Joint segmentation/registration model by shape alignment via weighted total variation minimization and nonlinear elasticity

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Image **segmentation** and image **registration** are fundamental tasks:

- **Image segmentation**: aims to **partition** a given image into meaningful components.

![Fig. 1 - Example of contour segmentation of a slice of a brain. (Permission of 'Laboratory Of NeuroImaging, UCLA'). On the left, the initial contour. On the right, the obtained result at the end of the process.](image)
Image registration:

- Given two images called Template and Reference ⇒ registration consists in determining an optimal diffeomorphic transformation $\varphi$ such that the deformed Template image is aligned with the Reference.

- ⇒ interest in clinical studies, when comparing an image to a database or for volumetric purposes.

- For images of the same modality: ⇒ correlate the geometrical features and the intensity level distribution of the Reference and those of the deformed Template.

- For images acquired through different mechanisms: ⇒ correlate both images while preserving the modality of the Template.
From left to right, top to bottom: Reference $R$; Template $T$ (mouse atlas and gene expression data); deformed Template; distortion map drawing the vectors from the grid points from the Reference image to non grid points after registration every 7 rows and columns; deformed grid.
In [13, 2013], Sotiras et al. provide an overview of the different existing registration methods and according to them, an image registration algorithm consists of three main components:

1. a deformation model;
2. an objective function;
3. an optimization method.
Deformation model

⇒ geometric transformations derived from physical models: elastic models, viscous fluid model, diffusion models, curvature-based models and flows of diffeomorphisms;

⇒ geometric transformations derived from interpolation theory: radial basis functions, elastic body splines, piecewise affine models, free-form deformations, etc.;

⇒ knowledge-based geometric transformations: statistically-constrained geometric transformations, biomechanical/biophysical models.
<table>
<thead>
<tr>
<th>Objective function</th>
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<td>⇒ <strong>geometric methods</strong>: landmark information;</td>
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<tr>
<td>⇒ <strong>iconic methods</strong>: intensity-based methods, attribute-based methods, information-theoretic approaches;</td>
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<td>⇒ <strong>hybrid methods</strong>.</td>
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### Optimization

- **continuous methods:** gradient descent, conjugate gradient, Quasi-Newton, stochastic gradient descent methods;

- **discrete methods:** graph-based, belief propagation, linear programming methods;

- **greedy approaches and evolutionary algorithms.**
**Proposed Method**: falls within the scope of
- *non-parametric* methods: deformation *not restricted* to a parameterizable set;
- *iconic methods*: introduction of an *intensity and shape-based* criterion.

**Scope**: 
- devise a *theoretically well-motivated* registration model in a *variational* formulation, authorizing *large and smooth* deformations, while keeping the deformation map *topology-preserving*.

**Physical arguments** often motivate the way the regularizer is designed ⇒ *classical regularizers* such as *linear elasticity* not *suitable* for problems involving *large deformations* since assuming *small strains* and the *validity* of Hooke’s law.
**Proposed Methodology:**

- introduction of a geometric dissimilarity measure based on shape comparisons and allowing for joint segmentation and registration.

- use of the weighted total variation in order to align the edges of the objects (*Bresson et al.* ([3, 2007]), *Chan et al.* ([5, 2006]), *Strang* ([14, 1983]));

- the algorithm produces both a smooth mapping between the two shapes and the segmentation of the object contained in the Reference.
Proposed Methodology: (continuation)

⇒ introduction of a nonlinear-elasticity-based smoother.

⇒ shapes to be matched viewed as isotropic, homogeneous, hyperelastic materials and more precisely as Saint Venant-Kirchhoff materials (see Ciarlet's book [7, 1985]).
⇒ Hyperelasticity is a suitable framework when dealing with large and nonlinear deformations.

