Reflector antenna problem

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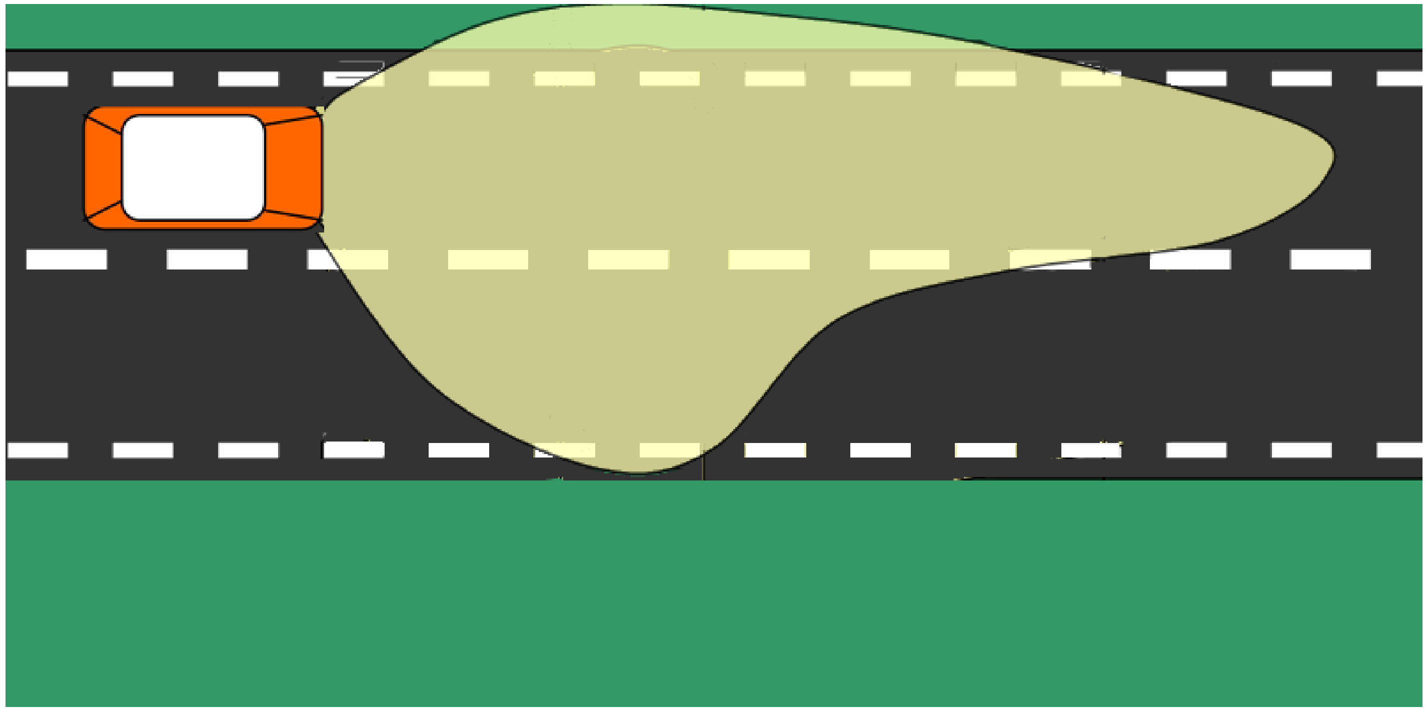
LJK Université de Grenoble

Joint work with Quentin Mérigot and Pedro Machado

Journées de Géométrie Algorithmique
December 16-20, 2013

Motivation



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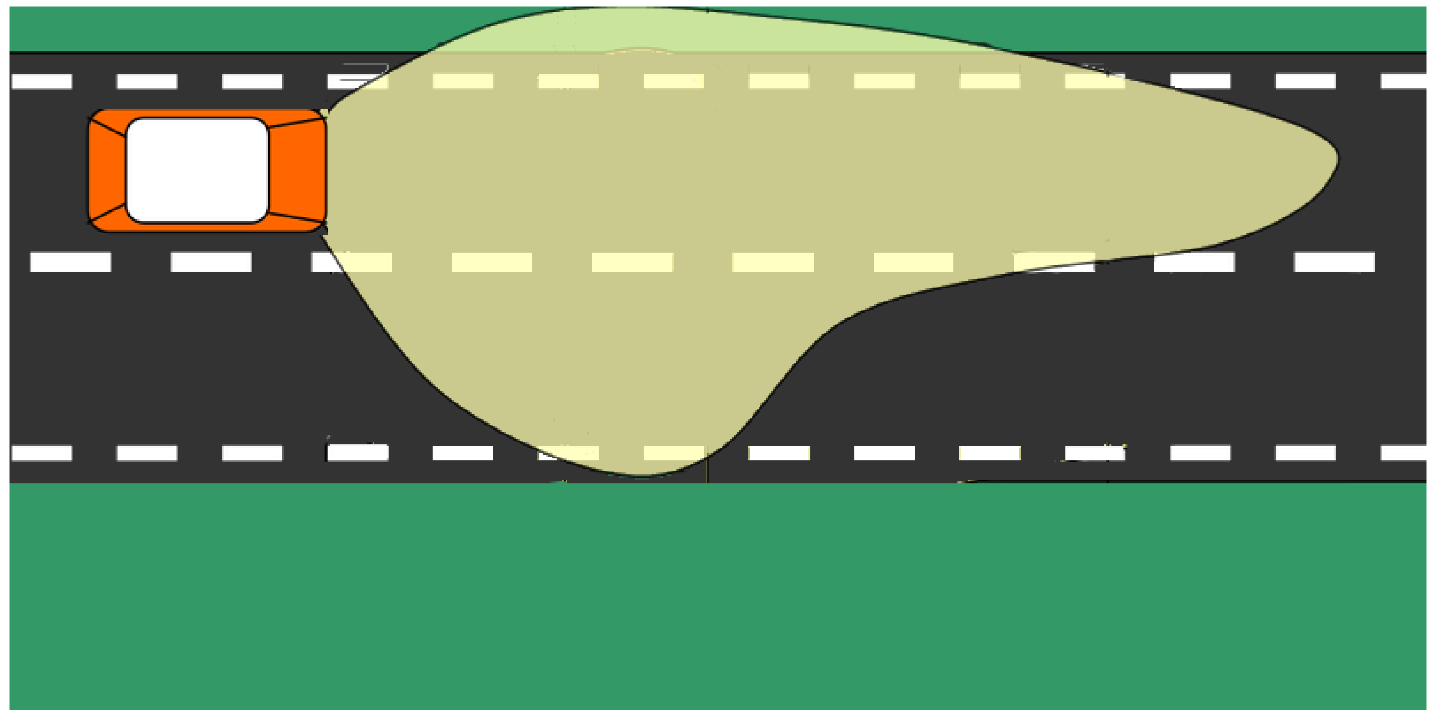

