Reflector antenna problem

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Motivation
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Pb : find the reflector surface
Far-Field Reflector Antenna Problem

Punctual light at origin \( o \), \( \mu \) measure on \( S^2_o \)
Prescribed far-field: \( \nu \) on \( S^2_\infty \)
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Prescribed far-field: \( \nu \) on \( S^\infty \)

**Goal:** Find a surface \( R \) which sends \((S^2_o, \mu)\) to \((S^\infty, \nu)\) under reflection by Snell’s law.
Far-Field Reflector Antenna Problem

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Prescribed far-field: $\nu$ on $S_\infty^2$

**Goal:** Find a surface $R$ which sends $(S_o^2, \mu)$ to $(S_\infty, \nu)$ under reflection by Snell’s law.

$R$ is parameterized over $S_o^2$
Far-Field Reflector Antenna Problem

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- \( R \) is parameterized over \( S^2_o \)
- Snell’s law
  \[
  T_R : x \in S^2_o \mapsto y = x - 2\langle x|n\rangle n
  \]
Far-Field Reflector Antenna Problem

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Brenier formulation $T_\# \mu = \nu$
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**Brenier formulation**
\[ T_\# \mu = \nu \]
i.e. for every borelian $B$
\[ \mu(T^{-1}(B)) = \nu(B) \]
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$$T_#\mu = \nu$$
i.e. for every borelian $B$
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**Monge-Ampere equation**
If $\mu(x) = f(x)dx$ and $\nu(y) = g(y)dy$
$$g(T(x)) \det(DT(x)) = f(x)$$
- highly non linear
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- highly non linear

**Existence**
Caffarelli & Oliker 94

**Regularity, uniqueness**
Wang 96, Guan & Wang 98
Reflector Problem: semi-discrete case

Punctual light at origin $o$, $\mu$ measure on $S^2_o$
Reflector Problem: semi-discrete case

Punctual light at origin $o$, $\mu$ measure on $S_o^2$

Prescribed far-field: $\nu = \nu_1 \delta_{y_1}$ on $S_\infty^2$
Reflector Problem: semi-discrete case

Punctual light at origin $o$, $\mu$ measure on $S^2_o$
Prescribed far-field: $\nu = \nu_1 \delta_{y_1}$ on $S^2_\infty$

$R$: paraboloid of direction $y_1$ and focal $O$
Reflector Problem : semi-discrete case

Punctual light at origin $o$, $\mu$ measure on $S^2_o$

Prescribed far-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $S^2_\infty$
Reflector Problem: semi-discrete case

Punctual light at origin $o$, $\mu$ measure on $S_o^2$

Prescribed far-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $S^2_\infty$

$P_i(\kappa_i) =$ solid paraboloid of revolution with focal $o$, direction $y_i$ and focal distance $\kappa_i$

$R(\kappa) = \partial \left( \bigcap_{i=1}^N P_i(\kappa_i) \right)$
Reflector Problem: semi-discrete case

Punctual light at origin $o$, $\mu$ measure on $S^2_o$

Prescribed far-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $S^2_\infty$

$P_i(\kappa_i) = \text{solid paraboloid of revolution with focal } o,$
$\text{direction } y_i \text{ and focal distance } \kappa_i$

$R(\vec{\kappa}) = \partial \left( \cap_{i=1}^N P_i(\kappa_i) \right)$

Decomposition of $S^2_o$: $\Pi_i(\vec{\kappa}) = \pi_{S^2_\infty}(R(\vec{\kappa}) \cap \partial P_i(\kappa_i))$
$= \text{directions that are reflected towards } y_i.$
Reflector Problem: semi-discrete case

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Decomposition of $S^2_o$: $\mathbf{P}I_i(\vec{\kappa}) = \pi_{S^2}(R(\vec{\kappa}) \cap \partial P_i(\kappa_i))$

$= \text{directions that are reflected towards } y_i.$

Problem (FF): Find $\kappa_1, \ldots, \kappa_N$ such that for every $i$, $\mu(\mathbf{P}I_i(\vec{\kappa})) = \nu_i.$
**Lemma:** With $c(x, y) = -\log(1 - \langle x | y \rangle)$, and $\psi_i := \log(\kappa_i)$,

$$\text{PI}_i(\kappa) = \{ x \in S_0^2, \ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \ \forall j \}. \quad \text{(Caffarelli-Oliker '94)}$$
**Lemma:** With \( c(x, y) = -\log(1 - \langle x|y \rangle) \), and \( \psi_i := \log(\kappa_i) \),

\[
\Pi_i(\kappa) = \{ x \in S_0^2, \ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \quad \forall j \}.
\]

**Proof:** \( \partial P_i(\kappa_i) \) is parameterized in radial coordinates by

\[
\rho_i : x \in S_o^2 \mapsto \frac{\kappa_i}{1-\langle x|y_i \rangle}
\]