Rubber, filled elastomers, biological tissues are often modeled within the hyperelastic framework.
Prior related works suggest to jointly treat segmentation and registration. Among others:

- **Yezzi et al.** ([16, 2001]): *curve evolution approach in a level set framework*;

- **Vemuri et al.** ([15, 2003]): *coupled PDE model to perform both segmentation and registration by evolving the level sets of the source image*;

- **Lord et al.** ([11, 2007]): *model based on metric structure comparisons*;

- **Le Guyader and Vese** ([10, 2011]): *combines a matching criterion founded on the active contour without edges model ([6, 2001]) and a nonlinear elasticity-based regularizer (Ciarlet-Geymonat stored energy function)*.
There are forward and backward transformations:

- the former is done in the Lagrangian framework where a forward transformation $\Psi$ is sought and grid points with intensity values $T(x)$ are moved and arrive at non-grid points $y = \Psi(x)$ with intensity values $T(x) = T(\Psi^{-1}(y))$;

- the latter is done in the Eulerian framework (considered here), where a backward transformation $\varphi = \Psi^{-1}$ is searched and grid points $y$ in the deformed image originate from non-grid points $x = \varphi(y) = \Psi^{-1}(y)$ and are assigned intensity values $T(\varphi(y)) = T(\Psi^{-1}(y)) = T(x)$. 

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A Joint Segmentation/Registration Model
\( \Omega \subset \mathbb{R}^2 \): connected bounded open subset of \( \mathbb{R}^2 \) with boundary \( \partial \Omega \) of class \( C^1 \).

\( R : \bar{\Omega} \to \mathbb{R} \) the Reference image and \( T : \bar{\Omega} \to \mathbb{R} \) the Template image.

For theoretical purposes:
- \( T \) compactly supported on \( \Omega \) to ensure that \( T \circ \varphi \) is well-defined.
- \( T \) is assumed to be Lipschitz continuous with Lipschitz constant \( \kappa > 0 \) \( \Rightarrow T \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}) \).

\( \varphi : \bar{\Omega} \to \mathbb{R}^2 \): deformation (or transformation). A deformation is a smooth mapping that is orientation-preserving and injective, except possibly on \( \partial \Omega \) (self-contact must be allowed (Ciarlet, [8, 1988])).

\( u \): associated displacement s.t. \( \varphi = \text{Id} + u \). The deformation gradient is defined by \( \nabla \varphi = I_2 + \nabla u, \bar{\Omega} \to M_2(\mathbb{R}) \).
• **Goal**: find a smooth deformation field \( \varphi \) s.t. the deformed Template matches the Reference.

⇒ model phrased as a functional minimization problem with unknown \( \varphi \),
\[
\min_{\varphi} I(\varphi) = E_{\text{dist}}(\varphi) + E_{\text{reg}}(\varphi)
\]
combining a distance measure criterion \( E_{\text{dist}}(\varphi) \) and a regularizer on \( \varphi \), \( E_{\text{reg}}(\varphi) \).

⇒ we start with the construction of the regularizer \( E_{\text{reg}}(\varphi) \) on the deformation \( \varphi \).
Fundamental concepts/notations for the construction of the nonlinear-elasticity-based regularizer:

- **Right Cauchy-Green strain tensor:**
  \[ C = \nabla \varphi^T \nabla \varphi = F^T F \in S^2 = \{ A \in M_2(\mathbb{R}), \ A = A^T \}. \]
  \[ \Rightarrow \] quantifier of the square of local change in distances due to deformation.

- **Green-Saint Venant strain tensor:**
  \[ E = \frac{1}{2} \left( \nabla u + \nabla u^T + \nabla u^T \nabla u \right). \]
  \[ \Rightarrow \] measure of the deviation between \( \varphi \) and a rigid deformation.

- **Notations:** \( A : B = \text{tr} A^T B \) and \( ||A|| = \sqrt{A : A} \) (Frobenius norm).
Fundamental concepts/notations for the construction of the nonlinear-elasticity-based regularizer:

- **Stored energy function of an isotropic, homogeneous, hyperelastic material:**

  \[
  W(F) = \hat{W}(E) = \frac{\lambda}{2} (\text{tr } E)^2 + \mu \text{tr } E^2 + o (\|E\|^2),
  \]

  \[
  F^T F = I + 2E.
  \]

- **Stored energy function of a Saint Venant-Kirchhoff material:**

  \[
  W_{\text{SVK}}(F) = \hat{W}(E) = \frac{\lambda}{2} (\text{tr } E)^2 + \mu \text{tr } E^2.
  \]

- **Saint Venant-Kirchhoff material:** ⇒ simplest one whose stored energy function agrees with expansion (1).
Construction of the nonlinear-elasticity-based regularizer:

⇒ proposed regularizer on \( \varphi \) based on the coupling of the stored energy function \( W_{SVK} \) of a Saint Venant-Kirchhoff material and on a term controlling that the Jacobian determinant remains close to 1.
⇒ thus the deformation map does not exhibit contractions or expansions that are too large.