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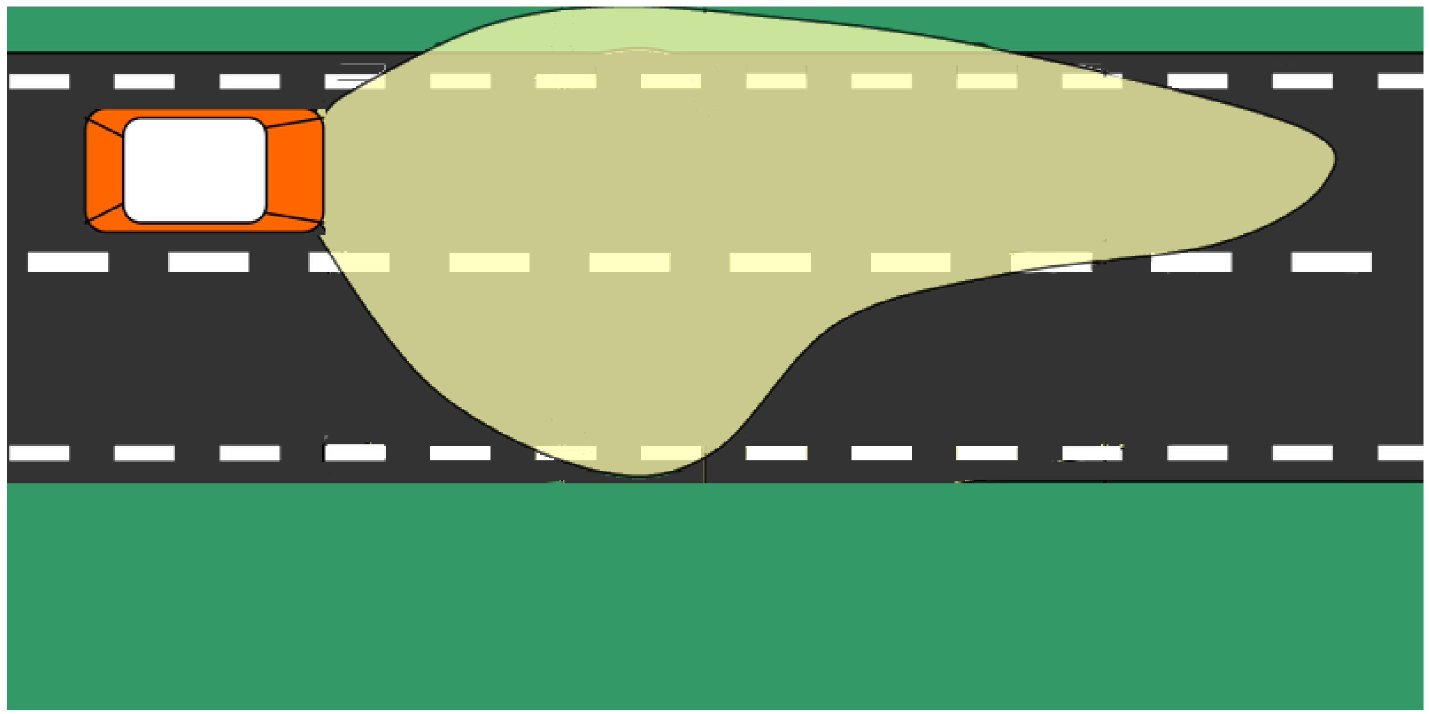
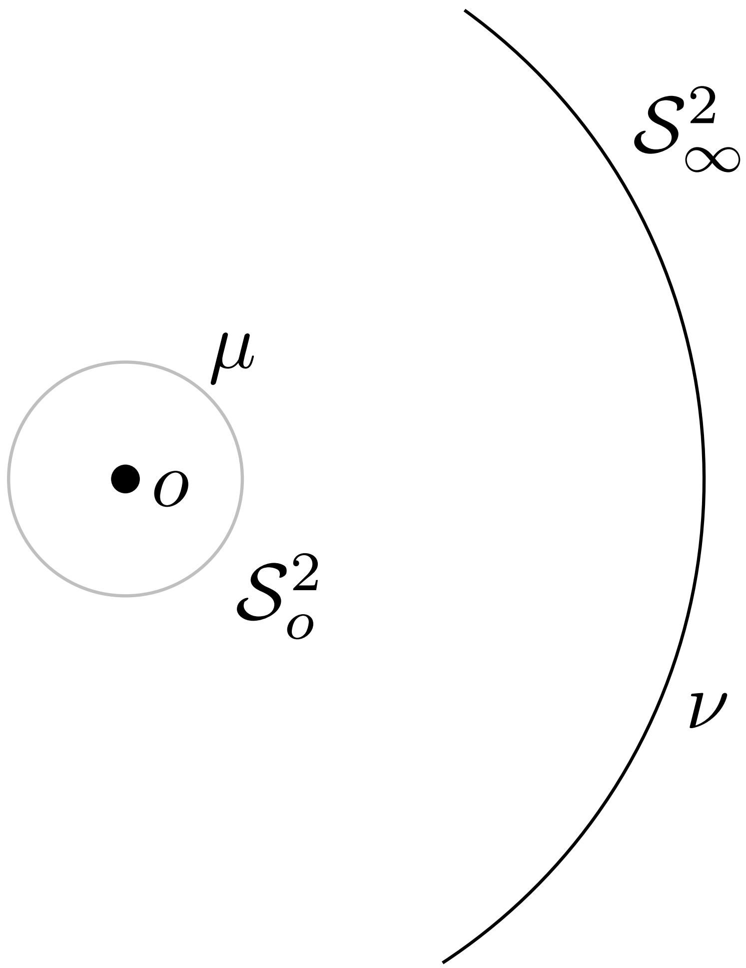


Pb : find the reflector surface

Far-Field Reflector Antenna Problem



Punctual light at origin o, μ measure on $\mathcal{S}^2_o$

Prescribed far-field: ν on $\mathcal{S}^2_\infty$

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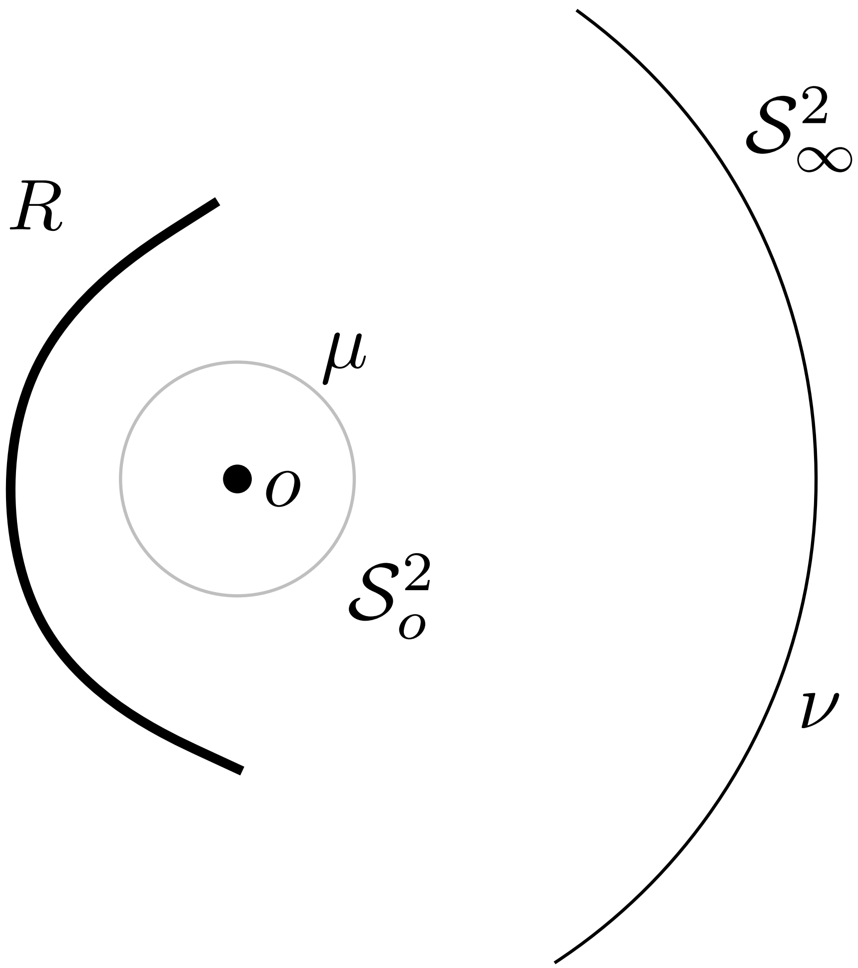

