A local frequency analysis framework for shape processing

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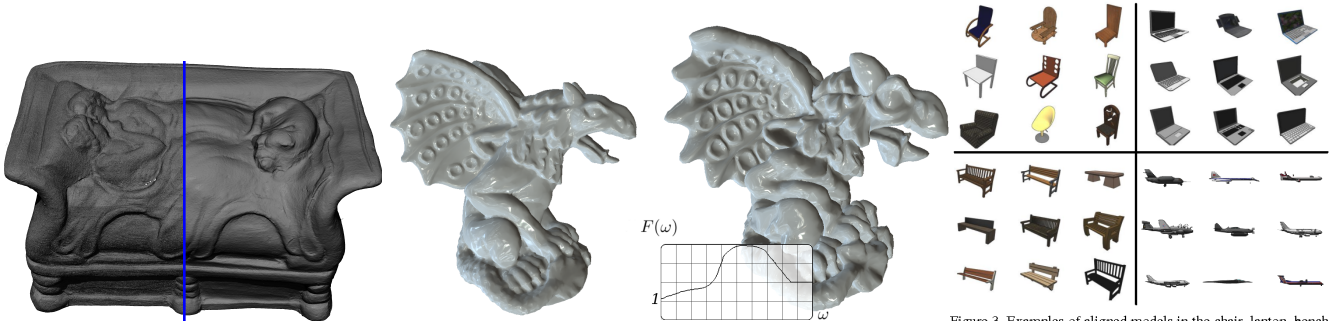
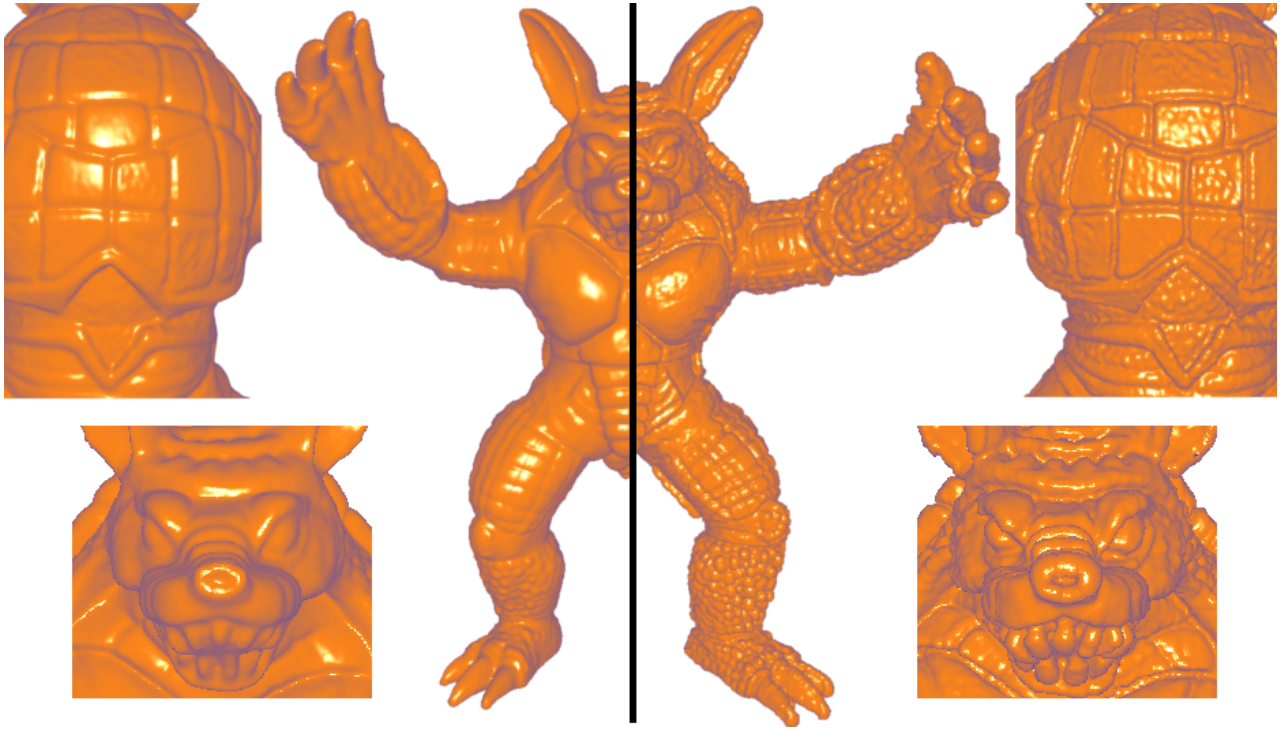
Shape processing

- Local analysis: Curvature estimation, patch-based analysis (denoising, super-resolution)...
- Global analysis: Manifold Harmonics Transform [Vallet, Lévy 2008], [Taubin 95]
- "More than global" analysis: Shape recognition in databases [ShapeNet - Chang 2015]



Application: Shape detail enhancement



Manifold Harmonics Transform and Inverse Transform

- Manifold Harmonics Transform (MHT): f: function defined on the vertices of a mesh $f = \sum_i x_i f_i$. Then:

$$\tilde{f}_i = \langle f, \phi_i \rangle = \sum_{j=1}^{n} x_i \langle f_i, \phi_j \rangle \text{ (MHT)}$$

$$f = \sum_{i=1}^{n} \tilde{f}_i \phi_i \text{ (Inverse MHT).}$$

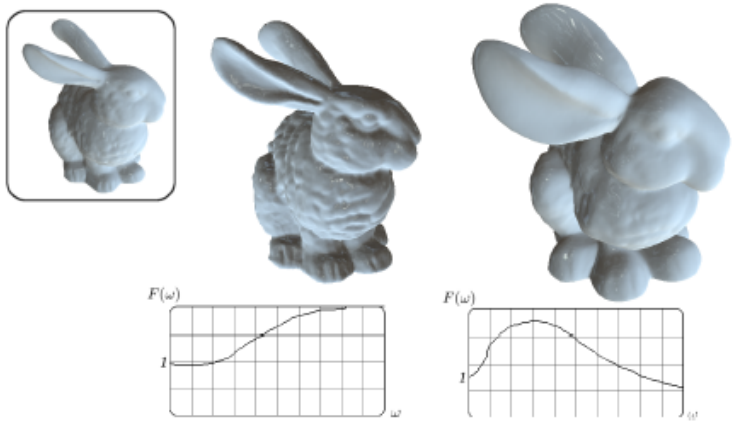


Figure 8: *Filtering Stanford's bunny. Results similar to geofilter are obtained, with the addition of interactivity, and without any shrinking effect.*

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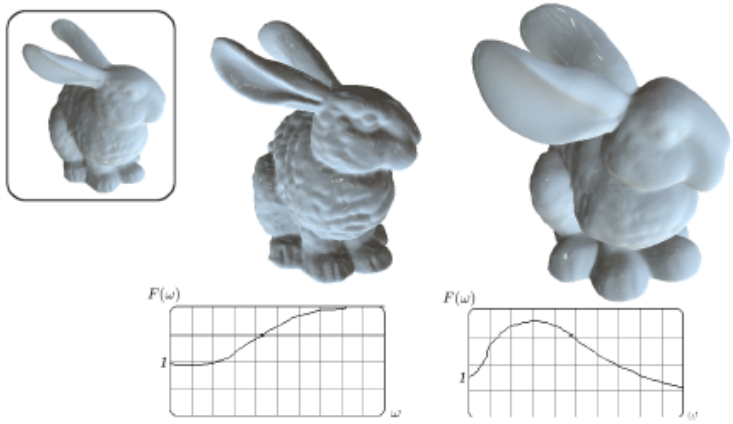


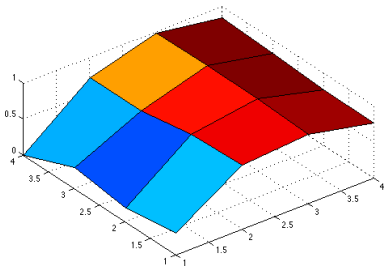
Figure 8: *Filtering Stanford's bunny. Results similar to geofilter are obtained, with the addition of interactivity, and without any shrinking effect.*

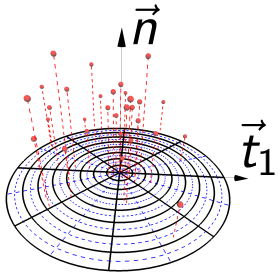
- Spherical Harmonics [Kazhdan03], Compressed Manifold Modes [Neumann14]

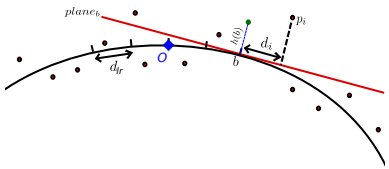
Local analysis: local height field

- Height field over a plane:

$$p(x, y, h = f(x, y))$$







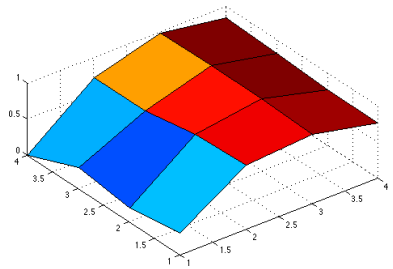
Local analysis: local height field

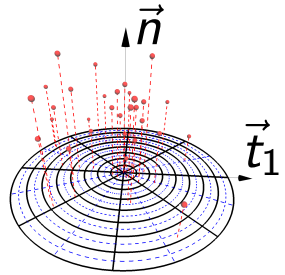
- Height field over a plane:

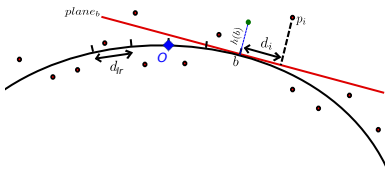
$$p(x, y, h = f(x, y))$$

- Or over a quadric [Hamdi-Cherif2017]

$$p(x+hn_x^q(x,y), y+hn_y^q(x,y), z+hn_z^q(x,y))$$







Local function basis for shape representation

Two strong assumptions on the surface S:

- S can be locally be expressed as a height field over a planar parameterization in neighborhoods of radius r
- S is smooth, $\mathcal{C}^\infty$

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Design a function basis taking into account both the local surface derivatives and the angular oscillations of the surface around one point of the surface.

