A local frequency analysis framework for shape processing

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Shape processing

- **Local analysis**: Curvature estimation, patch-based analysis (denoising, super-resolution)...
- **Global analysis**: Manifold Harmonics Transform [Vallet, Lévy 2008], [Taubin 95]
- "More than global" analysis: Shape recognition in databases [ShapeNet - Chang 2015]
Application: Shape detail enhancement
Manifold Harmonics Transform and Inverse Transform

- Manifold Harmonics Transform (MHT): $f$: function defined on the vertices of a mesh $f = \sum_i x_i f_i$. Then:

  $$\tilde{f}_i = \langle f, \phi_i \rangle = \sum_{j=1}^{n} x_i \langle f_i, \phi_j \rangle \text{ (MHT)}$$

  $$f = \sum_{i=1}^{n} \tilde{f}_i \phi_i \text{ (Inverse MHT).}$$

Figure 8: Filtering Stanford’s bunny. Results similar to geofilter are obtained, with the addition of interactivity, and without any shrinking effect.
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  Then:

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  \]

  \[
  f = \sum_{i=1}^{n} \tilde{f}_i \phi_i \quad \text{(Inverse MHT)}.
  \]

- **Spherical Harmonics** [Kazhdan03], **Compressed Manifold Modes** [Neumann14]

Figure 8: Filtering Stanford’s bunny. Results similar to geofilter are obtained, with the addition of interactivity, and without any shrinking effect.
Local analysis: local height field

- Height field over a plane:

\[ p(x, y, h = f(x, y)) \]
Local analysis: local height field

- Height field over a plane:
  \[ p(x, y, h = f(x, y)) \]

- Or over a quadric
  [Hamdi-Cherif2017]
  \[ p(x + h n_x^q(x, y), y + h n_y^q(x, y), z + h n_z^q(x, y)) \]
Local function basis for shape representation

Two strong assumptions on the surface $S$:
- $S$ can be locally be expressed as a height field over a planar parameterization in neighborhoods of radius $r$
- $S$ is smooth, $C^\infty$
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Goal
Design a function basis taking into account both the local surface derivatives and the angular oscillations of the surface around one point of the surface.
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Two strong assumptions on the surface $S$:

- $S$ can be locally be expressed as a height field over a planar parameterization in neighborhoods of radius $r$
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Goal

Design a function basis taking into account both the local surface derivatives and the angular oscillations of the surface around one point of the surface.

- Curvature computation using quadric regression [Chen92, Hamann93] or cubic regression [Goldfeather04]
Local function basis for shape representation: Jets [Cazals03]

- Surface parameterized w.r.t. $\mathcal{P}(p)$ Not necessarily equal to $\mathcal{T}(p)$ (tangent plane)

**Truncated Taylor expansion**

$S$ surface locally homeomorphic to a disk in a small neighborhood around a point $p$, expressed as $f(x, y)$ over a plane $\mathcal{P}(p)$ passing through $p$. The neighborhood of $p$ can be expressed as a Taylor Expansion:

\[
f(x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} f_{x^{k-j}y^j}(0, 0) \frac{1}{(k-j)!j!} x^{k-j} y^j
\]

(1)

where $f_{x^{k-j}y^j} = \frac{\partial^k f}{\partial x^{k-j} \partial y^j}$.
Local function basis for shape representation: Jets [Cazals03]

Accuracy theorem [Cazals03]

Given a Taylor expansion of order $K$ in a neighborhood of radius $r$, the precision of all $k$ order derivatives is $o(r^{K-k})$. 
Local function basis for shape representation: Jets [Cazals03]

Accuracy theorem [Cazals03]

Given a Taylor expansion of order $K$ in a neighborhood of radius $r$, the precision of all $k$ order derivatives is $o(r^{K-k})$.  