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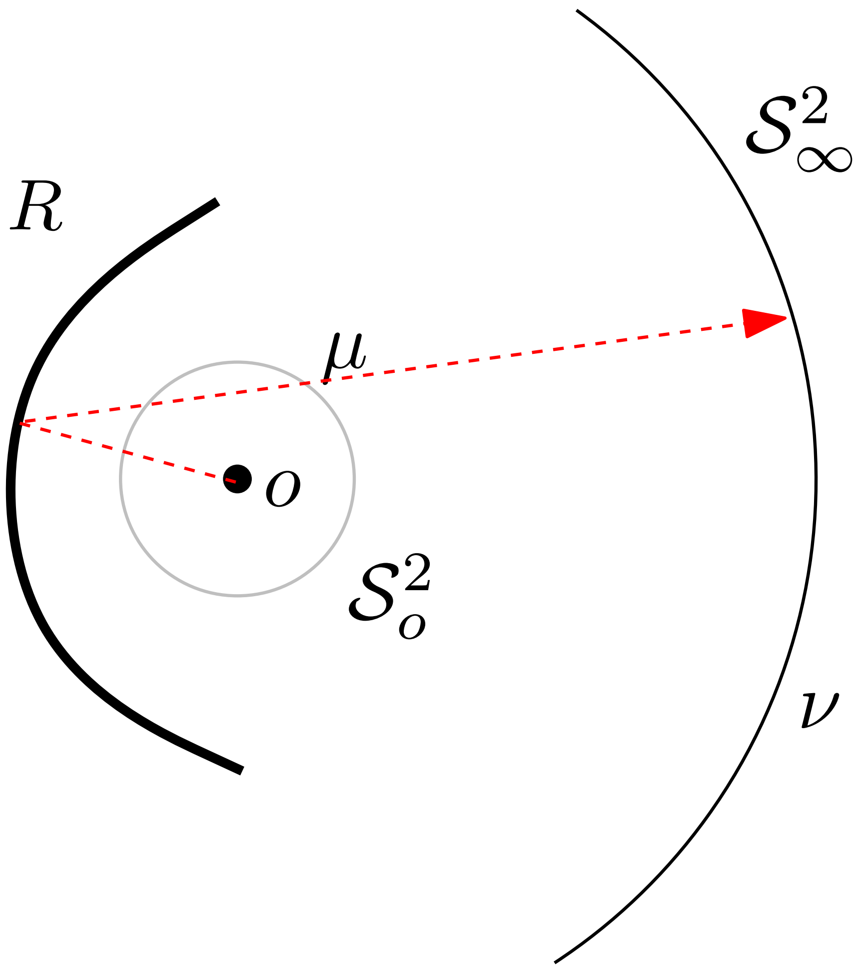

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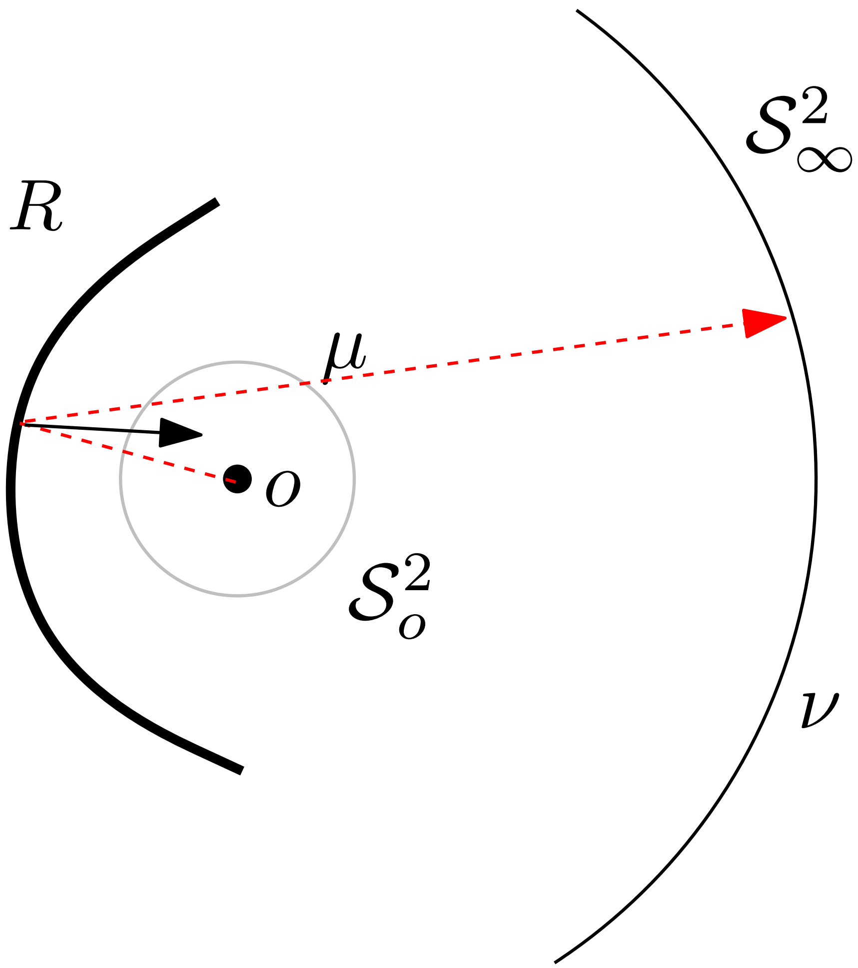

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  $T_R : x \in \mathcal{S}^2_0 \mapsto y = x - 2\langle x|n\rangle n$

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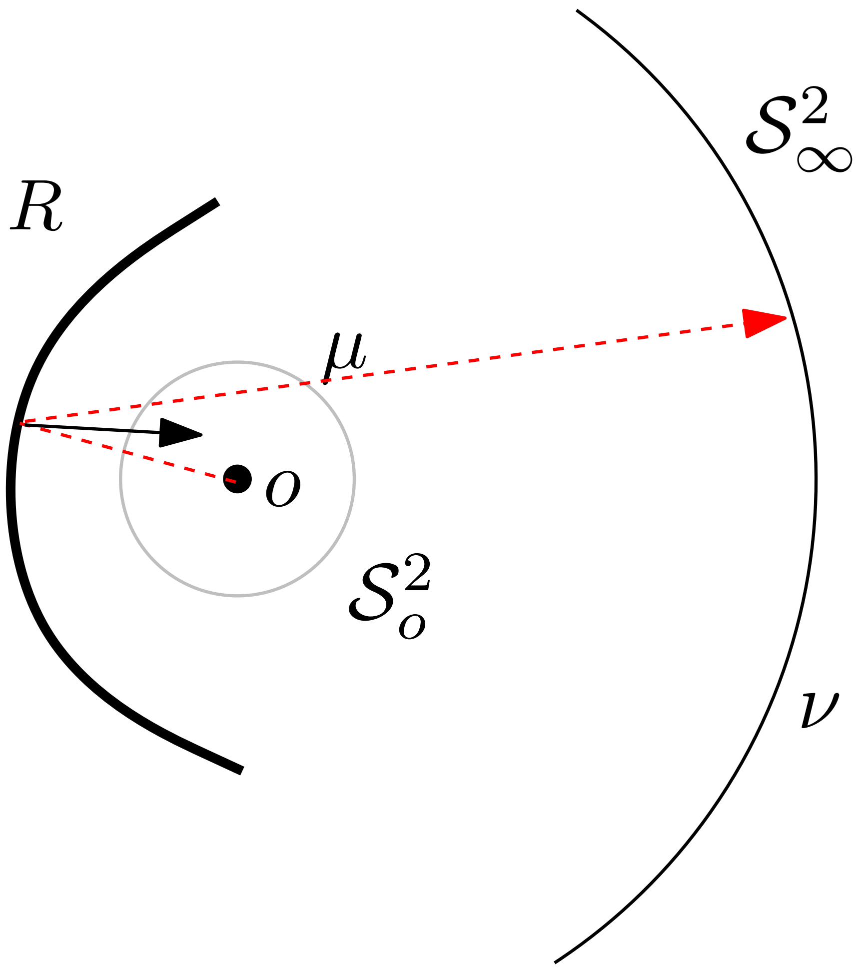