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Design a function basis taking into account both the local surface derivatives and the angular oscillations of the surface around one point of the surface.

- Curvature computation using quadric regression [Chen92,Hamann93] or cubic regression [Goldfeather04]

Local function basis for shape representation: Jets [Cazals03]

- Surface parameterized w.r.t. $\mathcal{P}(p)$ Not necessarily equal to $\mathcal{T}(p)$ (tangent plane)

Truncated Taylor expansion

$\mathcal{S}$ surface locally homeomorphic to a disk in a small neighborhood around a point $\boldsymbol{p}$, expressed as $f(x, y)$ over a plane $\mathcal{P}(\boldsymbol{p})$ passing through $\boldsymbol{p}$. The neighborhood of $\boldsymbol{p}$ can be expressed as a Taylor Expansion:

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{f_{x^{k-j}y^j}(0,0)}{(k-j)!j!} x^{k-j} y^j \tag{1}$$

where $f_{x^{k-j}y^j} = \frac{\partial^k f}{\partial x^{k-j} \partial y^j}$.

Local function basis for shape representation: Jets [Cazals03]

Accuracy theorem [Cazals03]

Given a Taylor expansion of order K in a neighborhood of radius r, the precision of all k order derivatives is $o(r^{K-k})$.

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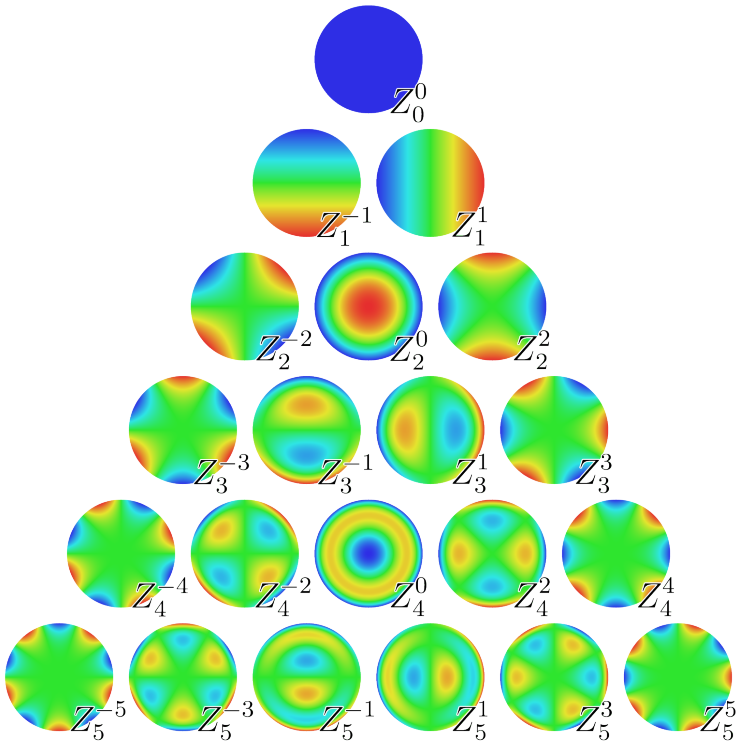
In practice:

- Computation of the coefficients at each vertex or point by linear system solve.

Local function basis for shape representation: Zernike polynomial [Zernike34]

$$V_p^q(\rho,\theta) = R_p^q(\rho)e^{\boldsymbol{i}q\theta}$$

- $(p,q) \in \{|q \in \mathbb{Z}, p \in \mathbb{Z}^{\geq 0}, |q| \leq p, p-q \text{ is even}\}$
- $R_p^q(\rho) = \sum_{\substack{k=|p| \\ p-q \text{ even}}} \frac{(-1)^{\frac{p-k}{2}} \frac{p+k}{2}}{\frac{p-k}{2}!\,\frac{k-q}{2}!\,\frac{k+q}{2}!}\rho^k$
- In practice: projection of neighborhoods (disks) on the basis.



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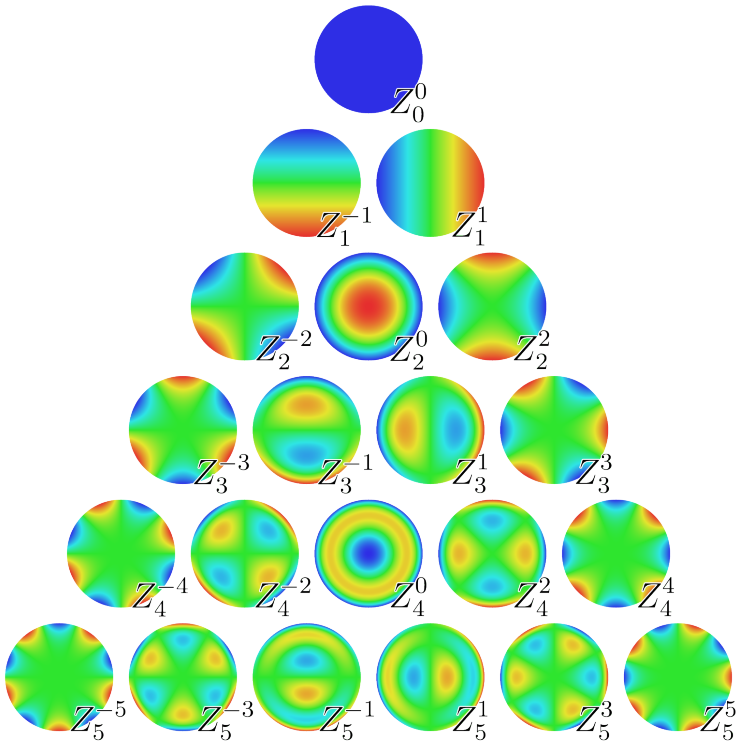


- Applied to image processing and surface processing [Khotanzad 88,Maximo 2011]

Definition

Wavejets

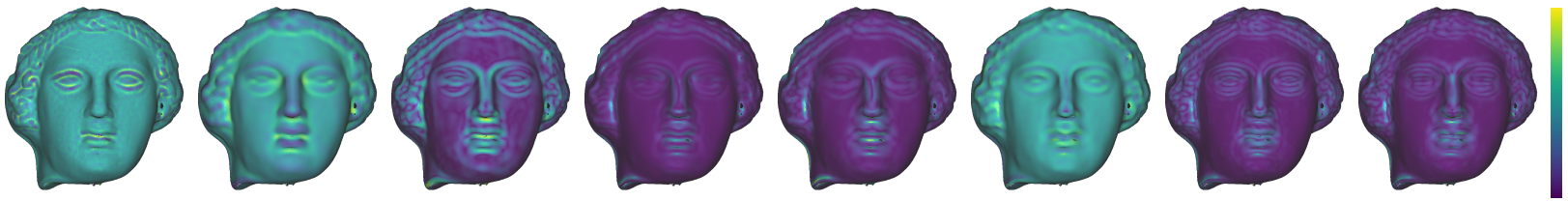
Locally we express the function as:

$$f(r,\theta) = \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \phi_{k,n} r^k e^{\boldsymbol{i} n\theta} \tag{2}$$

with $\phi_{k,n} = \sum_{j=0}^{k} \frac{1}{j!(k-j)!} b(k,j,n) f_{x^{k-j}y^j}(0,0)$.

- $b(k,j,n) = 0$ if k and n do not have the same parity
- $b(k,j,n) = \frac{1}{2^k \boldsymbol{i}^j} \sum_{h=0}^{\frac{n-k}{2}} \binom{k-j}{h} \binom{j}{\frac{n-k}{2} - h} (-1)^h$ otherwise.