In practice:

- Computation of the coefficients at each vertex or point by linear system solve.
Local function basis for shape representation: Zernike polynomial [Zernike34]

\[ V^q_p(\rho, \theta) = R^q_p(\rho)e^{iq\theta} \]

- \((p, q) \in \{|q| \in \mathbb{Z}, p \in \mathbb{Z}^0, \ |q| \leq p, \ p - q \text{ is even}\}\)

- \[ R^q_p(\rho) = \sum_{k=|p|}^{p-q \text{ even}} (-1)^{\frac{p-k}{2}}\frac{p+k}{2}!\frac{p-k}{2}!\frac{k-q}{2}!\rho^k \]

- In practice: projection of neighborhoods (disks) on the basis.
Local function basis for shape representation: Zernike polynomial \([\text{Zernike34}]\)

\[
V_p^q(\rho, \theta) = R_p^q(\rho)e^{iq\theta}
\]

- \((p, q) \in \{ |q| \in \mathbb{Z}, p \in \mathbb{Z}^\geq 0, |q| \leq p, p - q \text{ is even} \}\)

- \(R_p^q(\rho) = \sum_{k=|p|}^{p-q \text{ even}} \frac{(-1)^{p-k}}{2^{p-k}k-q!k+q!} \rho^k\)

- In practice: projection of neighborhoods (disks) on the basis.

- Applied to image processing and surface processing [Khotanzad 88, Maximo 2011]
Wavejets

Locally we express the function as:

\[ f(r, \theta) = \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \phi_{k,n} r^k e^{in\theta} \quad (2) \]

with \( \phi_{k,n} = \sum_{j=0}^{k} \frac{1}{j!(k-j)!} b(k, j, n) f_{x^{k-j}y^j}(0, 0) \).

- \( b(k, j, n) = 0 \) if \( k \) and \( n \) do not have the same parity
- \( b(k, j, n) = \frac{1}{2^k j^i} \sum_{h=0}^{\frac{n-k}{2}} \binom{k-j}{h} \left( \frac{j}{2} - h \right) (-1)^h \) otherwise.
$\phi_{k,n}$ coefficients
Properties

- **Gaussian Curvature**

\[
K(p) = \frac{4\phi_{2,0}^2 - 16\phi_{2,-2}\phi_{2,2}}{(1 + 4\phi_{1,-1}\phi_{1,1})^2}
\]  

(3)

- **Mean Curvature**

\[
H(p) = \frac{2\phi_{2,0} (1 + 4\phi_{1,-1}\phi_{1,1}) + 4\phi_{2,-2}\phi_{1,1} + 4\phi_{2,2}\phi_{1,-1}}{(1 + 4\phi_{1,-1}\phi_{1,1})^{\frac{3}{2}}}
\]  

(4)

Properties

- Gaussian Curvature

\[ K(p) = \frac{4\phi_{2,0}^2 - 16\phi_{2,-2}\phi_{2,2}}{(1 + 4\phi_{1,-1}\phi_{1,1})^2} \]  \hspace{1cm} (3)

- Mean Curvature

\[ H(p) = \frac{2\phi_{2,0} (1 + 4\phi_{1,-1}\phi_{1,1}) + 4\phi_{2,-2}\phi_{1,1} + 4\phi_{2,2}\phi_{1,-1}}{(1 + 4\phi_{1,-1}\phi_{1,1})^{3/2}} \] \hspace{1cm} (4)

Parameterization plane

If \( P(p) = T(p) \), the tangent plane to \( S \) at \( p \), then \( \phi_{1,1} = \phi_{1,-1} = 0 \), and:

\[ K(p) = 4 \left( \phi_{2,0}^2 - \phi_{2,-2}\phi_{2,2} \right) \hspace{1cm} H(p) = 2\phi_{2,0} \]  \hspace{1cm} (5)
Principal directions

- Principal curvatures:

\[ \kappa_1 = 2 (\phi_{2,0} + \phi_{2,2} + \phi_{2,-2}) \quad \text{and} \quad \kappa_2 = 2 (\phi_{2,0} - \phi_{2,2} - \phi_{2,-2}) \]  

(6)
Principal directions

- Principal curvatures:

\[ \kappa_1 = 2 (\phi_{2,0} + \phi_{2,2} + \phi_{2,-2}) \text{ and } \kappa_2 = 2 (\phi_{2,0} - \phi_{2,2} - \phi_{2,-2}) \]  \hspace{1cm} (6)