To sum up, the regularization can be written as

\[
E_{reg}(\varphi) = \int_{\Omega} W(\nabla \varphi) \, dx,
\]

with

\[
W(F) = W_{SVK}(F) + \mu (\det F - 1)^2.
\]

(The weighting of the determinant component by parameter \( \mu \) is for technical purpose.)

The regularizer on the deformation \( \varphi \) is complemented by a dissimilarity measure \( E_{dist} \) that we now address.
**Construction of the dissimilarity measure:**

- Based on the unified active contour model developed by Bresson *et al.*, [3, 2007] - designed to overcome the limitation of local minima and to deal with global minimum.

- Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an edge detector function satisfying $g(0) = 1$, $g$ strictly decreasing, and $\lim_{r \rightarrow +\infty} g(r) = 0$.

- Apply the edge detector function to the norm of the Reference image gradient: $g(|\nabla R|)$ with $|\nabla R| : \bar{\Omega} \rightarrow \mathbb{R}^+$. From now on, for the sake of conciseness, we set $g := g(|\nabla R|)$ and for theoretical purposes, we assume that $\exists c > 0$ such that $0 < c < g$ and that $g$ is Lipschitz continuous.
Construction of the dissimilarity measure:

Figure: Left: Image $R$; Right: $g(|\nabla R|)$
Construction of the dissimilarity measure:

- Use of the generalization of the notion of function of bounded variation to the setting of $BV$-spaces associated with a Muckenhoupt’s weight function (see Baldi [2, 2001]);

- Definition of the weighted $BV$-space related to weight $g$;

**Hypotheses on general weight $w$:** $\Omega_0$ being a neighborhood of $\bar{\Omega}$, $w \in L^1_{loc}(\Omega_0)$ assumed to belong to the global Muckenhoupt’s $A_1 = A_1(\Omega)$ class of weight functions, i.e., $w$ satisfies the condition:

$$C \, w(x) \geq \frac{1}{|B(x, r)|} \int_{B(x, r)} w(y) \, dy, \text{ a.e.}$$

in any ball $B(x, r) \subset \Omega_0$. 
Construction of the dissimilarity measure:

Definition 1

[2, Definition 2] Let \( w \) be a weight function in the class 
\( A_1^* = \{ v \in A_1, \ v \ lsc \} \). We denote by \( BV(\Omega, w) \) the set of functions 
\( u \in L^1(\Omega, w) \) (set of functions that are integrable with respect to the measure \( w(x)dx \)) such that:

\[
\sup \left\{ \int_{\Omega} u \ \text{div}(\phi) \ dx : |\phi| \leq w \ everywhere, \ \phi \in \text{Lip}_0(\Omega, \mathbb{R}^2) \right\} < \infty, \tag{3}
\]

with \( \text{Lip}_0(\Omega, \mathbb{R}^2) \) the space of Lipschitz functions with compact support. We denote by \( \text{var}_w u \) the quantity (3).
Construction of the dissimilarity measure:

Weighted Total Variation with weight $g$

$$\text{var}_g u = \sup \left\{ \int_{\Omega} u \div(\phi) \, dx : |\phi| \leq g \text{ everywhere}, \phi \in \text{Lip}_0 \right\},$$

with $g = \frac{1}{1 + |\nabla R|^2}$.

Interpretation of $\text{var}_g u$

If $u = 1_{\Omega_C}$ with closed set $\Omega_C \subset \Omega$, $C$ being the boundary of $\Omega_C$,

$$\text{var}_g (u = 1_{\Omega_C}) = \int_C g \, ds.$$
Distance measure criterion

\[ E_{dist}(\varphi) = \text{var}_g T \circ \varphi + \frac{\nu}{2} \int_\Omega (T \circ \varphi - R)^2 \, dx. \]