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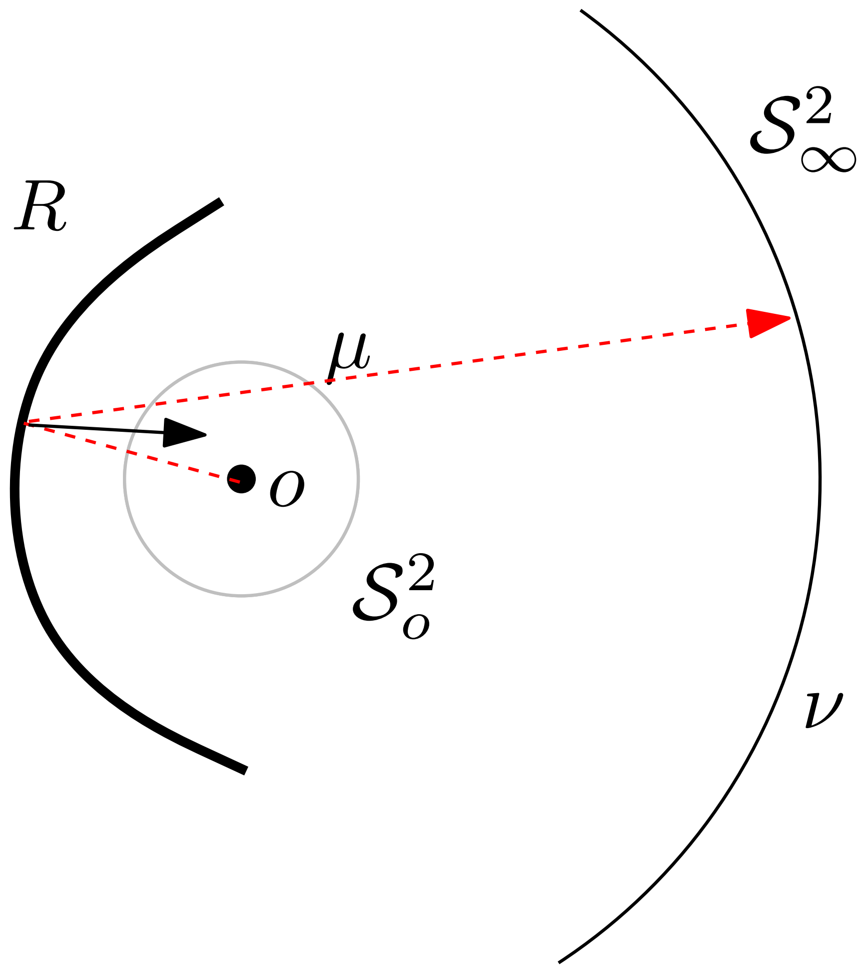

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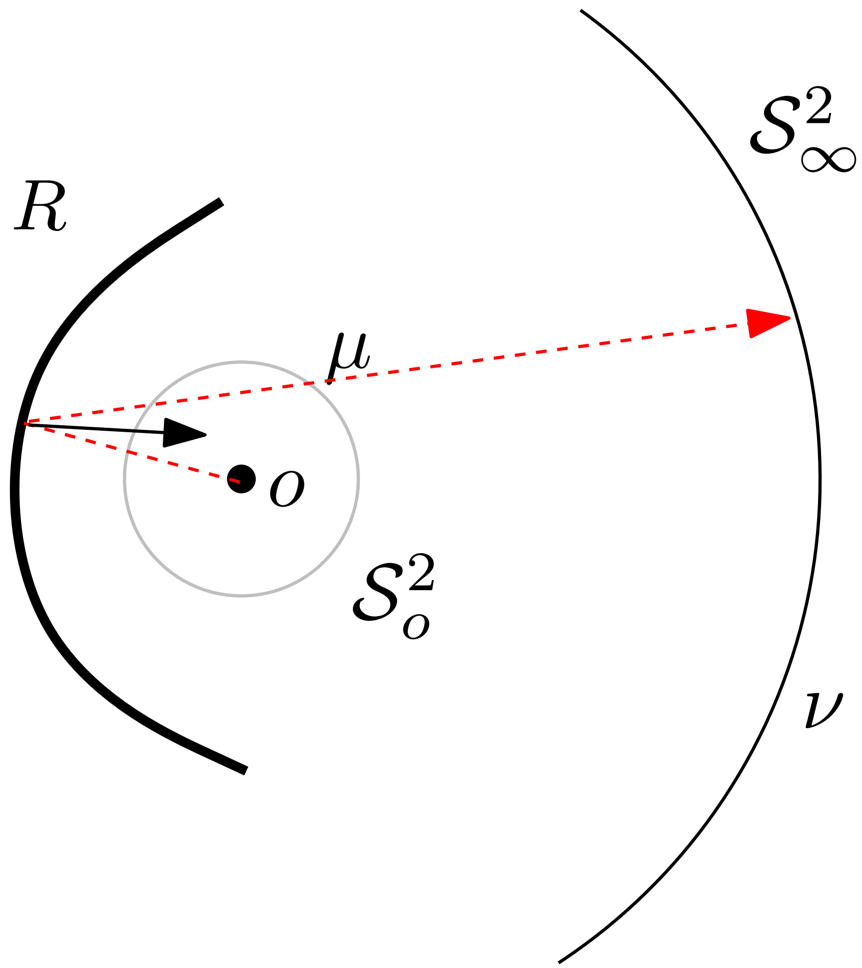