Caffarelli-Oliker '94
Far-Field Reflector Antenna Problem as OT

**Lemma:** With \( c(x, y) = -\log(1 - \langle x|y \rangle) \), and \( \psi_i := \log(\kappa_i) \),
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\Pi_i(\kappa) = \{ x \in S^2_0, \ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \quad \forall j \}.
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**Proof:** \( \partial P_i(\kappa_i) \) is parameterized in radial coordinates by
\[
\rho_i : x \in S^2_o \mapsto \frac{\kappa_i}{1 - \langle x|y_i \rangle}
\]
\[
x \in \Pi_i(\kappa) \iff i \in \arg \min_j \frac{\kappa_j}{1 - \langle x|y_j \rangle}
\]
Lemma: With \( c(x, y) = -\log(1 - \langle x | y \rangle) \), and \( \psi_i \equiv \log(\kappa_i) \),
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\Pi_i(\kappa) = \{ x \in S_0^2, \ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \quad \forall j \}.
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Proof: \( \partial P_i(\kappa_i) \) is parameterized in radial coordinates by
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\]
\[
x \in \Pi_i(\kappa) \iff i \in \text{arg min}_j \left\{ \frac{\kappa_j}{1-\langle x | y_j \rangle} \right\}
\]
\[
\iff i \in \text{arg min}_j \left( \log(\kappa_j) - \log(1 - \langle x | y_j \rangle) \right)
\]
**Far-Field Reflector Antenna Problem as OT**

**Lemma:** With $c(x, y) = -\log(1 - \langle x | y \rangle)$, and $\psi_i := \log(\kappa_i)$, $\text{PI}_i(\kbar) = \{ x \in S_0^2, \ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \quad \forall j \}$.  

**Proof:** $\partial P_i(\kappa_i)$ is parameterized in radial coordinates by $\rho_i : x \in S_o^2 \mapsto \frac{\kappa_i}{1 - \langle x | y_i \rangle}$

$x \in \text{PI}_i(\kbar) \iff i \in \arg \min_j \frac{\kappa_j}{1 - \langle x | y_j \rangle}$

$\iff i \in \arg \min_j \log(\kappa_j) - \log(1 - \langle x | y_j \rangle)$

$\iff i \in \arg \min_j \psi_j + c(x, y_j)$
Far-Field Reflector Antenna Problem as OT

Lemma: With \( c(x, y) = -\log(1 - \langle x | y \rangle) \), and \( \psi_i = \log(\kappa_i) \),
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\Pi_i(\vec{\kappa}) = \{ x \in S_0^2, \ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \ \forall j \}\.
\]

Proof: \( \partial P_i(\kappa_i) \) is parameterized in radial coordinates by
\[
\rho_i : x \in S_o^2 \mapsto \frac{\kappa_i}{1-\langle x | y_i \rangle}
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x \in \Pi_i(\vec{\kappa}) \iff i \in \arg \min_j \frac{\kappa_j}{1-\langle x | y_j \rangle}
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\iff i \in \arg \min_j \log(\kappa_j) - \log(1 - \langle x | y_j \rangle)
\]
\[
\iff i \in \arg \min_j \psi_j + c(x, y_j)
\]

▶ An optimal transport problem

Wang '04

Caffarelli-Oliker '94
Semi-discrete optimal transport

$\mu = \text{probability measure on } X$
with density, $X = \text{manifold}$

$\nu = \text{prob. measure on finite } Y$
$= \sum_{y \in Y} \nu_y \delta_y$
Semi-discrete optimal transport

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\[ \nu = \sum_{y \in Y} \nu_y \delta_y \]

\( \forall y \in Y, \mu(T^{-1}(\{y\})) = \nu_y \)

in short: \( T\#\mu = \nu \).
Semi-discrete optimal transport

\[ \mu = \text{probability measure on } X \]
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\[ \nu = \text{prob. measure on finite } Y \]
\[ = \sum_{y \in Y} \nu_\cdot \delta_y \]

Transport map: \( T : X \rightarrow Y \) s.t.
\[ \forall y \in Y, \mu(T^{-1}(y)) = \nu_y \]
in short: \( T_\# \mu = \nu \).

Cost function: \( c : X \times Y \rightarrow \mathbb{R} \)
\[ C_c(T) = \int_X c(x, T(x)) \, d\mu(x) \]
Semi-discrete optimal transport

\[ \mu = \text{probability measure on } X \]
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\[ C_c(T) = \int_X c(x, T(x)) \, d\mu(x) \]
\[ = \sum_y \int_{T^{-1}(y)} c(x, y) \, d\mu(x) \]
Semi-discrete optimal transport

\( \mu = \) probability measure on \( X \) with density, \( X = \) manifold

\( \nu = \) prob. measure on finite \( Y \)

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Monge problem: \( \mathcal{T}_c(\mu, \nu) := \min \{ C_c(T); T_\# \mu = \nu \} \)
We assume (Twist), i.e. \( c \in C^\infty \) and \( \forall x \in X \) the map \( y \in Y \mapsto \nabla_x c(x, y) \) is injective.