$\phi_{k,n}$ coefficients



$\phi_{0,0}$ $\phi_{2,0}$ $|\phi_{2,2}|$ $|\phi_{3,1}|$ $|\phi_{3,3}|$ $\phi_{4,0}$ $|\phi_{4,2}|$ $|\phi_{4,4}|$

Properties

- Gaussian Curvature

$$K(\boldsymbol{p}) = \frac{4\phi_{2,0}^2 - 16\phi_{2,-2}\phi_{2,2}}{\left(1 + 4\phi_{1,-1}\phi_{1,1}\right)^2} \tag{3}$$

- Mean Curvature

$$H(\boldsymbol{p}) = \frac{2\phi_{2,0}\left(1 + 4\phi_{1,-1}\phi_{1,1}\right) + 4\phi_{2,-2}\phi_{1,1} + 4\phi_{2,2}\phi_{1,-1}}{\left(1 + 4\phi_{1,-1}\phi_{1,1}\right)^{\frac{3}{2}}} \tag{4}$$

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Parameterization plane

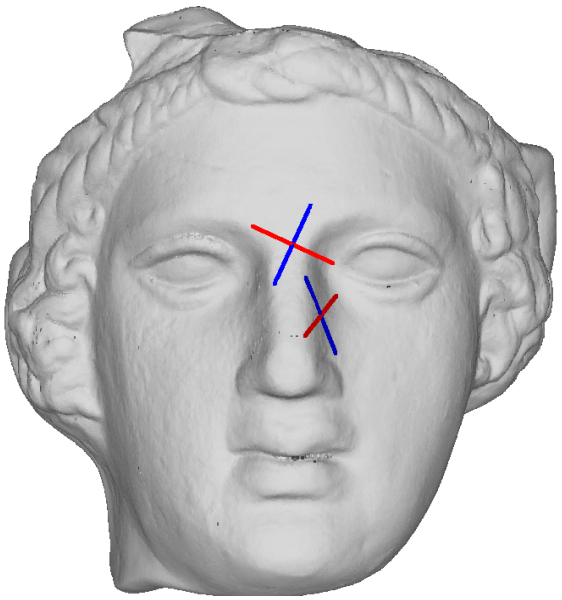
If $\mathcal{P}(\boldsymbol{p}) = \mathcal{T}(\boldsymbol{p})$, the tangent plane to $\mathcal{S}$ at $\boldsymbol{p}$, then $\phi_{1,1} = \phi_{1,-1} = 0$, and :

$$K(\boldsymbol{p}) = 4\left(\phi_{2,0}^2 - \phi_{2,-2}\phi_{2,2}\right) \ , \ H(\boldsymbol{p}) = 2\phi_{2,0} \tag{5}$$

Principal directions

- Principal curvatures:

$$\kappa_1 = 2\left(\phi_{2,0} + \phi_{2,2} + \phi_{2,-2}\right) \text{ and } \kappa_2 = 2\left(\phi_{2,0} - \phi_{2,2} - \phi_{2,-2}\right) \tag{6}$$



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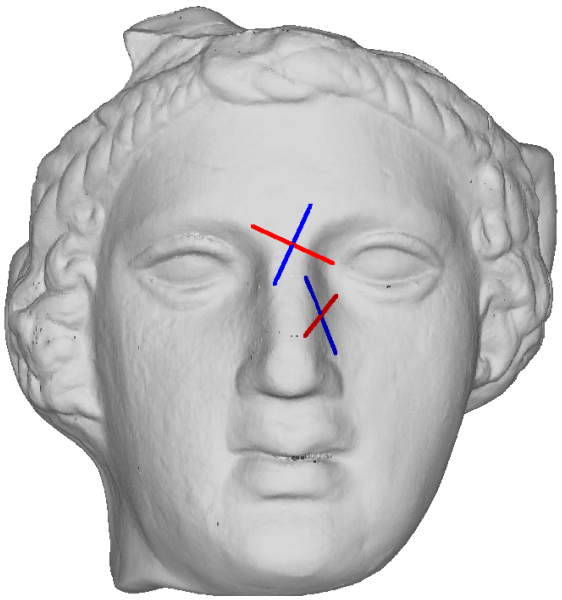
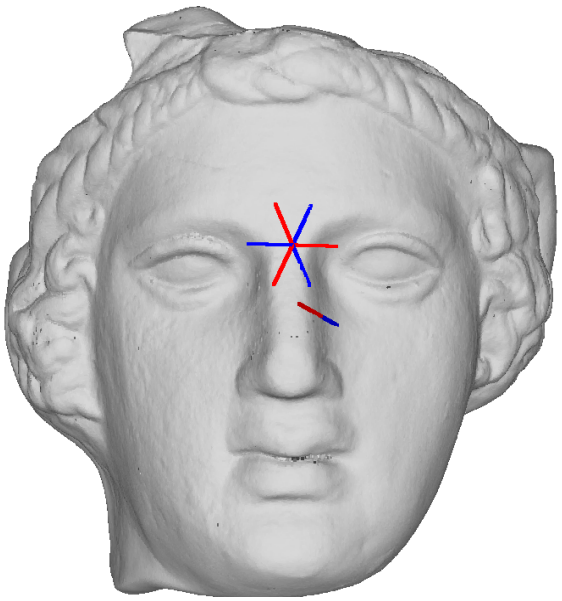
- $\sum_{\substack{-2 \leq n \leq 2 \\ \text{n even}}} \phi_{2,n} e^{\boldsymbol{i} n\theta} + \phi_{2,-n} e^{-\boldsymbol{i} n\theta}$ has 2 maxima aligned with the principal directions



Higher order principal directions

Order 3

- $\sum_{\substack{-n \leq 3 \\ \text{n odd}}} \phi_{3,n} e^{in\theta} + \phi_{3,-n} e^{-in\theta}$ has at most 3 maxima (either 1 or 3)



Order 3 maxima directions:

Higher order principal directions

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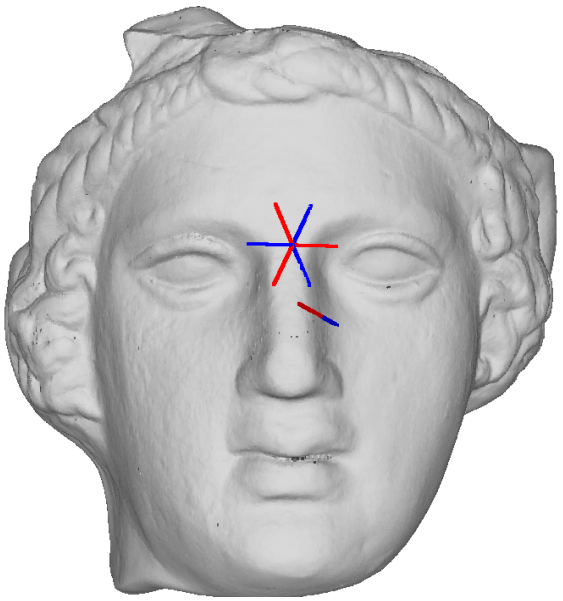
- See also [Joshi, Séquin 2010]



Order 3 maxima directions:

Stability with respect to the parameterization plane

- $(p, u) = \mathcal{T}(p) \bigcap \mathcal{P}(p)$
- γ: rotation angle along (p, u) to align $\mathcal{P}(p)$ to $\mathcal{T}(p)$
- Over $\mathcal{T}(p)$: $f(r, \theta) = \sum_{k=0}^{\infty} \sum_{n=-k}^{n=k} \phi_{k,n} r^k e^{\boldsymbol{i} n \theta}$;
- Over $\mathcal{P}(p)$: $f(R, \Theta) = \sum_{k=0}^{\infty} \sum_{n=-k}^{n=+k} \Phi_{k,n} R^k e^{\boldsymbol{i} n \Theta}$
- Assume θ and Θ are computed w.r.t. direction u

Stability Theorem

The coefficients $\Phi_{k,n}$ w.r.t to $\mathcal{P}(\boldsymbol{p})$ can be expressed with respect to the coefficients $\phi_{k,n}$ in the tangent plane $\mathcal{T}(\boldsymbol{p})$ as follows:

$$\begin{aligned} \Phi_{0,0} &= 0 \\ \Phi_{1,1} &= \Phi^*_{1,-1} = \frac{\gamma}{2} e^{-\boldsymbol{i}\frac{\pi}{2}} + o(\gamma) \\ \Phi_{k,n} &= \phi_{k,n} + \gamma F(k, n) + o(\gamma) \end{aligned} \tag{7}$$

where $F(k, n)$ is a function of the ϕ coefficients of order lower than k. It is independent of $R < R_\Phi$ and Θ.

Parameterization plane correction

Consequence (1)

$$\gamma = 2|\Phi_{1,1}| + o(|\Phi_{1,1}|) \text{ and } arg(\Phi_{1,1}) = \frac{\pi}{2} + o(\gamma)$$

The phase of $\Phi_{1,1}$ shifted by $\pi/2$ in the plane $\mathcal{P}(\boldsymbol{p})$ corresponds to the axis of rotation u. Therefore, it is possible to correct the parameterization by performing a rotation of $\mathcal{P}(\boldsymbol{p})$ along the axis u with rotation angle $2|\Phi_{1,1}|$.