- \[ \sum_{-2 \leq n \leq 2} \phi_{2,n} e^{in\theta} + \phi_{2,-n} e^{-in\theta} \] has 2 maxima aligned with the principal directions
Higher order principal directions

Order 3

\[ \sum_{-n \leq 3, \text{n odd}} \phi_3,n e^{in\theta} + \phi_3,-n e^{-in\theta} \] has at most 3 maxima (either 1 or 3)

Order 3 maxima directions:
Higher order principal directions

Order 3

- $\sum_{-n \leq 3, n \text{ odd}} \phi_3, n e^{in\theta} + \phi_3, -n e^{-in\theta}$ has at most 3 maxima (either 1 or 3)

- See also [Joshi, Séquin 2010]
Stability with respect to the parameterization plane

- \((p, u) = \mathcal{T}(p) \cap \mathcal{P}(p)\)
- \(\gamma\): rotation angle along \((p, u)\) to align \(\mathcal{P}(p)\) to \(\mathcal{T}(p)\)
- Over \(\mathcal{T}(p)\): \(f(r, \theta) = \sum_{k=0}^{\infty} \sum_{n=-k}^{n=k} \phi_{k,n} r^k e^{in\theta};\)
- Over \(\mathcal{P}(p)\): \(f(R, \Theta) = \sum_{k=0}^{\infty} \sum_{n=-k}^{n=k} \Phi_{k,n} R^k e^{i n \Theta}\)
- Assume \(\theta\) and \(\Theta\) are computed w.r.t. direction \(u\)

Stability Theorem

The coefficients \(\Phi_{k,n}\) w.r.t to \(\mathcal{P}(p)\) can be expressed with respect to the coefficients \(\phi_{k,n}\) in the tangent plane \(\mathcal{T}(p)\) as follows:

- \(\Phi_{0,0} = 0\)
- \(\Phi_{1,1} = \Phi_{1,-1} = \frac{\gamma}{2} e^{-i \frac{\pi}{2}} + o(\gamma)\)
- \(\Phi_{k,n} = \phi_{k,n} + \gamma F(k, n) + o(\gamma)\) \hspace{1cm} (7)

where \(F(k, n)\) is a function of the \(\phi\) coefficients of order lower than \(k\). It is independent of \(R < R_{\phi}\) and \(\Theta\).
Parameterization plane correction

Consequence (1)

\[ \gamma = 2|\Phi_{1,1}| + o(|\Phi_{1,1}|) \text{ and } \arg(\Phi_{1,1}) = \frac{\pi}{2} + o(\gamma) \]

The phase of \( \Phi_{1,1} \) shifted by \( \pi/2 \) in the plane \( \mathcal{P}(p) \) corresponds to the axis of rotation \( u \). Therefore, it is possible to correct the parameterization by performing a rotation of \( \mathcal{P}(p) \) along the axis \( u \) with rotation angle \( 2|\Phi_{1,1}| \).
Coefficient error correction

Consequence (2)

One can recover the true coefficients $\phi_{k,n}$, starting iteratively from the lowest order coefficients:

$$\phi_{k,n} = \Phi_{k,n} - \gamma \sum_{j=1}^{k-2} s_{j,k,n} + o(\gamma)$$

(8)

$$s_{j,k,n} = \sum_{\substack{p+m=n \\ |p| \leq k-j \\ |m| \leq j}} \frac{\phi_{k-j,p}}{2i} (\phi_{j+1,m+1}(m+j+2) + \phi_{j+1,m-1}(m-j-2))$$

(9)

Rk: In particular, $\phi_{2,0} = \Phi_{2,0} + o(\gamma)$, $\phi_{2,2} = \Phi_{2,2} + o(\gamma)$, $\phi_{2,-2} = \Phi_{2,-2} + o(\gamma)$
Remark on Zernike and Jets

- Normal error can be derived from order 1 jets coefficients, **BUT** from a nontrivial linear combinations of $V_k^{\pm 1}$
- Error to the surface can be derived from order 0 jets coefficients, **BUT** from a nontrivial linear combinations of $V_k^0$
- Integral invariants are easy to compute with Zernike polynomial **BUT** not easily written using jets.