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If $\mu(x) = f(x)dx$ and $\nu(y) = g(y)dy$

$$g(T(x))\det(DT(x)) = f(x)$$

- highly non linear

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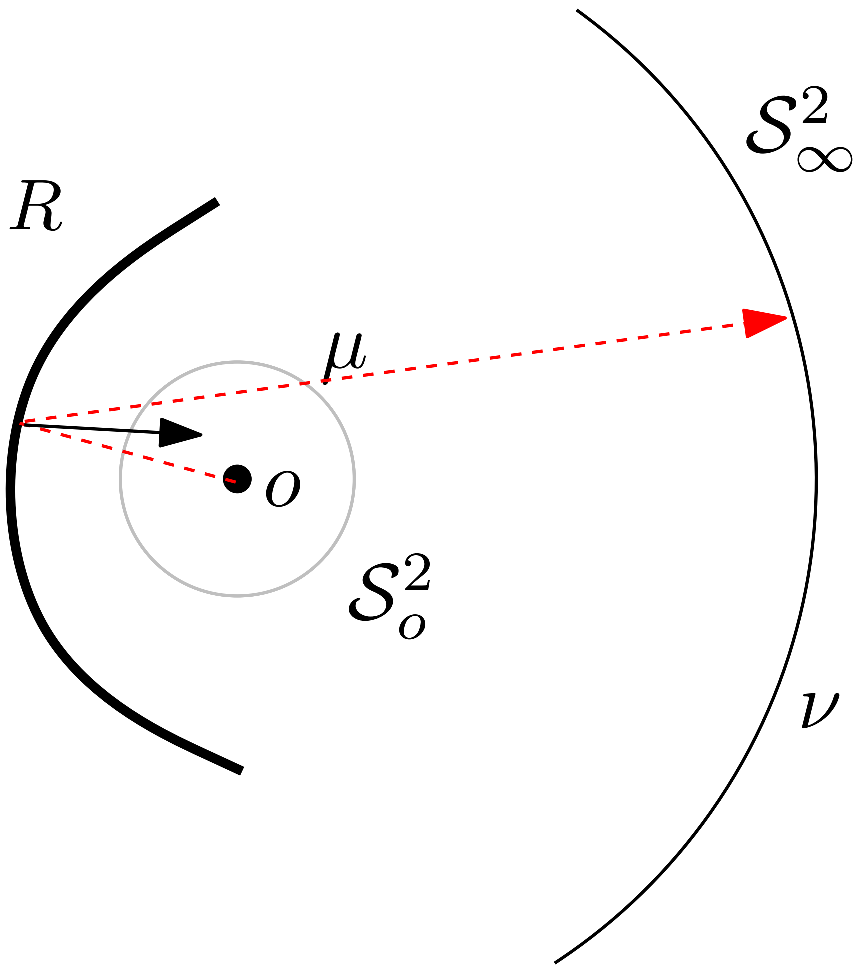

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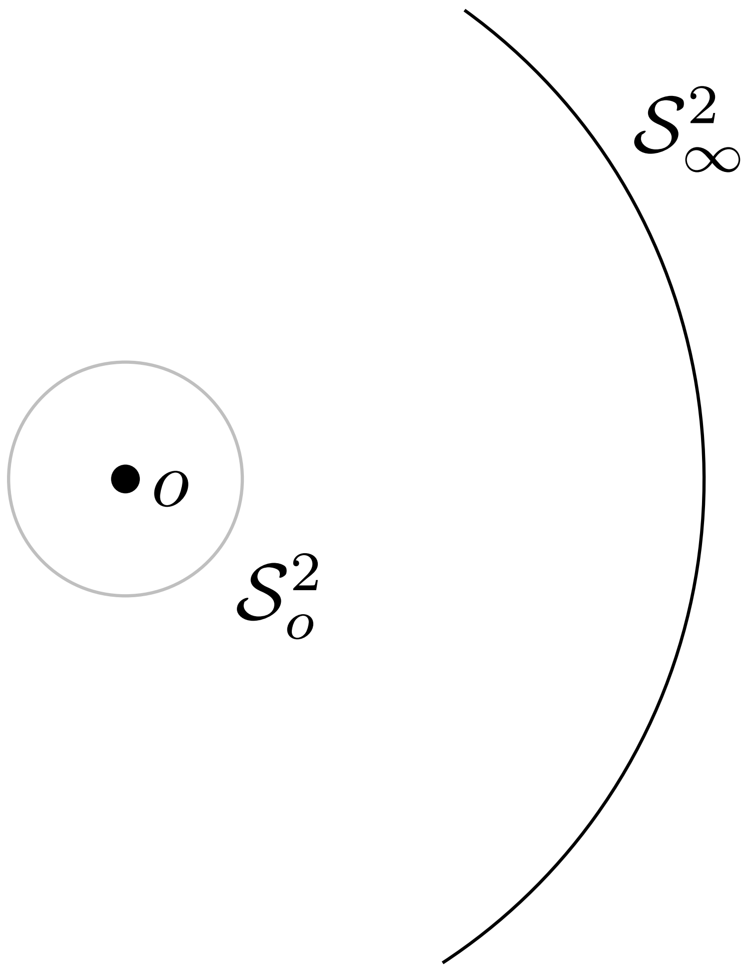
- Existence Caffarelli & Oliker 94
- Regularity, uniqueness Wang 96, Guan & Wang 98

Reflector Problem : semi-discrete case



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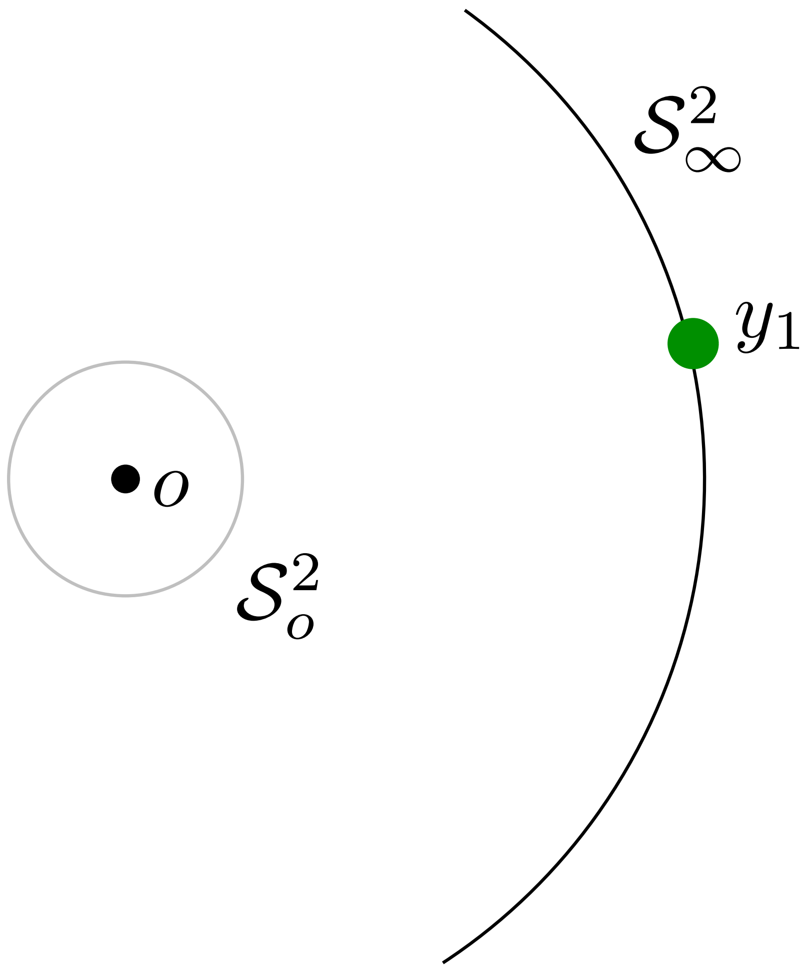