In the end, denoting by \( W^{1,4}(\Omega, \mathbb{R}^2) \) the Sobolev space of functions \( \varphi \in L^4(\Omega, \mathbb{R}^2) \) with distributional derivatives up to order 1 which also belong to \( L^4(\Omega) \), the considered minimization problem is:
Functional minimization problem

\[
\begin{align*}
\inf \left\{ I(\varphi) = \text{var}_g \ T \circ \varphi + \int_{\Omega} f(x, \varphi(x), \nabla \varphi(x)) \, dx \right\} \\
= \text{var}_g \ T \circ \varphi + \int_{\Omega} \left[ \frac{\nu}{2} \left( T(\varphi) - R \right)^2 + W(\nabla \varphi(x)) \right] \, dx,
\end{align*}
\]

with \( \varphi \in \text{Id} + W^{1,4}_0(\Omega, \mathbb{R}^2) \) and \( f(x, \varphi, \xi) = \frac{\nu}{2} \left( T(\varphi) - R \right)^2 + W(\xi) \).

Remarks

(i) \( \varphi \in \text{Id} + W^{1,4}_0(\Omega, \mathbb{R}^2) \) means that \( \varphi = \text{Id} \) on \( \partial \Omega \) and \( \varphi \in W^{1,4}(\Omega, \mathbb{R}^2) \).
Remarks (continuation)

(ii) From generalized Holdër’s inequality, if \( \varphi \in W^{1,4}(\Omega, \mathbb{R}^2) \), then \( \det \nabla \varphi \in L^2(\Omega) \).

(iii) From [1, Corollary 3.2; chain rule],
\[ T \circ \varphi \in W^{1,4}(\Omega) := W^{1,4}(\Omega, \mathbb{R}) \subset BV(\Omega) \], since \( \Omega \) is bounded.
As \( g \leq w^* \) with \( w^* = 1 \), \( BV(\Omega) \subset BV(\Omega, g) \) and
\[ T \circ \varphi \in BV(\Omega, g). \]
**Theoretical issue**: Function $f$ in (P) fails to be quasiconvex (*weaker definition of convexity*) $\Rightarrow$ raises a drawback of theoretical nature since we cannot obtain the weak lower semi-continuity of the functional.

**Idea**: replace problem (P) by an associated relaxed problem (QP) formulated in terms of the quasiconvex envelope $Qf$ of $f$.

Note that the introduced problem (QP) is not the relaxed problem associated to (P) in the sense of Dacorogna ([9, 2008]). One has $\inf QP \leq \inf P$

We start by establishing the explicit expression of the quasiconvex envelope of $f$ and derive the associated relaxed problem (QP).
Proposition 2

The quasiconvex envelope $Qf$ of $f$ is defined by

$$Qf(x, \varphi, \xi) = \frac{\nu}{2} (T(\varphi) - R)^2 + QW(\xi),$$

with

$$QW(\xi) = \begin{cases} W(\xi) & \text{if } \|\xi\|^2 \geq 2\frac{\lambda+\mu}{\lambda+2\mu}, \\ \Psi(\text{det } \xi) & \text{if } \|\xi\|^2 < 2\frac{\lambda+\mu}{\lambda+2\mu} \end{cases}$$

and $\Psi$, the convex mapping such that $\Psi : t \mapsto -\frac{\mu}{2} t^2 + \mu (t - 1)^2 + \frac{\mu(\lambda+\mu)}{2(\lambda+2\mu)}$.

The relaxed problem $(QP)$ is chosen to be:

$$\inf \left\{ \bar{I}(\varphi) = \text{var}_g T \circ \varphi + \int_{\Omega} Qf(x, \varphi(x), \nabla \varphi(x)) \right\}, \quad (QP)$$

with $\varphi \in Id + W^{1,4}_0(\Omega, \mathbb{R}^2)$. 

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A Joint Segmentation/Registration Model
Remarks

(i) The judicious rewriting of $W(\xi)$ into
$W(\xi) = \beta (\|\xi\|^2 - \alpha)^2 + \Psi(\det \xi)$ allows to see that $W^{1,4}(\Omega, \mathbb{R}^2)$ is a suitable functional space.

(ii) We can prove a stronger result, namely, that the polyconvex envelope of $W$, $PW$, coincides with the quasiconvex envelope of $W$: $PW = QW$.