Coefficient error correction

Consequence (2)

One can recover the *true* coefficients $\phi_{k,n}$, starting iteratively from the lowest order coefficients:

$$\phi_{k,n} = \Phi_{k,n} - \gamma \sum_{j=1}^{k-2} s_{j,k,n} + o(\gamma) \tag{8}$$

$$s_{j,k,n} = \sum_{\substack{p+m=n \\ |p|\leq k-j \\ |m|\leq j}} \frac{\phi_{k-j,p}}{2\boldsymbol{i}} (\phi_{j+1,m+1}(m+j+2) + \phi_{j+1,m-1}(m-j-2)) \tag{9}$$

- Rk: In particular, $\phi_{2,0} = \Phi_{2,0} + o(\gamma)$, $\phi_{2,2} = \Phi_{2,2} + o(\gamma)$, $\phi_{2,-2} = \Phi_{2,-2} + o(\gamma)$

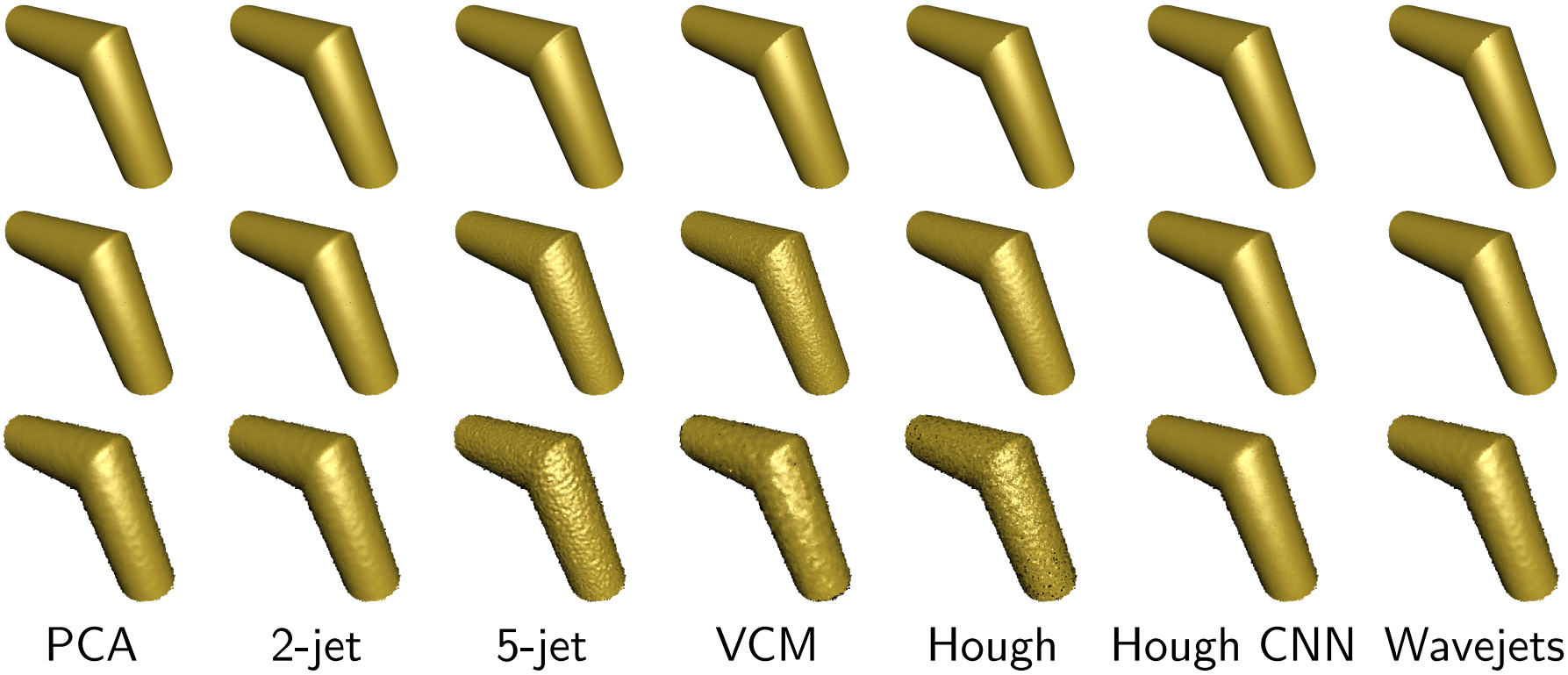
Remark on Zernike and Jets

- Normal error can be derived from order 1 jets coefficients, BUT from a nontrivial linear combinations of $V_k^{\pm 1}$
- Error to the surface can be derived from order 0 jets coefficients, BUT from a nontrivial linear combinations of V_k^0
- Integral invariants are easy to compute with Zernike polynomial BUT not easily written using jets.

Wavejets

Normal error, error to the surface, integral invariants are easy to write with Wavejets

First application: normal correction



Normal estimation on two intersecting cylinders creating a sharp edge. First row : Noise free, Second row : Gaussian noise 1.2% - Third row : Gaussian noise 3.6%

Link with Integral Invariants

- $\boldsymbol{V}(s)$ signed volume between the surface and $\mathcal{P}(p)$ in a small radius $s < R_\phi$ around one point $\boldsymbol{p}$.

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- $\boldsymbol{V}(s) = \int_0^{2\pi} A(\theta, s) d\theta$ with

$$A(\theta, s) = \int_0^s \left(\sum_{k=0}^{\infty} \sum_{n=-k}^{k} r^k \phi_{k,n} e^{\boldsymbol{i} n\theta} \right) r dr = \sum_{n=-\infty}^{\infty} a_n(s) e^{\boldsymbol{i} n\theta} \tag{10}$$

With $a_n(s) = \sum_{k=|n|}^{\infty} \frac{\phi_{k,n} s^{k+2}}{k+2}$

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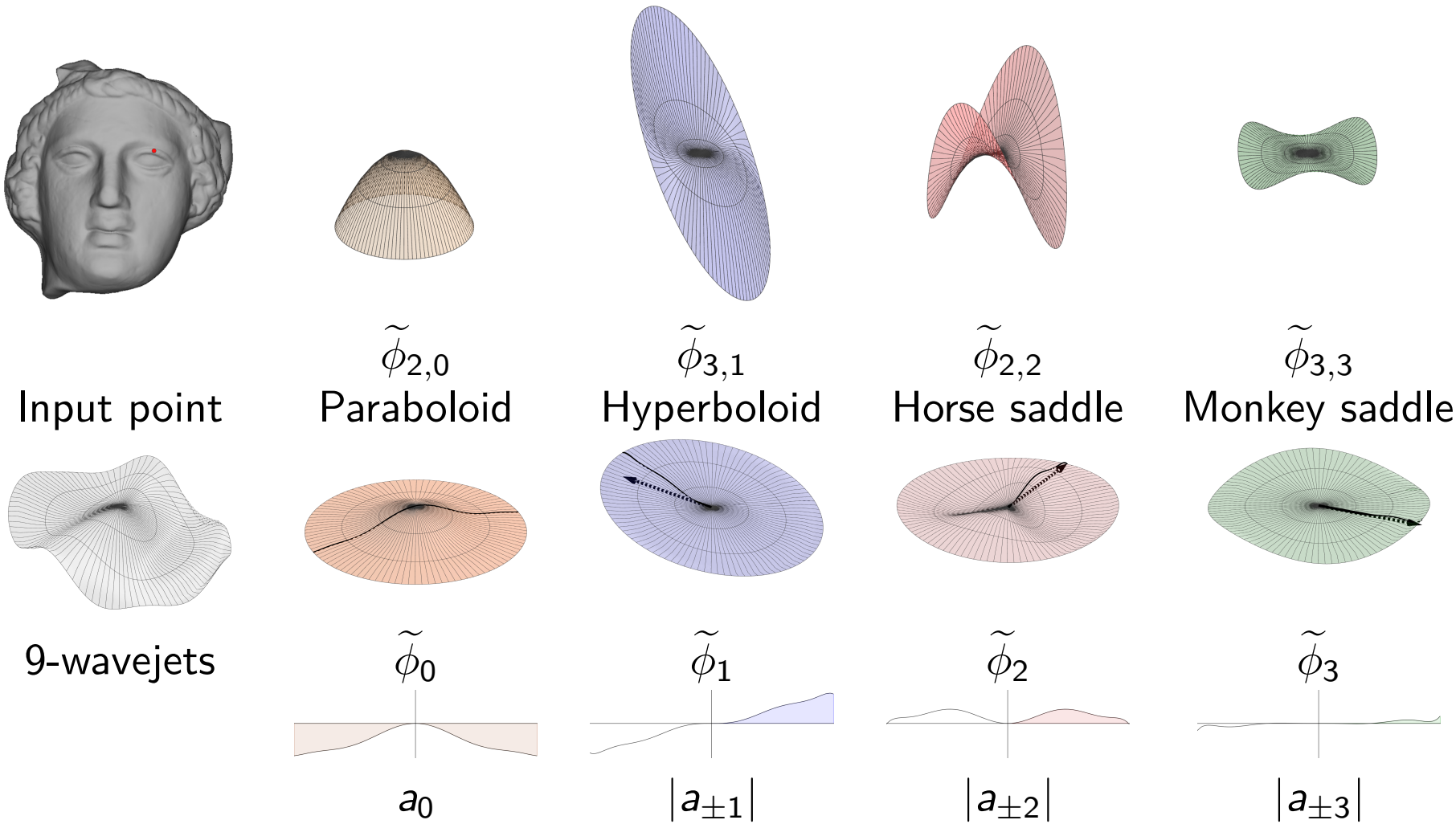
Link with Integral Invariants