Wavejets

Normal error, error to the surface, integral invariants are easy to write with Wavejets
First application: normal correction

Normal estimation on two intersecting cylinders creating a sharp edge. First row: Noise free, Second row: Gaussian noise 1.2% - Third row: Gaussian noise 3.6%
Link with Integral Invariants

- $V(s)$ signed volume between the surface and $P(p)$ in a small radius $s < R_\phi$ around one point $p$. 
Link with Integral Invariants

- $V(s)$ signed volume between the surface and $\mathcal{P}(p)$ in a small radius $s < R_\phi$ around one point $p$.
- $V(s) = \int_0^{2\pi} A(\theta, s) d\theta$ with

\[
A(\theta, s) = \int_0^s \left( \sum_{k=0}^{\infty} \sum_{n=-k}^{k} r^k \phi_{k,n} e^{in\theta} \right) rdr = \sum_{n=-\infty}^{\infty} a_n(s)e^{in\theta} \tag{10}
\]

With $a_n(s) = \sum_{k=|n|}^{\infty} \frac{\phi_{k,n}s^{k+2}}{k+2}$
Link with Integral Invariants

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\]

With \( a_n(s) = \sum_{k=|n|}^{\infty} \phi_{k,n} s^{k+2} \frac{1}{k+2} \)

Link with Integral Invariants

\( V_s(p) \): volume of the intersection of a sphere and the interior of the surface (e.g.) Manay (2006), Pottmann (2007, 2009):

\[
V_s(p) - 2\pi a_0 \approx \frac{2}{3} \pi s^3 . \quad (11)
\]
Wavejets (order 9) decomposition of a real surface.

\[ \tilde{\phi}_{k,n}(r, \theta) = r^k \left( \phi_{k,n} e^{in\theta} + \phi_{k,-n} e^{-in\theta} \right) \] and
\[ \tilde{\phi}_n = \sum_{k=0}^{\infty} \tilde{\phi}_{k,n}. \]
Computing Wavejets on point sets

Wavejets regression

Assume we have $L$ neighboring points $(r_l, \theta_l, z_l)$ then, we minimize:

$$E(\Phi) = \sum_{l=1}^{L} \left\| z_l - \sum_{k=0}^{K-1} \sum_{n=-k}^{k} r_l^k e^{in\theta_l} \phi_{k,n} \right\|_2^2$$
Computing Wavejets on point sets

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- To remove outliers we add a weight and use Iteratively Reweighted Least Squares
Computing Wavejets on point sets

Wavejets regression

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- To remove outliers we add a weight and use Iteratively Reweighted Least Squares
- Solve performed by QR decomposition.
Results

Noisy normals

\( \phi_{0,0} \)  \( \phi_{1,1} \)  \( \phi_{2,0} \)  \( \phi_{2,2} \)  \( \phi_{3,1} \)  \( \phi_{3,3} \)
Position enhancement filter

- *Unsharp Masking*, inverse curvature motion, [Gabor 1965]

**Position update**

Move \( p \) to its new position \( p' \):

\[
p' = p + (\phi_{0,0} - 2\pi(\alpha_0 - 1)a_0(s)) \, n
\]
Position enhancement filter

- *Unsharp Masking, inverse curvature motion*, [Gabor 1965]

### Position update

Move \( \mathbf{p} \) to its new position \( \mathbf{p}' \):

\[
\mathbf{p}' = \mathbf{p} + (\phi_{0,0} - 2\pi(\alpha_0 - 1)a_0(s)) \mathbf{n}
\]

- Continuous motion, assuming \( \mathcal{P}(\mathbf{p}) \) is corrected to \( \mathcal{T}(\mathbf{p}) \) beforehand.
Normal enhancement filter

- Principle: Modify $\phi_{1,1}$ and $\phi_{1,-1}$ and deduce the false normal
Normal enhancement filter