(iii) We understand better through the proof of Proposition 2. the choice of the weighting parameter balancing the component $(\det \xi - 1)^2$; it has been chosen in order that the mapping $\Psi$ is convex.
We emphasize that the extension of the model to the 3D case is not straightforward. Indeed, in three dimensions, the expression of $W_{SVK}(\xi)$ involves the cofactor matrix denoted by $\text{Cof} \xi$ as follows:

$$W_{SVK}(\xi) = \frac{\lambda}{8} \left( \|\xi\|^2 - \left( 3 + \frac{2\mu}{\lambda} \right) \right)^2 + \frac{\mu}{4} \left( \|\xi\|^4 - 2\|\text{Cof} \xi\|^2 \right)$$

$$- \frac{\mu}{4\lambda} (2\mu + 3\lambda).$$

$\Rightarrow$ it is not clear that one can derive the explicit expression of the quasiconvex envelope $QW$ of $W$. 
Theoretical Results: Existence of Minimizers

Theorem 3 (Existence of Minimizers)

Assume that there exists \( \hat{\varphi} \in \text{Id} + W_0^{1,4}(\Omega, \mathbb{R}^2) \) such that \( \overline{I}(\hat{\varphi}) < +\infty \). Then the infimum of (QP) is attained.

We now concentrate upon the relation between \( \inf \text{QP}(= \min \text{QP}) \) and \( \inf P \) when additional hypotheses are assumed. Consider the auxiliary problem (AP):

\[
\inf \left\{ \mathcal{F}(\varphi) = \int_{\Omega} f(x, \varphi(x), \nabla \varphi(x)) \, dx : \varphi \in \text{Id} + W_0^{1,4}(\Omega, \mathbb{R}^2) \right\},
\]

(AP)

with \( f \) given in (P).

The following results can be established successively:
Proposition 4

The relaxed problem in the sense of Dacorogna associated to (AP) is defined by:

\[
\inf \left\{ \bar{\mathcal{F}}(\varphi) = \int_{\Omega} Qf(x, \varphi(x), \nabla \varphi(x)) \, dx \mid \varphi \in Id + W_{0}^{1,4}(\Omega, \mathbb{R}^{2}) \right\},
\]

(QAP)

with \(Qf\) given in Proposition 2.

Proposition 5

The infimum of (QAP) is attained. Let then \(\varphi^{*} \in W^{1,4}(\Omega, \mathbb{R}^{2})\) be a minimizer of the relaxed problem (QAP). Then there exists a sequence \(\{\varphi_{\nu}\}_{\nu=1}^{\infty} \subset \varphi^{*} + W_{0}^{1,4}(\Omega, \mathbb{R}^{2})\) such that \(\varphi_{\nu} \to \varphi^{*}\) in \(L^{4}(\Omega, \mathbb{R}^{2})\) as \(\nu \to +\infty\) and \(\mathcal{F}(\varphi_{\nu}) \to \bar{\mathcal{F}}(\varphi^{*})\) as \(\nu \to +\infty\), yielding to \(\min(QAP) = \inf(AP)\).

Moreover, the following holds: \(\varphi_{\nu} \rightharpoonup \varphi^{*}\) in \(W^{1,4}(\Omega, \mathbb{R}^{2})\).
Proposition 6 (Relaxation result)

Let us assume that $T \in W^{2,\infty}(\mathbb{R}^2, \mathbb{R})$, $\nabla T$ being Lipschitz continuous with Lipschitz constant $\kappa'$. Let $\bar{\varphi} \in Id + W^{1,4}_0(\Omega, \mathbb{R}^2)$ be a minimizer of the relaxed problem (QP). Then there exists a sequence $\{\varphi_\nu\}_{\nu=1}^\infty \subset \bar{\varphi} + W^{1,4}_0(\Omega, \mathbb{R}^2)$ such that $\varphi_\nu \rightharpoonup \bar{\varphi}$ in $W^{1,4}(\Omega, \mathbb{R}^2)$ as $\nu \to \infty$ and

$$\int_\Omega f(x, \varphi_\nu(x), \nabla \varphi_\nu(x)) \, dx \to \int_\Omega Qf(x, \bar{\varphi}(x), \nabla \bar{\varphi}(x)) \, dx.$$ If moreover $\nabla \varphi_\nu$ strongly converges to $\nabla \bar{\varphi}$ in $L^1(\Omega, M_2(\mathbb{R}))$, then one has $I(\varphi_\nu) \to \bar{I}(\bar{\varphi})$ as $\nu \to \infty$ and therefore

$$\inf(QP) = \min(QP) = \inf(P).$$
Motivations and Prior related work:

In [12, 1990], Negrón Marrero describes and analyzes a numerical method to detect singular minimizers and to avoid the Lavrentiev phenomenon for three dimensional problems in nonlinear elasticity.