$V_s(p)$: volume of the intersection of a sphere and the interior of the surface (e.g.) Manay (2006), Pottmann (2007, 2009):

$$V_s(\boldsymbol{p}) - 2\pi a_0 \approx \frac{2}{3}\pi s^3 \ . \tag{11}$$



Wavejets (order 9) decomposition of a real surface.
$\widetilde{\phi}_{k,n}(r,\theta) = r^k \left(\phi_{k,n} e^{\boldsymbol{i} n\theta} + \phi_{k,-n} e^{-\boldsymbol{i} n\theta}\right)$ and $\widetilde{\phi}_n = \sum_{k=0}^{\infty} \widetilde{\phi}_{k,n}$.

Computing Wavejets on point sets

Wavejets regression

Assume we have L neighboring points (r_l, θ_l, z_l) then, we minimize:

$$E(\Phi) = \sum_{l=1}^{L} \left\| z_l - \sum_{k=0}^{K-1} \sum_{n=-k}^{k} {r_l}^k e^{\boldsymbol{i} n \theta_l} \phi_{k,n} \right\|_2^2$$

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- Solve performed by QR decomposition.

Results

Noisy normals | $\phi_{0,0}$ | $|\phi_{1,1}|$ | $\phi_{2,0}$ | $|\phi_{2,2}|$ | $|\phi_{3,1}|$ | $|\phi_{3,3}|$

Position enhancement filter

- *Unsharp Masking*, *inverse curvature motion*, [Gabor 1965]

Position update

Move $\boldsymbol{p}$ to its new position $\boldsymbol{p}'$:

$$\boldsymbol{p}' = \boldsymbol{p} + (\phi_{0,0} - 2\pi(\alpha_0 - 1)a_0(s))\,\boldsymbol{n}$$

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- Continuous motion, assuming $\mathcal{P}(p)$ is corrected to $\mathcal{T}(p)$ beforehand.

Normal enhancement filter

- Principle: Modify $\phi_{1,1}$ and $\phi_{1,-1}$ and deduce the false normal

Normal enhancement filter

- Principle: Modify $\phi_{1,1}$ and $\phi_{1,-1}$ and deduce the false normal

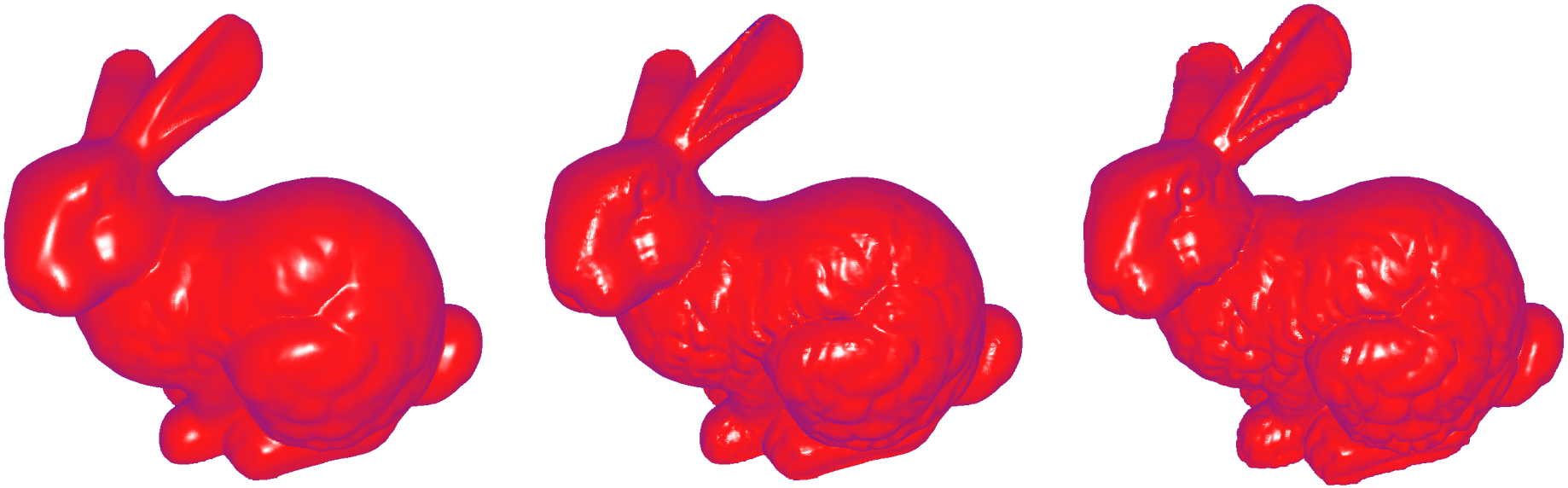
Method

Compute $\phi_{1,\pm 1}$:

$$\phi_{1,\pm 1} = -\pi(\alpha_{\pm 1} - 1)a_{\pm 1}(s)$$

Deduce the normal by rotating the current normal of angle $2|\phi'_{1,1}|$ and axis given by $arg(\phi'_{1,1}) + \frac{\pi}{2}$.