\( Y \) finite set, \( \psi : Y \to \mathbb{R} \)
We assume \((\textbf{Twist})\), i.e. \(c \in C^\infty\) and \(\forall x \in X\) the map \(y \in Y \mapsto \nabla_x c(x, y)\) is injective.

\[
T_c^\psi(x) = \arg \min_{y \in Y} c(x, y) + \psi(y)
\]
Weighted Voronoi and Optimal Transport

We assume \((\text{Twist})\), i.e. \(c \in C^\infty\) and \(\forall x \in X\) the map \(y \in Y \mapsto \nabla_x c(x, y)\) is injective.

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T^\psi_c(x) = \arg \min_{y \in Y} c(x, y) + \psi(y)
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\[
\text{Vor}^\psi_c(y) = \{x \in \mathbb{R}^d; T^\psi_c(x) = y\}
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= generalized weighted Voronoi cell

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\textbf{NB:} Under \((\textbf{Twist})\), \((\text{Vor}^\psi_c(y))_{y \in Y}\) partitions \(X\) and \(T^\psi_c\) well-defined a.e.
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**NB:** Under \((\text{Twist})\), \((\text{Vor}_c^\psi(y))_{y \in Y}\) partitions \(X\) and \(T_c^\psi\) well-defined a.e.

**Lemma:** Given a measure \(\mu\) with density and \(\psi : Y \to \mathbb{R}\), the map \(T_c^\psi\) is a \(c\)-optimal transport between \(\mu\) and \(T_c^\psi \# \mu\).
Weighted Voronoi and Optimal Transport

Aurenhammer, Hoffman, Aronov '98 Merigot '2010

We assume \((\text{Twist})\), i.e. \(c \in C^\infty\) and \(\forall x \in X\) the map \(y \in Y \mapsto \nabla_x c(x, y)\) is injective.

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**Lemma:** Given a measure \(\mu\) with density and \(\psi : Y \to \mathbb{R}\), the map \(T^\psi_c\) is a \(c\)-optimal transport between \(\mu\) and \(T^\psi_c \# \mu\).

\[T^\psi_c \# \mu = \sum_{y \in Y} \mu(\text{Vor}_c^\psi(y)) \delta_y.\]
Weighted Voronoi and Optimal Transport

Aurenhammer, Hoffmann, Aronov '98 Merigot '2010

We assume \( \text{(Twist)} \), i.e. \( c \in C^\infty \) and \( \forall x \in X \) the map \( y \in Y \mapsto \nabla_x c(x, y) \) is injective.

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\[
\text{Note: } T^\psi_c \# \mu = \sum_{y \in Y} \mu(\text{Vor}^\psi_c(y)) \delta_y.
\]

\[
\text{Converse?}
\]
Lemma: With $c(x, y) = -\log(1 - \langle x | y \rangle)$, and $\psi_i := \log(\kappa_i)$,
$\Pi_i(\kappa) = \{x \in S_0^2, c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \quad \forall j\}$. 
Lemma: With $c(x, y) = -\log(1 - \langle x | y \rangle)$, and $\psi_i := \log(\kappa_i)$,

$$\text{PI}_i(\kappa) = \{ x \in S_0^2, \ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \ \forall j \}.$$
**Lemma:** With \( c(x, y) = -\log(1 - \langle x | y \rangle) \), and \( \psi_i := \log(\kappa_i) \),

\[
\Pi_i(\kappa) = \{ x \in S^2_0, \ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \ \forall j \}.
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The map \( T^\psi_c(x) \) is a \( c \)-optimal transport between \( \mu \) and \( T^\psi_c \# \mu \).
Lemma: With \( c(x, y) = -\log(1 - \langle x | y \rangle) \), and \( \psi_i := \log(\kappa_i) \),
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\pi_i(\kappa) = \{ x \in S_0^2, \ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \ \forall j \}.
\]

Optimal transport formulation
- \( \pi_i(\kappa) = \text{Vor}_c^{\psi}(y_i) \).
- \( T_c^{\psi}(x) = \arg \min_{y \in Y} c(x, y) + \psi(y) \)

The map \( T_c^{\psi} \) is a \( c \)-optimal transport between \( \mu \) and \( T_c^{\psi} \# \mu \).

Problem (FF): Find \( \psi_1, \ldots, \psi_N \) such that \( T_c^{\psi} \# \mu = \nu \).
Supporting paraboloids algorithm’ 99

Cafarelli-Kochengin-Oliker’99: coordinate-wise ascent, with minimum increment
Supporting paraboloids algorithm’ 99

Cafarelli-Kochengin-Oliker’99:
coordinate-wise ascent, with minimum increment

**Initialization:** Fix $y_0 \in Y$, let $\delta = \varepsilon/N$ and compute $\psi$ s.t.

$$\forall y \in Y \setminus \{y_0\}, \quad \mu(\text{Vor}_c^\psi(p)) \leq \nu_y + \delta$$

**While** $\exists y \neq y_0$ such that $\mu(\text{Vor}_c^\psi(y)) \leq \nu_y - \delta$, **do:**

decrease $\psi(y)$ s.t. $\mu(\text{Vor}_c^\psi(y)) \in [\nu_y, \nu_y + \delta]$,

**Result:** $\psi$ s.t. for all $y$, $|\mu(\text{Vor}_c^\psi(y)) - \nu_y| \leq \varepsilon$. 


Cafarelli-Kochengin-Oliker’99:
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Supporting paraboloids algorithm’ 99

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While $\exists y \neq y_0$ such that $\mu(\text{Vor}_c^\psi(y)) \leq \nu_y - \delta$, do:
\[ \text{decrease } \psi(y) \text{ s.t. } \mu(\text{Vor}_c^\psi(y)) \in [\nu_y, \nu_y + \delta], \]

Result: $\psi$ s.t. for all $y$, $|\mu(\text{Vor}_c^\psi(y)) - \nu_y| \leq \varepsilon$.