⇒ consists in decoupling function \( \varphi \) from its gradient and in formulating a related decoupled problem under inequality constraint.

Idea:

⇒ introduction of an auxiliary variable \( V \) simulating the Jacobian deformation field \( \nabla \varphi \). The underlying idea is to remove the nonlinearity in the derivatives of the deformation;

⇒ introduction of an auxiliary variable \( \tilde{T} \) simulating the deformed Template \( T \circ \varphi \);

⇒ derivation of a functional minimization problem phrased in terms of \( \varphi, V \) and \( \tilde{T} \).
Differences with [12]:

- We do not formulate a minimization problem under constraints but incorporate $L^p$-type penalizations.

- In [12], the author focuses on the decoupled discretized problem (discretized with the finite element method) for which the existence of minimizers is guaranteed ⇒ we consider the continuous problem.

- The author Assumes that the finite element approximations satisfy some convergence hypotheses.

- In our case, less regularity is required for the formulation of the inequality constraint.
Decoupled Problem

\[ \bar{I}_\gamma(\varphi, V, \tilde{T}) = \text{var}_g \tilde{T} + \frac{\nu}{2} \| T(\varphi) - R \|_{L^2(\Omega)}^2 + \int_{\Omega} Q\hat{W}(V) \, dx \]

\[ + \frac{\gamma}{2} \| V - \nabla \varphi \|_{L^2(\Omega, M^2)}^2 + \gamma \| \tilde{T} - T \circ \varphi \|_{L^1(\Omega)}. \]  

We set \( \hat{W} = \text{Id} + W^{1,2}_0(\Omega, \mathbb{R}^2) \) and \( \hat{\chi} = \{ V \in L^4(\Omega, M_2(\mathbb{R})) \} \).

The decoupled problem consists in minimizing (4) on \( \hat{W} \times \hat{\chi} \times BV(\Omega, g) \). Then the following theorem holds.
Theorem 7 (Γ-convergence)

Let \((\gamma_j)\) be an increasing sequence of positive real numbers such that \(\lim_{j \to +\infty} \gamma_j = +\infty\). Let also \(\left(\varphi_k(\gamma_j), V_k(\gamma_j), \tilde{T}_k(\gamma_j)\right)\) be a minimizing sequence of the decoupled problem with \(\gamma = \gamma_j\). Then there exist a subsequence denoted by

\[
\left(\varphi_N(\gamma_{\zeta(j)})(\gamma_{\zeta(j)}), V_N(\gamma_{\zeta(j)})(\gamma_{\zeta(j)}), \tilde{T}_N(\gamma_{\zeta(j)})(\gamma_{\zeta(j)})\right)
\]

of \(\left(\varphi_k(\gamma_j), V_k(\gamma_j), \tilde{T}_k(\gamma_j)\right)\) and a minimizer \(\varphi\) of \(\bar{I}\)

\((\varphi \in Id + W_0^{1,4}(\Omega, \mathbb{R}^2))\) such that:

\[
\lim_{j \to +\infty} \bar{I}_{\gamma_{\zeta(j)}} \left(\varphi_N(\gamma_{\zeta(j)})(\gamma_{\zeta(j)}), V_N(\gamma_{\zeta(j)})(\gamma_{\zeta(j)}), \tilde{T}_N(\gamma_{\zeta(j)})(\gamma_{\zeta(j)})\right) = \bar{I}(\varphi).
\]
Alternating algorithm

We first solve

$$\inf_{\tilde{T}} E(\tilde{T}) = \operatorname{var}_{g} \tilde{T} + \gamma \|\tilde{T} - T \circ \varphi\|_{L^1(\Omega)} \quad (P_1)$$

for fixed $\varphi$ and $V$.