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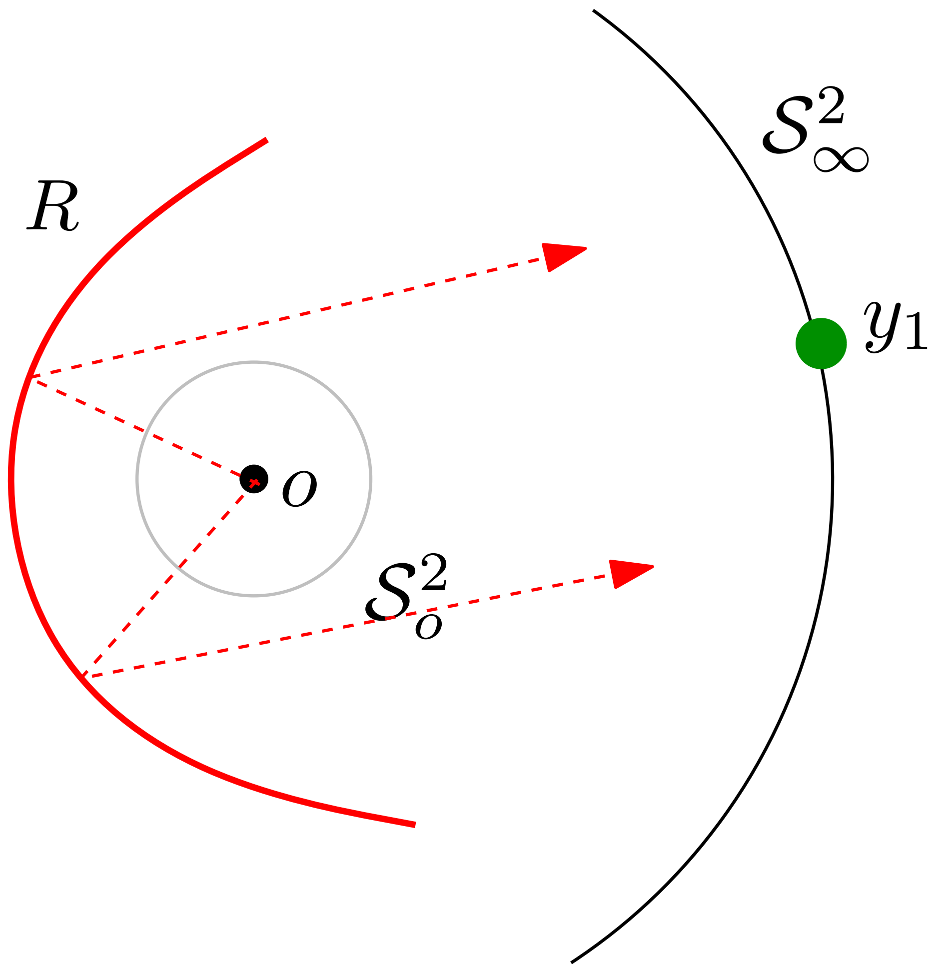

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R : paraboloid of direction y_1 and focal O

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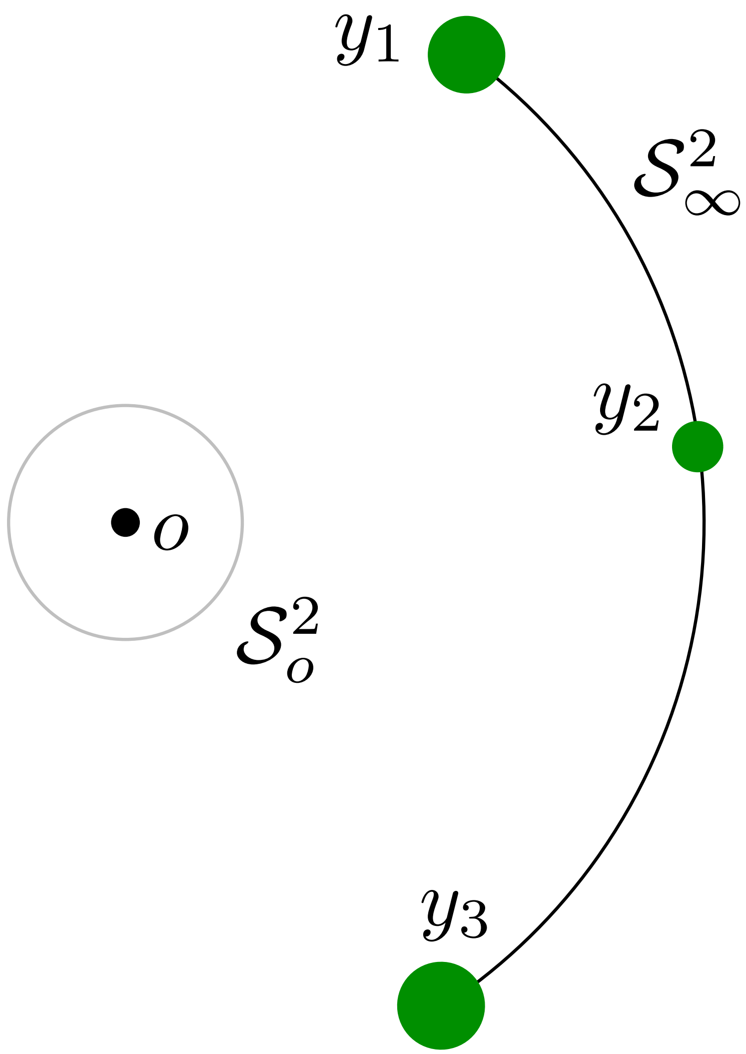

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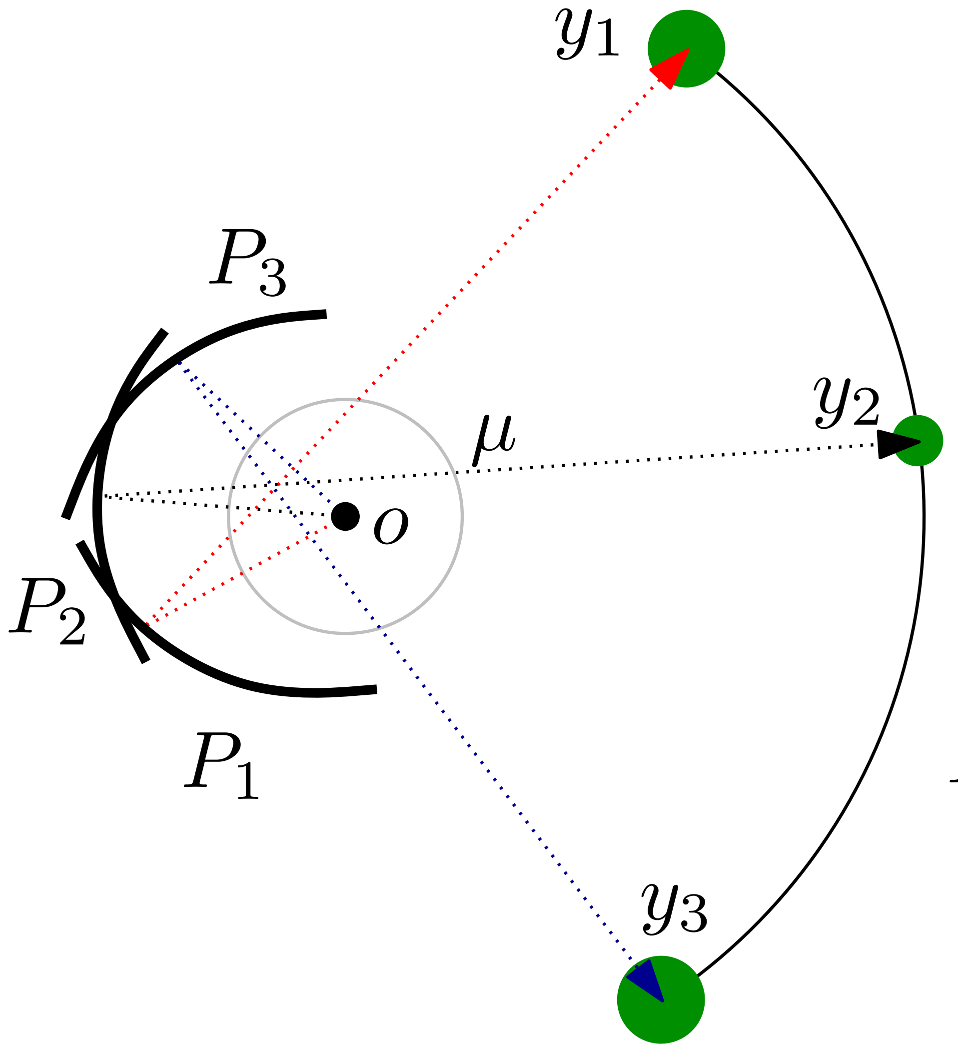