- Complexity of SP: $N^2 / \varepsilon$ steps
Supporting paraboloids algorithm’ 99

Cafarelli-Kochengin-Oliker’99:
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While \( \exists y \neq y_0 \) such that \( \mu(\text{Vor}_c^\psi(y)) \leq \nu_y - \delta \), do:
decrease \( \psi(y) \) s.t. \( \mu(\text{Vor}_c^\psi(y)) \in [\nu_y, \nu_y + \delta] \),

Result: \( \psi \) s.t. for all \( y \), \( |\mu(\text{Vor}_c^\psi(y)) - \nu_y| \leq \varepsilon \).

- Complexity of SP: \( N^2 / \varepsilon \) steps
- Generalization of Oliker–Prussner in \( \mathbb{R}^2 \) with \( c(x, y) = \|x - y\|^2 \)
Cafarelli-Kochengin-Oliker’99:
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Initialization: Fix $y_0 \in Y$, let $\delta = \varepsilon/N$ and compute $\psi$ s.t.
$$\forall y \in Y \setminus \{y_0\}, \quad \mu(\text{Vor}_c^\psi(p)) \leq \nu_y + \delta$$

While $\exists y \neq y_0$ such that $\mu(\text{Vor}_c^\psi(y)) \leq \nu_y - \delta$, do:
- decrease $\psi(y)$ s.t. $\mu(\text{Vor}_c^\psi(y)) \in [\nu_y, \nu_y + \delta]$.

Result: $\psi$ s.t. for all $y$, $|\mu(\text{Vor}_c^\psi(y)) - \nu_y| \leq \varepsilon$.

- Complexity of SP: $\mathcal{O}(N^2/\varepsilon)$ steps
- Generalization of Oliker–Prussner in $\mathbb{R}^2$ with $c(x, y) = \|x - y\|^2$
- Generalization: MTW$^+$ costs Kitagawa ’12
Concave maximization

**Theorem:** $\kappa$ solves (FF) iff $\psi = \log(\kappa)$ maximizes

$$\Phi(\psi) := \sum_i \int_{\text{Vor}_c(y_i)} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum_i \psi_i \nu_i$$

with $c(x, y) = -\log(1 - \langle x \mid y \rangle)$.

Aurenhammer, Hoffman, Aronov '98
**Theorem:** \( \vec{\kappa} \) solves \((\text{FF})\) iff \( \vec{\psi} = \log(\vec{\kappa}) \) maximizes

\[
\Phi(\psi) := \sum_i \int_{\text{Vor}_C(y_i)} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum_i \psi_i \nu_i
\]

with \( c(x, y) = -\log(1 - \langle x | y \rangle) \).

A consequence of Kantorovich duality.

Aurenhammer, Hoffman, Aronov ’98
Proof of concave maximization thm

**Supdifferentials.** Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ and $\psi \in \mathbb{R}^d$.

$\triangleright \quad \partial^+ \Phi(\psi) = \{ v \in \mathbb{R}^d, \quad \Phi(\varphi) \leq \Phi(\psi) + \langle \varphi - \psi | v \rangle \quad \forall \varphi \in \mathbb{R}^d \}.$
Proof of concave maximization thm

Supdifferentials. Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ and $\psi \in \mathbb{R}^d$.

- $\partial^+ \Phi(\psi) = \{v \in \mathbb{R}^d, \quad \Phi(\varphi) \leq \Phi(\psi) + \langle \varphi - \psi | v \rangle \quad \forall \varphi \in \mathbb{R}^d\}$.  
- $\Phi$ concave $\iff \forall \psi \in \mathbb{R}^d \partial^+ \Phi(\psi) \neq \emptyset$.  
- In this case: $\partial^+ \Phi(\psi) = \{\nabla \Phi(\psi)\}$ a.e.  
- $\psi$ maximum of $\Phi \iff 0 \in \partial^+ \Phi(\psi)$

![Graph of a concave function with supdifferentials](image-url)
Proof of concave maximization thm

\[ \Phi(\psi) := \sum_i \int_{\text{Vor}_c(y_i)} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum_i \psi_i \nu_i \]
Proof of concave maximization thm

\[ \Phi(\psi) := \sum_i \int_{\text{Vor}_c(y_i)} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum_i \psi_i \nu_i \]

\[ = \int_{S^{d-1}} \min_{1 \leq i \leq N} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum_i \psi_i \nu_i \]
Proof of concave maximization thm

\[ \Phi(\psi) := \sum_i \int_{\text{Vor}_c(y_i)} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum_i \psi_i \nu_i \]