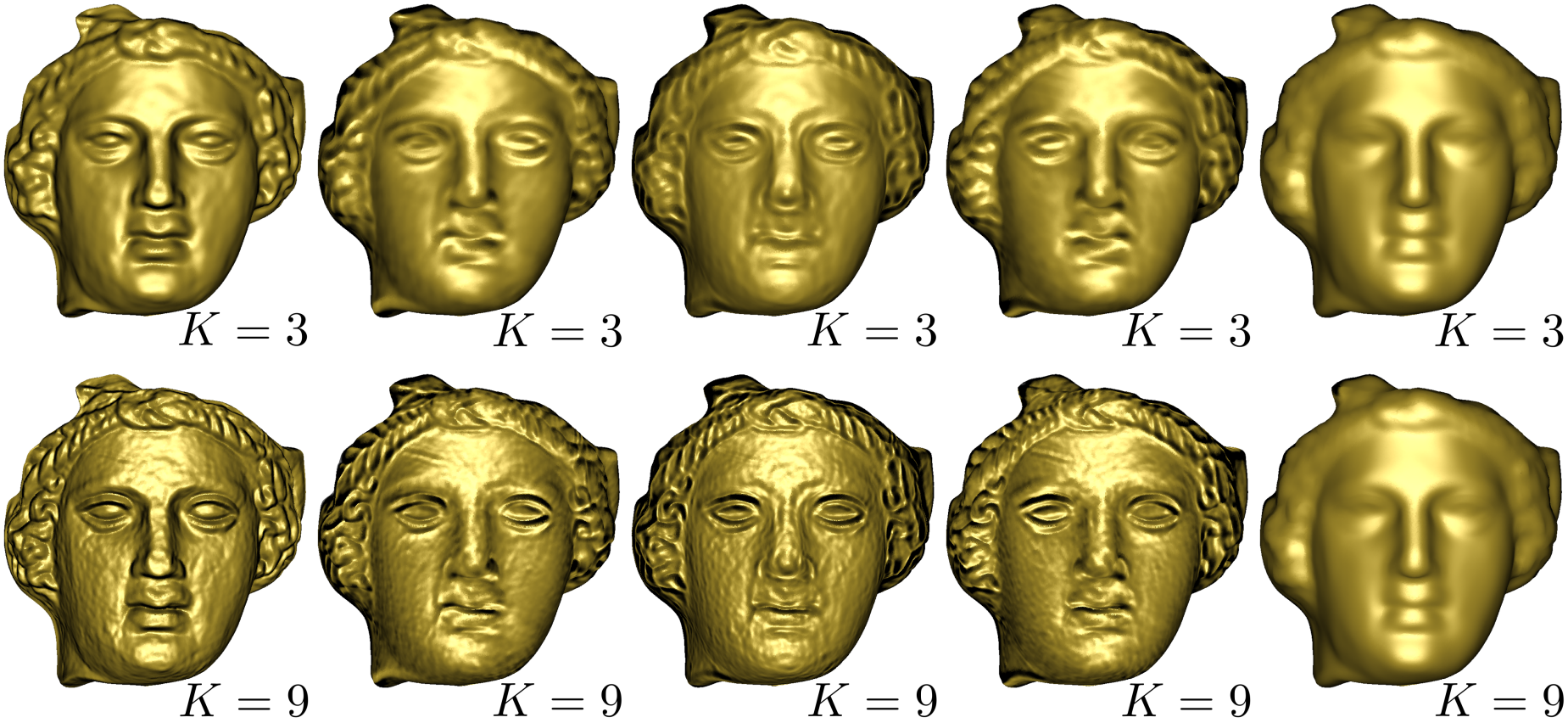
Results



Original | Enhanced normals | Enhanced positions

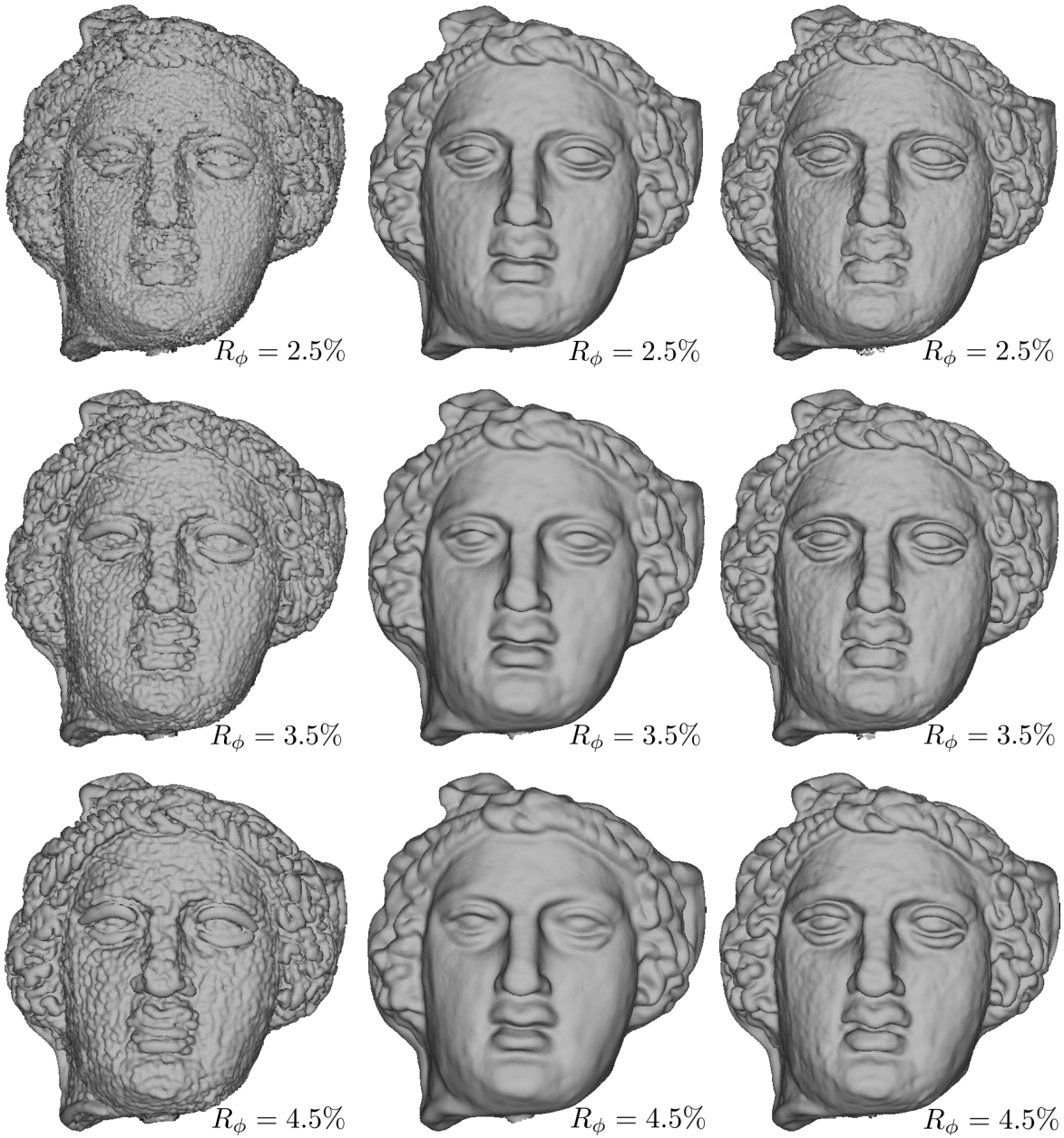
Normal and position enhancement on a bunny with 6-Wavejets. R_ϕ is equal to 3% of the shape diameter, and $\alpha_0 = \alpha_{\pm 1} = 2$.