- Convex regularization, Dual formulation of the weighted total variation, Chambolle's projection algorithm ([4, 2004]);
- **Important remark**: If $T \circ \varphi$ is a characteristic function, if $\tilde{T}$ is a minimizer of $E$, then for almost every $\mu_0 \in [0, 1]$, one has that the characteristic function $1_{\{x \mid \tilde{T}(x) > \mu_0\}}$ is a global minimizer of $E$. 
Alternating algorithm (Continuation)

We then solve (for fixed $\tilde{T}$):

$$\inf_{\varphi, \tilde{T}} \mathcal{J}_\epsilon(\varphi, V) + \frac{\gamma}{2} \left\| \nabla \varphi - V \right\|^2_{L^2(\Omega, M_2(\mathbb{R}))} + \gamma \left\| \tilde{T} - T \circ \varphi \right\|_{L^1(\Omega)},$$

with

$$\mathcal{J}_\epsilon(\varphi, V) = \int_\Omega W(V) H_\epsilon \left( \| V \|^2 - \alpha \right) \, dx$$

$$+ \int_\Omega \Psi(\det V) H_\epsilon \left( \alpha - \| V \|^2 \right) \, dx$$

$$+ \nu \frac{\nu}{2} \int_\Omega (T(\varphi) - R)^2 \, dx. \quad (P_2)$$

$H_\epsilon : z \mapsto \frac{1}{2} \left( 1 + \frac{2}{\pi} \arctan \frac{z}{\epsilon} \right)$: regularized one-dimensional Heaviside function.
Solving of \((P_1)\)

- Following the same strategy as Bresson’s in [3, 2007], we introduce an auxiliary variable \(f\) such that the problem amounts to minimizing:

\[
\inf_{\tilde{T}, f} \left\{ \text{var} \tilde{T} + \gamma \| f \|_{L^1(\Omega)} + \frac{1}{2\theta} \| \tilde{T} - T \circ \varphi + f \|_{L^2(\Omega)}^2 \right\},
\]

problem decoupled into the two following subproblems:

\[
\begin{align*}
\inf_{\tilde{T}} \left\{ \text{var} \tilde{T} + \frac{1}{2\theta} \| \tilde{T} - T \circ \varphi + f \|_{L^2(\Omega)}^2 \right\}; & \quad (P'_1) \\
\inf_{f} \left\{ \gamma \| f \|_{L^1(\Omega)} + \frac{1}{2\theta} \| \tilde{T} - T \circ \varphi + f \|_{L^2(\Omega)}^2 \right\}. & \quad (P''_1)
\end{align*}
\]
Solving of \((P_1)\) (Continuation)

Proposition 8 (Adaptation of Chambolle’s projection algorithm [4])

The solution of \((P'_1)\) is given by

\[
\tilde{T} = T \circ \varphi - f - \Pi_{\theta K} (T \circ \varphi - f).
\]

where \(\Pi\) represents the \textit{orthogonal projection}, and \(K\) is the closure of

\[
\{ \text{div} \xi : \xi \in C_c^1(\Omega, \mathbb{R}^2), |\xi(x)| \leq g(x), \forall x \in \Omega \}, \ | \cdot | \ \text{being the Euclidean norm in} \ \mathbb{R}^2.
\]
Solving of \((P_1)\) (Continuation)

Proposition 9

The solution of \((P''_1)\) is given by

\[
f = \begin{cases} 
T \circ \varphi - \bar{T} - \theta \gamma & \text{if } T \circ \varphi - \bar{T} \geq \theta \gamma \\
T \circ \varphi - \bar{T} + \theta \gamma & \text{if } T \circ \varphi - \bar{T} \leq -\theta \gamma \\
0 & \text{otherwise}
\end{cases}
\]
Reminder

- For fixed $\varphi$ and $V$,
  $$\inf_{\tilde{T}, f} \left\{ \text{var}_{g} \tilde{T} + \gamma \| f \|_{L^1(\Omega)} + \frac{1}{2\theta} \| \tilde{T} - T \circ \varphi + f \|_{L^2(\Omega)}^2 \right\};$$