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For all \( \varphi \in \mathbb{R}^d \)
\[ \min_{1 \leq i \leq N} [c(x, y_i) + \varphi_i] \leq [c(x, y_{T\psi}(x)) + \varphi_{T\psi}(x)] \]
Proof of concave maximization thm

\[
\Phi(\psi) := \sum_i \int_{\text{Vor}_c^\psi(y_i)} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum_i \psi_i \nu_i \\
= \int_{S^{d-1}} \min_{1 \leq i \leq N} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum_i \psi_i \nu_i
\]

For all \( \varphi \in \mathbb{R}^d \)
\[
\min_{1 \leq i \leq N} [c(x, y_i) + \varphi_i] \leq [c(x, y_{T_\psi(x)}) + \varphi_{T_\psi(x)}]
\]

\( T_\psi(x) = i \iff x \in \text{Vor}_c^\psi(y_i) \)
Proof of concave maximization thm

\[ \Phi(\psi) := \sum_i \int_{\text{Vor}_c(\psi_i)} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum_i \psi_i \nu_i \]
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For all \( \varphi \in \mathbb{R}^d \)
\[ \min_{1 \leq i \leq N} [c(x, y_i) + \varphi_i] \leq [c(x, y_{T_\psi}(x)) + \varphi_{T_\psi}(x)] \]
\[ \leq [c(x, y_{T_\psi}(x)) + \psi_{T_\psi}(x)] + \varphi_{T_\psi}(x) - \psi_{T_\psi}(x) \]

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Proof of concave maximization thm

\[ \Phi(\psi) := \sum_i \int_{\text{Vor}_c(y_i)} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum_i \psi_i \nu_i \]
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For all \( \varphi \in \mathbb{R}^d \)
\[ \min_{1 \leq i \leq N} [c(x, y_i) + \varphi_i] \leq [c(x, y_{T_\psi}(x)) + \varphi_{T_\psi}(x)] \]
\[ \leq [c(x, y_{T_\psi}(x)) + \psi_{T_\psi}(x)] + \varphi_{T_\psi}(x) - \psi_{T_\psi}(x) \]
\[ \int_{S^{d-1}} \Phi(\varphi) + \sum_i \varphi_i \nu_i \]
\[ \Phi(\psi) + \sum_i \psi_i \nu_i \]
\[ \int_{S^{d-1}} \varphi_{T_\psi}(x) - \psi_{T_\psi}(x) \, d\mu(x) \]

\( T_\psi(x) = i \iff x \in \text{Vor}_c(y_i) \)
Proof of concave maximization thm

\[ \Phi(\psi) := \sum_i \int_{\text{Vor}_c^\psi(y_i)} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum_i \psi_i \nu_i \]
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\[ T_\psi(x) = i \iff x \in \text{Vor}_c^\psi(y_i) \]

\[ \Phi(\varphi) - \Phi(\psi) \leq \int_{S^{d-1}} \varphi_{T_\psi(x)} - \psi_{T_\psi(x)} \, d\mu(x) - \sum_i (\varphi_i - \psi_i) \nu_i \]
Proof of concave maximization thm

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\Phi(\psi) := \sum_i \int_{\text{Vor}_c^\psi(y_i)} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum_i \psi_i \nu_i \\
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T_\psi(x) = i \iff x \in \text{Vor}_c^\psi(y_i)
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\[
\Phi(\varphi) - \Phi(\psi) \leq \int_{S^{d-1}} \varphi_{T_\psi(x)} - \psi_{T_\psi(x)} \, d\mu(x) - \sum_i (\varphi_i - \psi_i) \nu_i \\
\leq \sum_{1 \leq i \leq N} \left[ \int_{\text{Vor}_c^\psi(y_i)} d\mu(x) - \nu_i \right] (\varphi_i - \psi_i) \\
= \langle D\Phi(\psi) | \varphi - \psi \rangle \\
\text{with } D\Phi(\psi) = \left( \mu(\text{Vor}_c^\psi(y_i)) - \nu_i \right)
\]
Proof of concave maximization thm

\[
\Phi(\psi) := \sum \int_{\text{Vor}_c^\psi(y_i)} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum \psi_i \nu_i \\
= \int_{S^{d-1}} \min_{1 \leq i \leq N} [c(x, y_i) + \psi_i] \, d\mu(x) - \sum \psi_i \nu_i
\]

\[
\Phi(\varphi) \leq \Phi(\psi) + \langle D\Phi(\psi) | \varphi - \psi \rangle
\]

with \( D\Phi(\psi) = (\mu(\text{Vor}_c^\psi(y_i)) - \nu_i) \)
Proof of concave maximization thm

\[ \Phi(\psi) := \sum_i \int_{\text{Vor}_c^\psi(y_i)} \left[ c(x, y_i) + \psi_i \right] \, d\mu(x) - \sum_i \psi_i \nu_i \]
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\[ T_\psi(x) = i \iff x \in \text{Vor}_c^\psi(y_i) \]

\[ \Phi(\varphi) \leq \Phi(\psi) + \langle D\Phi(\psi) | \varphi - \psi \rangle \]

with \[ D\Phi(\psi) = \left( \mu(\text{Vor}_c^\psi(y_i)) - \nu_i \right) \]

