $\mathcal{N}(\partial K, L/4) = \mathrm{O}(1/L^d) = \mathrm{O}(\delta^{-hd})$,

$$\begin{aligned}\mathcal{H}^d(\Delta_L(K,K') \cap K^R) &\leqslant \mathcal{N}(\partial K, L/4)\,\mathrm{O}(\delta^{-h(d-1)-\frac{1}{2}(d-1)+\frac{1}{2}d}) \\ &\leqslant \mathrm{O}\left(\delta^{\frac{1}{2}-h(2d-1)}\right)\end{aligned}$$

Hence, following the proof of Proposition 3.2,

$$\begin{aligned}\|\mathrm{p}_{K'} - \mathrm{p}_K\|_{\mathrm{L}^1(E)} &\leqslant L\mathcal{H}^d(E) + \mathcal{H}^d(\Delta_L(K,K') \cap E)\,\mathrm{diam}(K \cup K') \\ &= \mathrm{O}(\delta^h + \delta^{1/2-h(2d-1)})\end{aligned}$$

The second term converges to zero as $\delta = \mathrm{d}_H(K,K')$ does if $h < \frac{1}{2(2d-1)}$. This concludes the proof. □

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