- Principle: Modify $\phi_{1,1}$ and $\phi_{1,-1}$ and deduce the false normal

**Method**

Compute $\phi_{1,\pm 1}$:

$$\phi_{1,\pm 1} = -\pi(\alpha_{\pm 1} - 1)a_{\pm 1}(s)$$

Deduce the normal by rotating the current normal of angle $2|\phi'_{1,1}|$ and axis given by $\arg(\phi'_{1,1}) + \frac{\pi}{2}$. 
Results

Normal and position enhancement on a bunny with 6-Wavejets. $R_\phi$ is equal to 3% of the shape diameter, and $\alpha_0 = \alpha_{\pm1} = 2$. 
Results

\[ \alpha_{\pm 1} = 2 \]
\[ \alpha_{\pm 1} = \pm 2i \]
\[ \alpha_{\pm 1} = -2 \]
\[ \alpha_{\pm 1} = \mp 2i \]
\[ \alpha_{\pm 1} = 0 \]

Normal amplification

\( K = 3 \)
\( K = 3 \)
\( K = 3 \)
\( K = 3 \)
\( K = 3 \)
\( K = 9 \)
\( K = 9 \)
\( K = 9 \)
\( K = 9 \)
\( K = 9 \)
Comparisons with Unsharp masking

$R_\phi = 2.5\%$

$K = 8$

$R_\phi = 3.5\%$

$K = 2$

$R_\phi = 4.5\%$

$K = 8$
Results

\[
\begin{align*}
\alpha_0 &= -2 & K &= 8 \\
\alpha_0 &= -1 & K &= 8 \\
\alpha_0 &= 0 & K &= 8 \\
\alpha_0 &= 1 & K &= 8 \\
\alpha_0 &= 2 & K &= 8 \\
\end{align*}
\]

Position amplification
Resilience to noise

\[ \alpha_0 = 2 \]
\[ \alpha_{\pm1} = 0 \]
\[ \alpha_{\pm1} = 2 \]

\[ \sigma = 0.1\% \]
\[ \sigma = 0.2\% \]
\[ \sigma = 0.5\% \]
Results

Outputs of our procedures on an armadillo (Left: original; Middle: normal-based detail enhancement with $K = 7, \alpha_{\pm 1} = 3$; Right: position-based detail enhancement with $K = 6, \alpha_0 = 2$).
Comparisons

Top row: wavejets. Bottom row: high-boost filter in the manifold harmonics basis [Vallet-Lévy 2008].
Comparisons

\( \alpha_1 = 24 \)
\( \alpha_1 = 24e^{i\frac{\pi}{4}} \)
\( \alpha_1 = 24i \)
\( \alpha_1 = 24e^{i\frac{3\pi}{4}} \)

\( \alpha_1 = -24e^{i\frac{3\pi}{4}} \)
\( \alpha_1 = -24i \)
\( \alpha_1 = -24e^{i\frac{\pi}{4}} \)
\( \alpha_1 = -24 \)

Original  Ours (\( \alpha_1 = 24 \))  [Cignoni05]  [Rusinkiewicz06]
Normal enhancement on a golf ball with \( K = 9 \). (nb: \( \alpha_{-1} = \alpha_1^* \))
Comparisons

Normal enhancement with different methods.

Original [Rusinkiewicz06] [Cignoni05] \( (K = 5, \alpha_{\pm 1} = 3) \)
$K = 7$ (normal exaggeration) and order $K = 6$ filters (position filter)

$\alpha_0 = \alpha \pm 1 = 2.$
Applying order 9 (normal exaggeration) and order 8 filters (position filter) to the Anubis datasets with $\alpha_0 = \alpha_{\pm 1} = 2$. 

More results
Complexity per point

- Given a set of $N$ neighbors and a Wavejet order $K$: $O(NK^4)$
- Once the wavejet is computed, applying the filter amounts to summing $K$ terms: $O(K)$
- 1.5M points, order $K = 6$: Decomposition time 1 min 40s; Filtering time: 0.6s.
Conclusion

- A new basis for representing the local variations of the surface
- Interpretation for the third order principal directions as tensor eigenvectors (as defined by Qi 2007)
- Filtering higher order derivatives using $a_2, a_3$...
- Filter design in this basis is still a work in progress.

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