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$P_i(\kappa_i) =$ solid paraboloid of revolution with focal o, direction y_i and focal distance κ_i

$R(\vec{\kappa}) = \partial \left(\cap_{i=1}^{N} P_i(\kappa_i) \right)$

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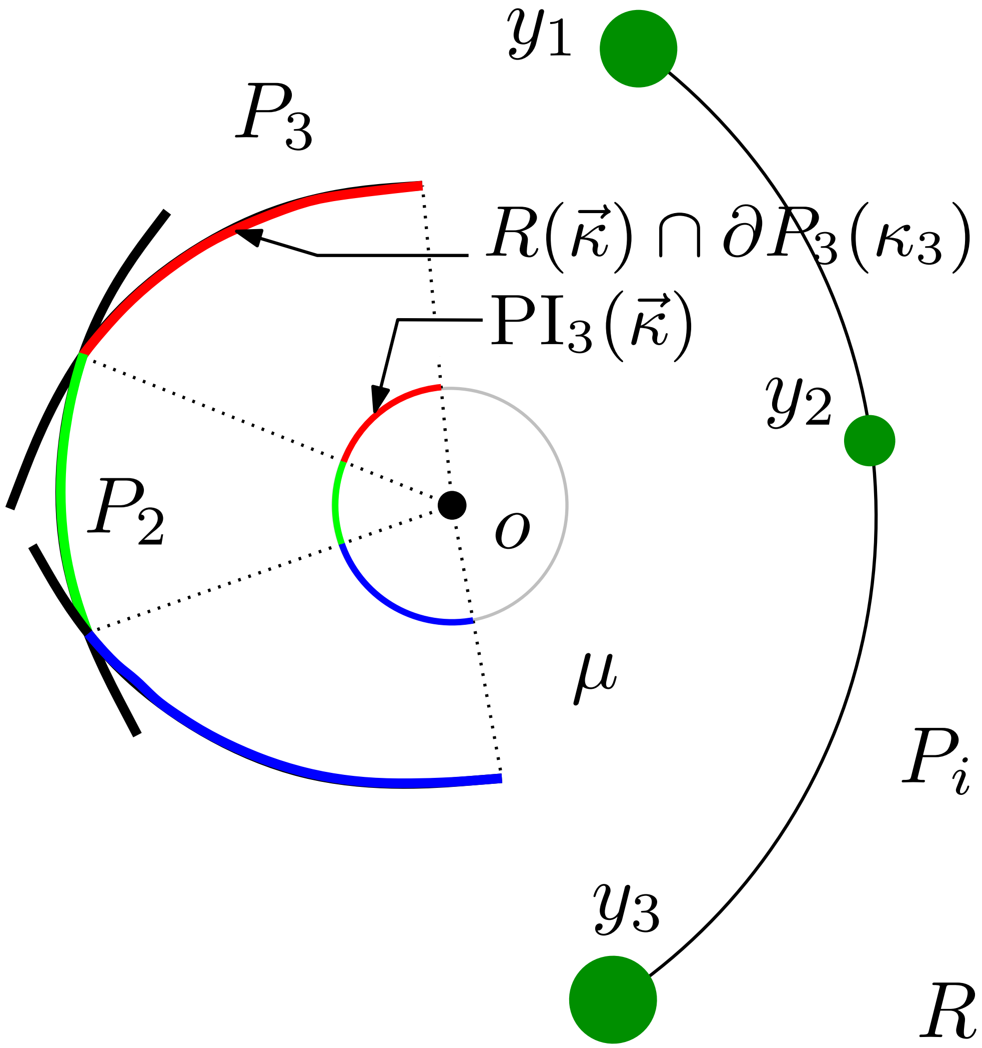

Punctual light at origin o, μ measure on $\mathcal{S}_o^2$

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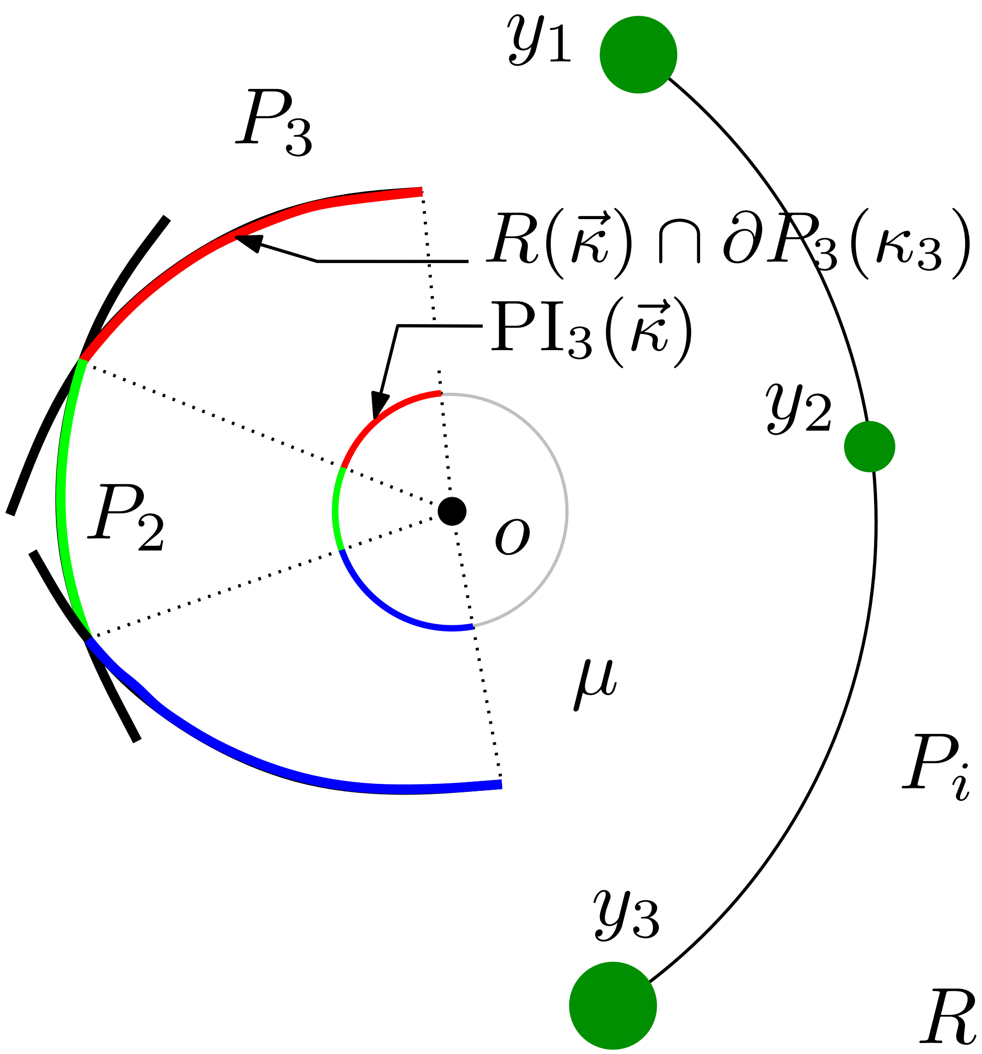
Decomposition of $\mathcal{S}_o^2$: $\mathrm{PI}_i(\vec{\kappa}) = \pi_{\mathcal{S}_o^2}(R(\vec{\kappa}) \cap \partial P_i(\kappa_i))$

$=$ directions that are reflected towards y_i.