Results



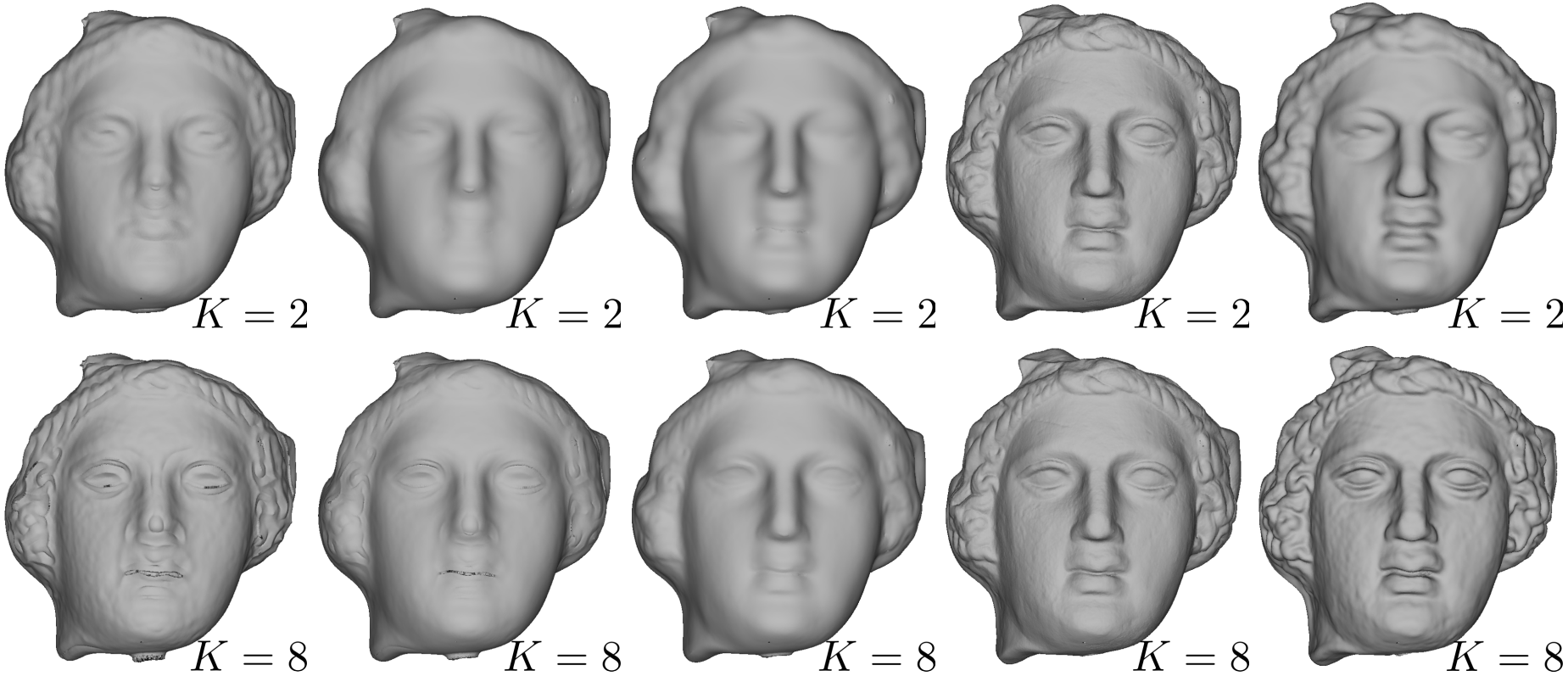
Normal amplification

Comparisons with Unsharp masking



$K = 8$ $K = 2$ $K = 8$

Results



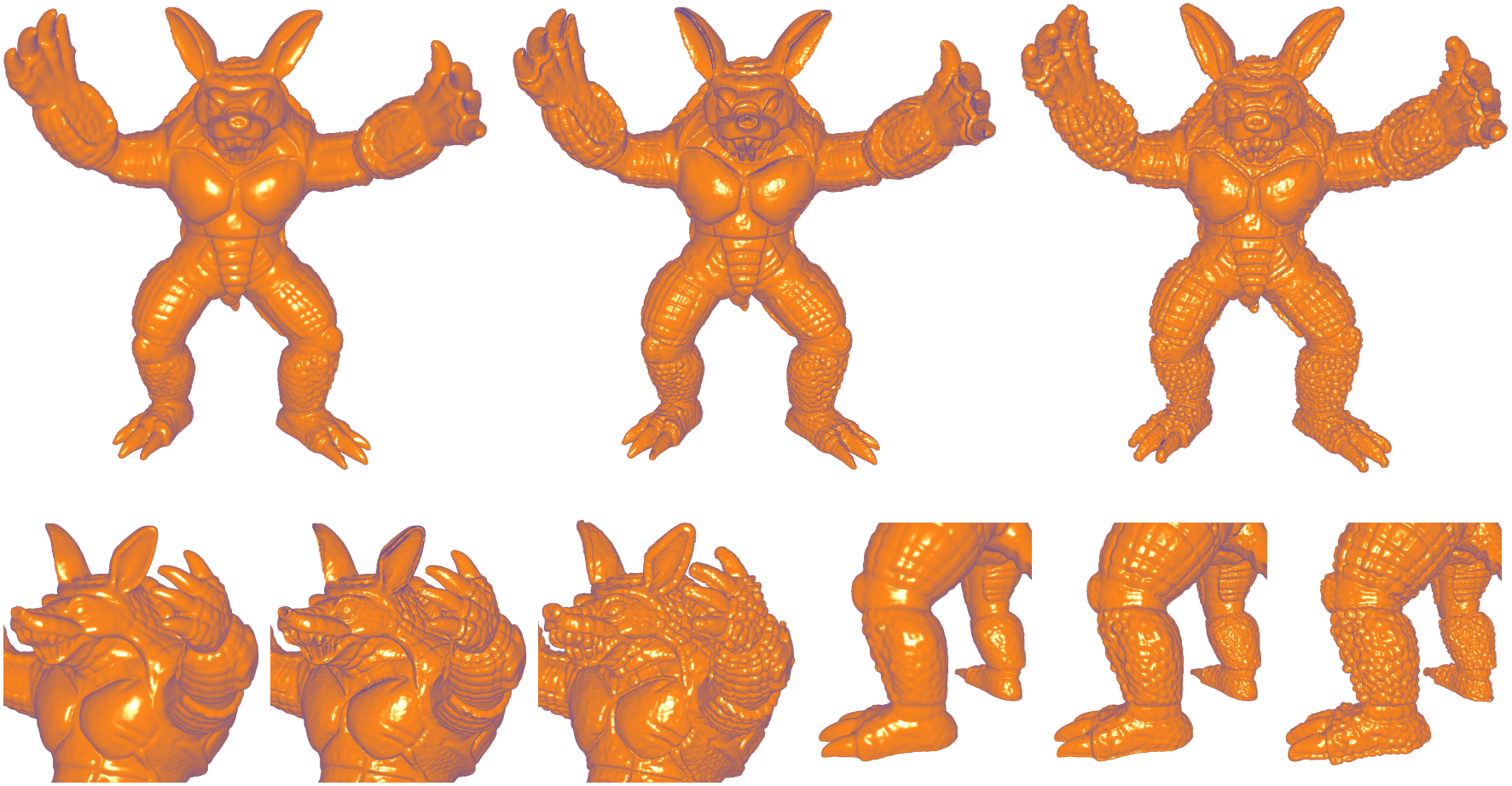
$\alpha_0 = -2$ $\alpha_0 = -1$ $\alpha_0 = 0$ $\alpha_0 = 1$ $\alpha_0 = 2$

Position amplification

Resilience to noise

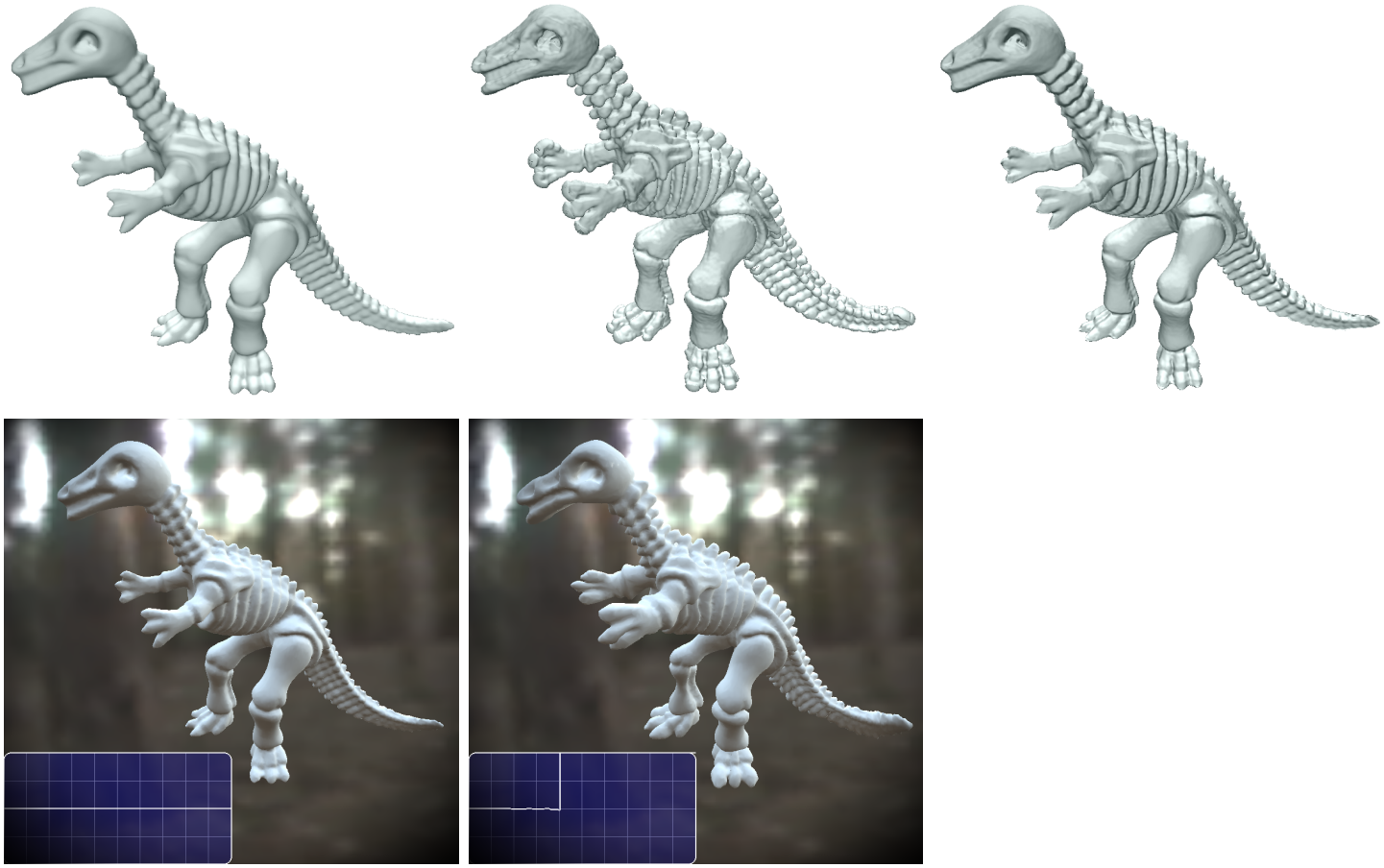
Input | $\alpha_0 = 2$ | $\alpha_{\pm 1} = 0$ | $\alpha_{\pm 1} = 2$

Results



Outputs of our procedures on an armadillo (Left:original; Middle: normal-based detail enhancement with $K = 7, \alpha_{\pm 1} = 3$; Right: position-based detail enhancement with $K = 6, \alpha_0 = 2$).