- \( D\Phi(\psi) \in \partial^+ \Phi(\psi) \Rightarrow \Phi \) concave.
- \( D\Phi(\psi) \) depends continuously on \( \psi \Rightarrow \Phi \) of class \( C^1 \).
- \( \psi \) maximum of \( \Phi \) \iff \( \mu(\text{Vor}_c^\psi(y_i)) = \nu_i \ \forall i \)
2. Implementation
Implementation of Convex Programming ($-\Phi$)

- Quasi-Newton scheme:
  
  Computation of descent direction / time step
  
  LBFGS: low-storage version of the BFGS quasi-Newton scheme
Implementation of Convex Programming ($-\Phi$)

- **Quasi-Newton scheme:**
  Computation of descent direction / time step
  LBFGS: low-storage version of the BFGS quasi-Newton scheme

- **Evaluation of $\Phi$ and $\nabla \Phi$:**
  \[
  \int_{\text{Vor}_c(p)} \, d\mu(x) \\
  \int_{\text{Vor}_c(y)} c(x, y) \, d\mu(x)
  \]

  **Main difficulty:** computation of $\text{Vor}_c(y)$
Implementation of Convex Programming ($-\Phi$)

- Quasi-Newton scheme:
  Computation of descent direction / time step
  LBFGS: low-storage version of the BFGS quasi-Newton scheme

- Evaluation of $\Phi$ and $\nabla\Phi$:

$$\int_{\text{Vor}_c^\psi(p)} d\mu(x)$$
$$\int_{\text{Vor}_c^\psi(y)} c(x, y) d\mu(x)$$

Main difficulty: computation of $\text{Vor}_c^\psi(y)$
Computation of the generalized Voronoi cells

Definition: Given \( P = \{ p_i \}_{1 \leq i \leq N} \subseteq \mathbb{R}^d \) and \((\omega_i)_{1 \leq i \leq N} \in \mathbb{R}^N\),

\[
\text{Pow}_P^\omega(p_i) := \{ x \in \mathbb{R}^d; i = \arg \min_j \| x - p_j \|^2 + \omega_j \}
\]
**Computation of the generalized Voronoi cells**

**Definition:** Given $P = \{p_i\}_{1 \leq i \leq N} \subseteq \mathbb{R}^d$ and $(\omega_i)_{1 \leq i \leq N} \in \mathbb{R}^N$

\[
\text{Pow}^\omega_P(p_i) := \{x \in \mathbb{R}^d; i = \arg \min_j \|x - p_j\|^2 + \omega_j\}
\]

- Efficient computation of $(\text{Pow}^\omega_P(p_i))_i$ using **CGAL** ($d = 2, 3$)
Computation of the generalized Voronoi cells

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$$\text{Pow}_P^\omega(p_i) := \{x \in \mathbb{R}^d; i = \arg \min_{j} \|x - p_j\|^2 + \omega_j\}$$

- Efficient computation of $(\text{Pow}_P^\omega(p_i))_i$ using **CGAL** ($d = 2, 3$)

**Lemma:** With $\psi = \log(\kappa)$, $p_i := -\frac{y_j}{2\kappa_j}$ and $\omega_i := -\|\frac{y_j}{2\kappa_j}\|^2 - \frac{1}{\kappa_j}$,

$$\text{Vor}_{c}^\psi(y_i) = \text{Pow}_P^\omega(p_i) \cap S^2$$
Computation of the generalized Voronoi cells

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$\text{Pow}_P^\omega(p_i) := \{x \in \mathbb{R}^d ; i = \arg \min_j \|x - p_j\|^2 + \omega_j\}$

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Lemma: With $\bar{\psi} = \log(\kappa)$, $p_i := -\frac{y_j}{2\kappa_j}$ and $\omega_i := -\|\frac{y_j}{2\kappa_j}\|^2 - \frac{1}{\kappa_j}$,

$\text{Vor}_c^\psi(y_i) = \text{Pow}_P^\omega(p_i) \cap S^2$

Proof: $x \in \text{Vor}_c^\psi(y_i) \subseteq S^2$

$\iff i \in \arg \min_j \frac{\kappa_j}{1 - \langle x | y_j \rangle}$

$\iff i \in \arg \min_j \langle x | \frac{y_j}{\kappa_j} \rangle - \frac{1}{\kappa_j}$

$\iff i \in \arg \min_j \|x + \frac{y_j}{2\kappa_j} - p_j\|^2 - \|\frac{y_j}{2\kappa_j}\|^2 - \frac{1}{\kappa_j}$

$\iff x \in \text{Pow}_P^\omega(p_i) \cap S^2$
Computation of the generalized Voronoi cells

- in general, the cells $C_i := \text{Pow}_{P}^{\omega}(p_i) \cap S^2$ can be disconnected, have holes, etc.
Computation of the generalized Voronoi cells

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- **boundary representation**: a family of oriented cycles composed of circular arcs per cell.
Computation of the generalized Voronoi cells