- Then for fixed $\tilde{T}$ and $f$,
  $$\inf_{\varphi, V} \left\{ \frac{\nu}{2} \int_{\Omega} (T(\varphi) - R)^2 \, dx + \int_{\Omega} W(V) H_\epsilon (\| V \|_2^2 - \alpha) \, dx ight. $$
  $$\left. + \int_{\Omega} \psi(\det V) H_\epsilon (\alpha - \| V \|_2^2) \, dx ight. $$
  $$\left. + \frac{\gamma}{2} \| \nabla \varphi - V \|_{L^2(\Omega, M_2(\mathbb{R}))}^2 + \frac{1}{2\theta} \int_{\Omega} (\tilde{T} - T \circ \varphi + f)^2 \, dx \right\}.$$
Solving of \((P_2)\)

The Euler-Lagrange equation for \(\varphi\) is given by:

\[
0 = \nu(T \circ \varphi - R) \nabla T(\varphi) - \gamma \Delta \varphi + \gamma \left( \frac{\text{div} \ V_1}{\text{div} \ V_2} \right) \frac{1}{\theta} (f - T \circ \varphi + \tilde{T}) \nabla T(\varphi),
\]

and the system of equations for \(V\) is:

\[
0 = 2\beta c_0 \ V_{11} \left( 2H_\varepsilon(c_0) + c_0 \delta_\varepsilon(c_0) \right) + \mu V_{22}(\det V - 2) + \gamma \left( V_{11} - \frac{\partial \varphi_1}{\partial x} \right);
\]

\[
0 = 2\beta c_0 \ V_{12} \left( 2H_\varepsilon(c_0) + c_0 \delta_\varepsilon(c_0) \right) + \mu V_{21}(\det V - 2) + \gamma \left( V_{12} - \frac{\partial \varphi_1}{\partial y} \right);
\]
Solving of \((P_2)\) (Continuation)

\[
0 = 2\beta c_0 V_{21} (2H_\varepsilon(c_0) + c_0 \delta_\varepsilon(c_0)) + \mu V_{12} (\text{det } V - 2) \\
+ \gamma (V_{21} - \frac{\partial \varphi_2}{\partial x});
\]

\[
0 = 2\beta c_0 V_{22} (2H_\varepsilon(c_0) + c_0 \delta_\varepsilon(c_0)) + \mu V_{11} (\text{det } V - 2) \\
+ \gamma (V_{22} - \frac{\partial \varphi_2}{\partial y}),
\]

with \(c_0 = (\| V \|^2 - \alpha)\).
Orientation Preservation

Figure: Obtained global deformation field when topology preservation is not enforced.
Regridding Algorithm

**Algorithm 1**: Regridding.

- **Initialization**: $V^0 = I$, $\varphi^0 = Id$, regrid_count=0.
- For $k = 0, 1, \ldots$

$$ (\varphi^{k+1}, V^{k+1}) = \arg \min_{\varphi, V} \tilde{J}_{\varepsilon, \gamma} $$

If $\det \nabla \varphi^{k+1} < tol$

- regrid_count=regrid_count+1
- $T = T \circ \varphi^k$
- save $\varphi^k(\text{regrid\_count})$, $\varphi^{k+1} = Id$, $V^{k+1} = 0$

- If regrid_count>0
  $$ \varphi^{final} = \varphi(1) \circ \ldots \circ \varphi(\text{regrid\_count}) $$
A Joint Segmentation/Registration Model
5 - Numerical Simulations

Solène Ozeré and Carole Le Guyader

A Joint Segmentation/Registration Model
$V_{11}$  
$V_{12}$  
$V_{21}$  
$V_{22}$

\[
\left\| V_{11} - \frac{\partial \varphi_1}{\partial x} \right\|_\infty = 0.09, \quad \left\| V_{12} - \frac{\partial \varphi_1}{\partial y} \right\|_\infty = 0.03, \quad \left\| V_{21} - \frac{\partial \varphi_2}{\partial x} \right\|_\infty = 0.06, \\
\left\| V_{22} - \frac{\partial \varphi_2}{\partial y} \right\|_\infty = 0.07
\]
5 - Numerical Simulations

- Template
- Reference
- Deformation

- Deformed Template
- Segmented Reference
- Vector field
5 - Numerical Simulations

MRI images of cardiac cycle

Template  Reference  Deformation

Deformed Template  Segmented Reference  Vector field
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<th>$\theta$</th>
<th>$\gamma_1$</th>
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<th>$\mu$</th>
<th>$\lambda$</th>
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Thank you for your attention.