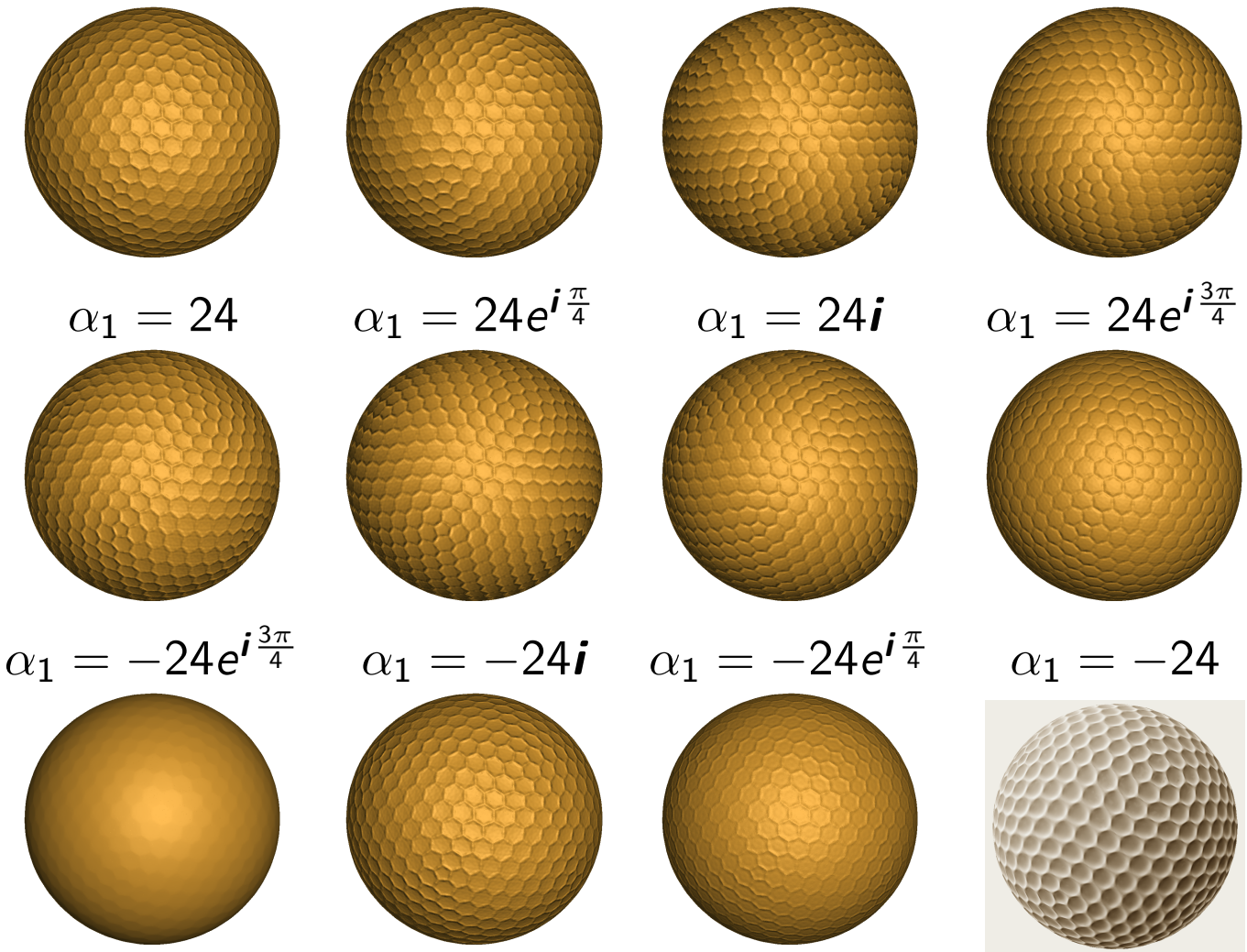
Comparisons



Original Positions enhanced Normals enhanced

Top row: wavejets. Bottom row: high-boost filter in the manifold harmonics basis [Vallet-Lévy 2008].

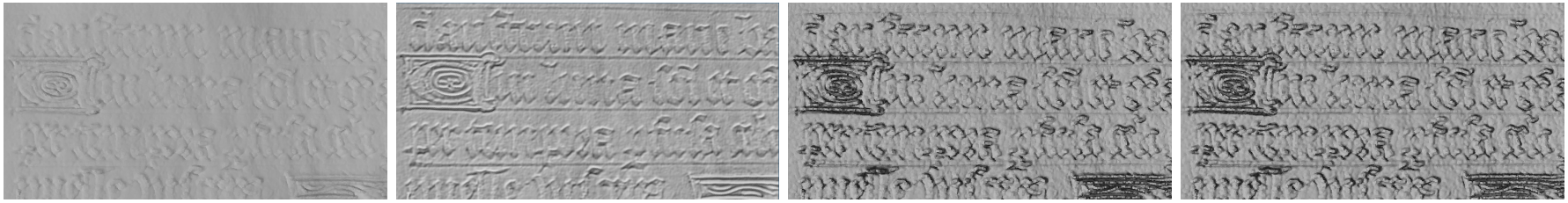
Comparisons



Original Ours ($\alpha_1 = 24$) [Cignoni05] [Rusinkiewicz06]

Normal enhancement on a golf ball with $K = 9$. (nb: $\alpha_{-1} = \alpha_1^*$)

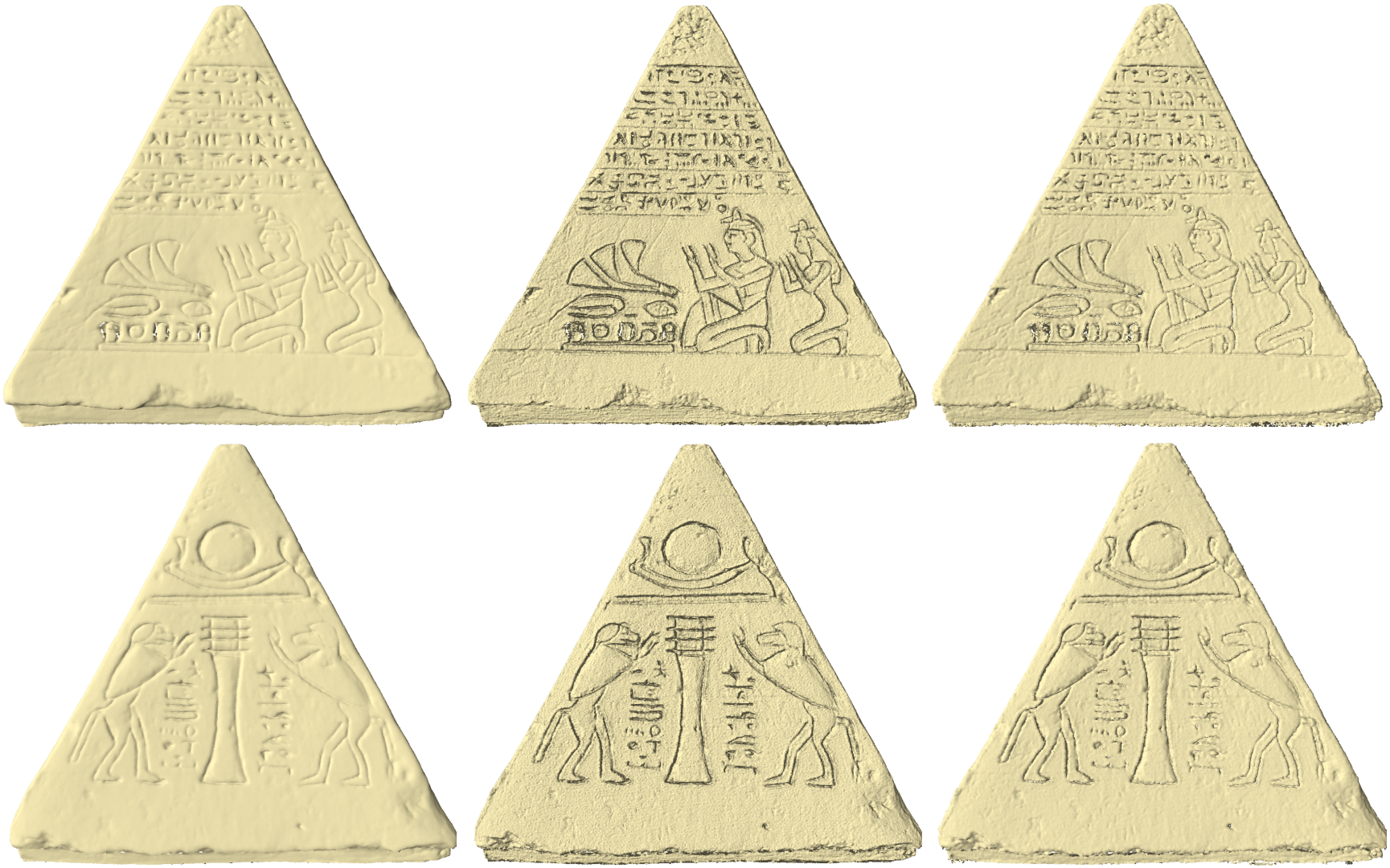
Comparisons



Original [Rusinkiewicz06] [Cignoni05] $(K = 5, \alpha_{\pm 1} = 3)$

Normal enhancement with different methods.

More results



Original Normal enhanced Position enhanced

$K = 7$ (normal exaggeration) and order $K = 6$ filters (position filter)

$\alpha_0 = \alpha \pm 1 = 2.$

More results

Original Normal enhanced Position enhanced

Applying order 9 (normal exaggeration) and order 8 filters (position filter) to the Anubis datasets with $\alpha_0 = \alpha_{\pm 1} = 2$.

Complexity per point

- Given a set of N neighbors and a Wavejet order K: $O(NK^4)$
- Once the wavejet is computed, applying the filter amounts to summing K terms: $O(K)$
- $1.5M$ points, order $K = 6$: Decomposition time $1min40s$; Filtering time: $0.6s$.

Conclusion

- A new basis for representing the local variations of the surface
- Interpretation for the third order principal directions as tensor eigenvectors (as defined by Qi 2007)
- Filtering higher order derivatives using a_2, a_3...
- Filter design in this basis is still a work in progress.

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Wavejets: A Local Frequency Framework for Shape Detail Amplification, Yohann Béarzi, Julie Digne, Raphaëlle Chaine, conditionally accepted to Computer Graphics Forum (Eurographics 2018).