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- Lower complexity bound: $\Omega(N \log N)$. 
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**Algorithm:** for each cell \( C_i = \text{Pow}_P^\omega(p_i) \cap S^2 \)
Computation of the generalized Voronoi cells

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**Algorithm:** for each cell $C_i = \text{Pow}_{P}^{\omega}(p_i) \cap S^2$
1. Compute *implicitly* the intersection between every edge of $\text{Pow}_{P}^{\omega}(p_i)$ and $S^2$. Set vertices $V := \{ \bullet \}$
Computation of the generalized Voronoi cells

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1. Compute **implicitly** the intersection between every edge of $\text{Pow}_P^\omega(p_i)$ and $S^2$. Set vertices $V := \{ \bullet \}$
2. Scan the **edges** of every 2-facet in clockwise order and construct oriented edges $E$ between vertices.
Computation of the generalized Voronoi cells

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3. Extract oriented cycles from $G = (V, E)$. 
Computation of the generalized Voronoi cells

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4. Handle circular arcs without vertex separately.
Computation of the generalized Voronoi cells

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4. Handle circular arcs without vertex separately.

**Complexity:** $O(N \log N + C)$ where $C =$ complexity of the Power diagram.
Numerical results (1)

\[ \nu = \sum_{i=1}^{N} \nu_i \delta_{x_i} \] obtained by discretizing a picture of G. Monge.

\[ \mu = \text{uniform measure on half-sphere } S^2_+ \]

\( N = 1000 \)

drawing of \( (\text{Vor}_c^{\psi}(y_i)) \) (on \( S^2_+ \)) for \( \psi = 0 \)
Numerical results (1)
normalized measure by discretizing a picture of G. Monge.

\[ \nu = \sum_{i=1}^{N} \nu_i \delta_{x_i} \]

\[ \mu = \text{uniform measure on half-sphere } S_+^2 \]

\[ N = 1000 \]

drawing of \( \text{Vor}_c^{\psi}(y_i) \) (on \( S_+^2 \)) for \( \psi_{sol} \)
Numerical results (1)

\[ \nu = \sum_{i=1}^{N} \nu_i \delta_{x_i} \] obtained by discretizing a picture of G. Monge.

\[ \mu = \text{uniform measure on half-sphere } S^2_+ \]

\[ N = 1000 \]

rendering of the image reflected at infinity (using LuxRender)
Numerical results (2)

\[ \nu = \sum_{i=1}^{N} \nu_i \delta_{x_i} \] obtained by discretizing a picture of G. Monge.

\[ \mu = \text{uniform measure on half-sphere } S_+^2 \quad N = 15000 \]

drawing of \((Vor^\psi_c(y_i))\) (on \(S_+^2\)) for \(\psi_{sol}\)
Numerical results (2)

\[ \nu = \sum_{i=1}^{N} \nu_i \delta_{x_i} \] obtained by discretizing a picture of G. Monge.

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\( N = 15000 \)

solution to the far-field reflector problem: \( R(\kappa_{sol}) \)
Numerical results (2)

\[ \nu = \sum_{i=1}^{N} \nu_i \delta_{x_i} \] obtained by discretizing a picture of G. Monge.

\[ \mu = \text{uniform measure on half-sphere } S^2_+ \]

\[ N = 15000 \]

rendering of the image reflected at infinity (using LuxRender)
3. Complexity of paraboloid intersection
Theorem: For $N$ paraboloids, the complexity of the diagram $(\text{PI}_i(\kappa))_{1 \leq i \leq N}$ is $O(N)$. 
Complexity of the paraboloid intersection (PI)

**Theorem:** For $N$ paraboloids, the complexity of the diagram $(\text{PI}_i(\vec{\kappa}))_{1 \leq i \leq N}$ is $O(N)$.

**Complexity:** $E + F + V$, where

- $E = \#$ edges
- $V = \#$ vertices
- $F = \text{total} \ # \ of \ connected \ components$
Complexity of the paraboloid intersection (PI)

**Theorem:** For $N$ paraboloids, the complexity of the diagram $(\text{PI}_i(\kappa))_{1 \leq i \leq N}$ is $O(N)$.

**Proof:**

- $F \leq N$
Complexity of the paraboloid intersection (PI)

**Theorem:** For \( N \) paraboloids, the complexity of the diagram \((\text{PI}_i(\kappa))^1_{i \leq N}\) is \( O(N) \).

**Proof:**

\[ F \leq N \]

**Lemma:** The projection of \( \partial P_i \cap P_j \) onto the plane \( \{ y_i \} \) is a disc.

\( \text{PI}_3(\kappa) \cap \partial P_3(\kappa_3) \)

\( \{ y_3 \} \)

\( \{ y_i \} \)

\( \partial P_3 \)

\( \text{PI}_3(\kappa) \)

\( P_3 \)

\( P_2 \)

\( P_1 \)

\( O \)

\( \text{PI}_i(\kappa) \)

\( \{ y_i \} \)

\( P_i \)

\( P_j \)
Complexity of the paraboloid intersection (PI)

**Theorem:** For $N$ paraboloids, the complexity of the diagram $(\text{PI}_i(\overline{\kappa}))_{1 \leq i \leq N}$ is $O(N)$.

**Proof:**

$F \leq N$

$\{y_3\} \perp$

$\{y\} \perp$

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Complexity of the paraboloid intersection (PI)

**Theorem:** For $N$ paraboloids, the complexity of the diagram $(\text{PI}_i(\vec{\kappa}))_{1 \leq i \leq N}$ is $O(N)$.

**Proof:**

$\implies F \leq N$

$\implies$ the projection of $R(\vec{\kappa}) \cap \partial P_i$ on $\{y_i\}^\perp$ is convex

$\implies \text{PI}_i(\vec{\kappa})$ is connected.

**Lemma:** The projection of $\partial P_i \cap P_j$ onto the plane $\{y_i\}^\perp$ is a disc.
Theorem: For \( N \) paraboloids, the complexity of the diagram 
\((\text{PI}_i(\bar{\kappa}))_{1 \leq i \leq N}\) is \( O(N) \).

Proof:

\( F \leq N \)

Every vertex has 3 edges \( \Rightarrow 3V \leq 2E \).
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**Proof:**

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- Every vertex has 3 edges $\Rightarrow 3V \leq 2E$.

- Euler’s formula $V - E + F = 2$ implies $V \leq 2F - 4$ and $E \leq 3F - 6$. 
4. Other types of reflectors
Other type: paraboloid union (PU)

Punctual light at origin $o$, $\mu$ measure on $S_o^2$

Prescribed far-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $S_\infty^2$
Other type: paraboloid union (PU)

Punctual light at origin $o$, $\mu$ measure on $S^2_o$.
Prescribed far-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $S^2_\infty$.

$P_i(\kappa_i) = \text{convex hull of paraboloid with focal } o$, direction $y_i$ and focal distance $\kappa_i$.
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\[
R(\vec{\kappa}) = \partial \left( \bigcup_{i=1}^N P_i(\kappa_i) \right)
\]

\[
\text{PU}_i(\vec{\kappa}) = \pi_{S^2_o}(R(\vec{\kappa}) \cap \partial P_i(\kappa_i))
\]
Other type: paraboloid union (PU)

Punctual light at origin $o$, $\mu$ measure on $S_0^2$

Prescribed far-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $S_2^\infty$

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$R(\vec{\kappa}) = \partial \left( \bigcup_{i=1}^N P_i(\kappa_i) \right)$

$PU_i(\vec{\kappa}) = \pi_{S_0^2}(R(\vec{\kappa}) \cap \partial P_i(\kappa_i))$

Far-field reflector antenna problem:

**Problem (FF')**: Find $\kappa_1, \ldots, \kappa_N$ such that for every $i$, $\mu(\text{PU}_i(\vec{\kappa})) = \nu_i$. 
Near-Field Reflector Antenna Problem

Punctual light at origin $o$, $\mu$ measure on $S^2_o$

Prescribed near-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $\mathbb{R}^3$
Near-Field Reflector Antenna Problem

Punctual light at origin $o$, $\mu$ measure on $S_o^2$

Prescribed near-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $\mathbb{R}^3$

$E_i(e_i) = \text{convex hull of ellipsoid with focals } o \text{ and } y_i, \text{ and eccentricity } e_i$
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Punctual light at origin $o$, $\mu$ measure on $S_o^2$
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Punctual light at origin $o$, $\mu$ measure on $S^2_o$

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- $E_i(e_i) = \text{convex hull of ellipsoid with focals } o \text{ and } y_i, \text{ and eccentricity } e_i$
- $R(e) = \partial (\cap_{i=1}^N E_i(e_i))$
- $EI_i(e) = \pi S_o^2 (R(e) \cap \partial E_i(\kappa_i))$

Near-field reflector antenna problem:

Problem (NF): Find $e_1, \ldots, e_N$ such that for every $i$, $\mu(EI_i(e)) = \nu_i$.

amount of light reflected to the point $y_i$. 

Oliker '04
Complexity of a single iteration

Complexity of union/intersection of solid confocal quadric of revolutions in $\mathbb{R}^3$:

<table>
<thead>
<tr>
<th>Solid Type</th>
<th>Combinatorial complexity</th>
<th>Computational c.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paraboloid intersection</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n \log n)$</td>
</tr>
<tr>
<td>Paraboloid union</td>
<td>$\Omega(n)$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>Ellipsoid intersection</td>
<td>$\Theta(n^2)$</td>
<td>$\Theta(n^2)$</td>
</tr>
<tr>
<td>Ellipsoid union</td>
<td>$\Theta(n^2)$</td>
<td>$\Theta(n^2)$</td>
</tr>
</tbody>
</table>

$\Omega(n^2)$ for ellipsoids

# faces + points + edges
Conclusion

A simple quasi-Newton scheme can be used to solve rather large (15k points) geometric instances of optimal transport.
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Power diagrams can be used to compute efficiently the $c$-Voronoi cells.
Conclusion

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Future work:

- Near field reflector problem
- complexity of paraboloid union?
- quantitative stability results?
Conclusion

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Power diagrams can be used to compute efficiently the $c$-Voronoi cells.

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Thank you!