Reflector Problem : semi-discrete case



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Problem (FF): Find $\kappa_1, \ldots, \kappa_N$ such that for every i, $\mu(\mathrm{PI}_i(\vec{\kappa})) = \nu_i$.

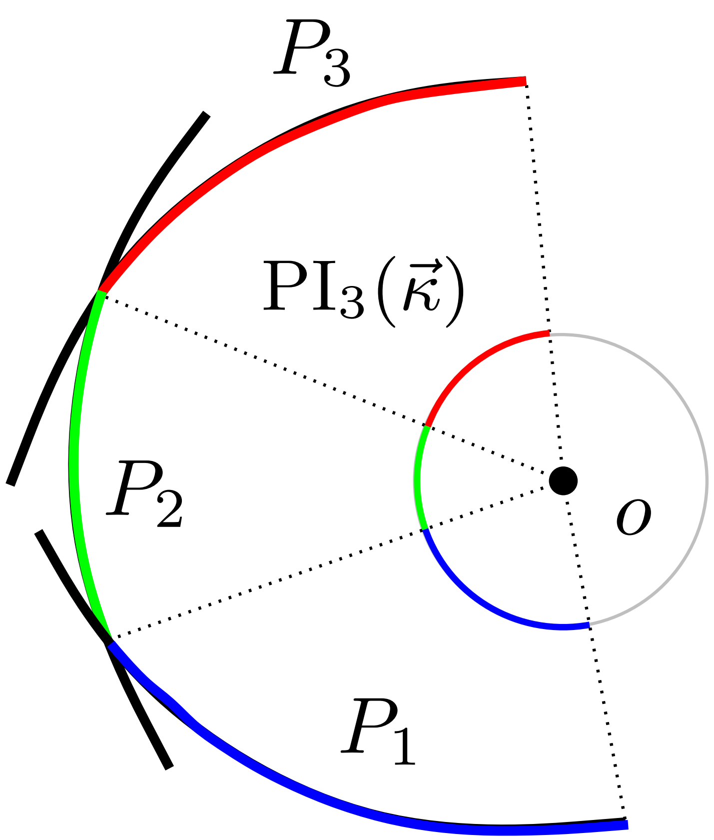
amount of light reflected in direction y_i.

Far-Field Reflector Antenna Problem as OT

Lemma: With $c(x, y) = -\log(1 - \langle x|y\rangle)$, and $\psi_i := \log(\kappa_i)$,

$$\mathrm{PI}_i(\vec{\kappa}) = \{x \in \mathcal{S}_0^2,\ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \quad \forall j\}.$$

Caffarelli-Oliker '94



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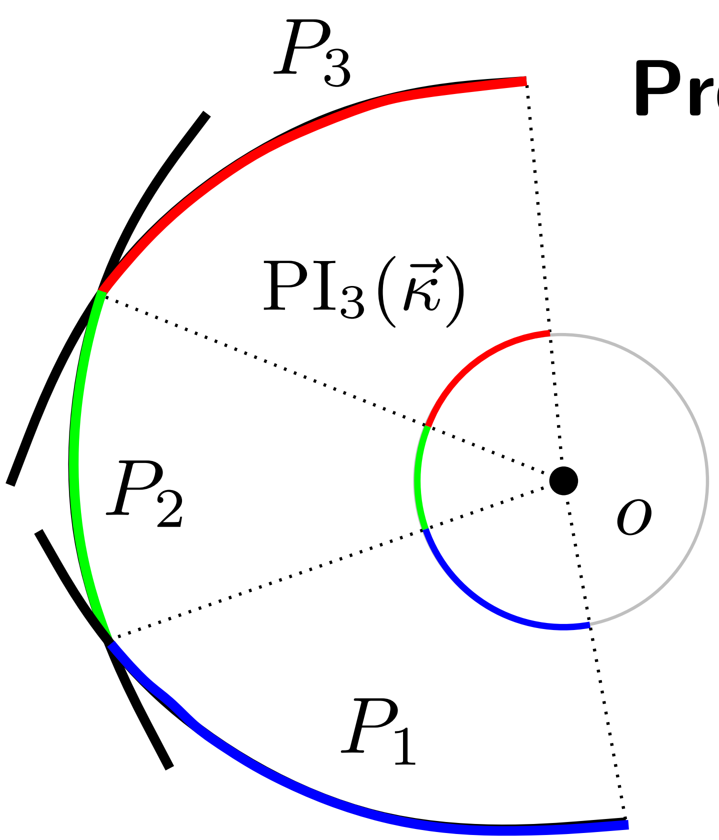

Proof: $\partial P_i(\kappa_i)$ is parameterized in radial coordinates by
$$\rho_i : x \in \mathcal{S}^2_o \mapsto \frac{\kappa_i}{1 - \langle x|y_i\rangle}$$

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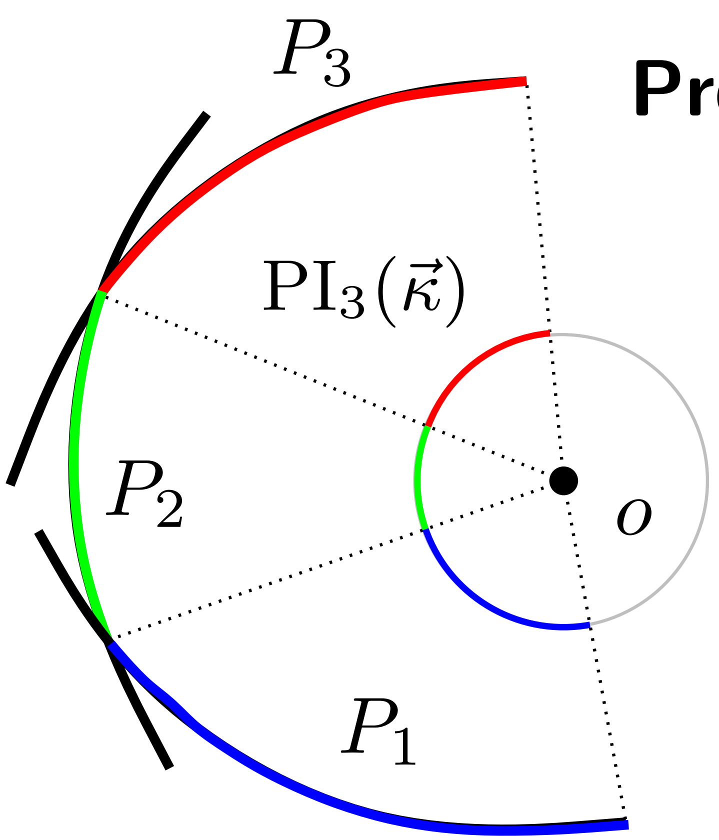

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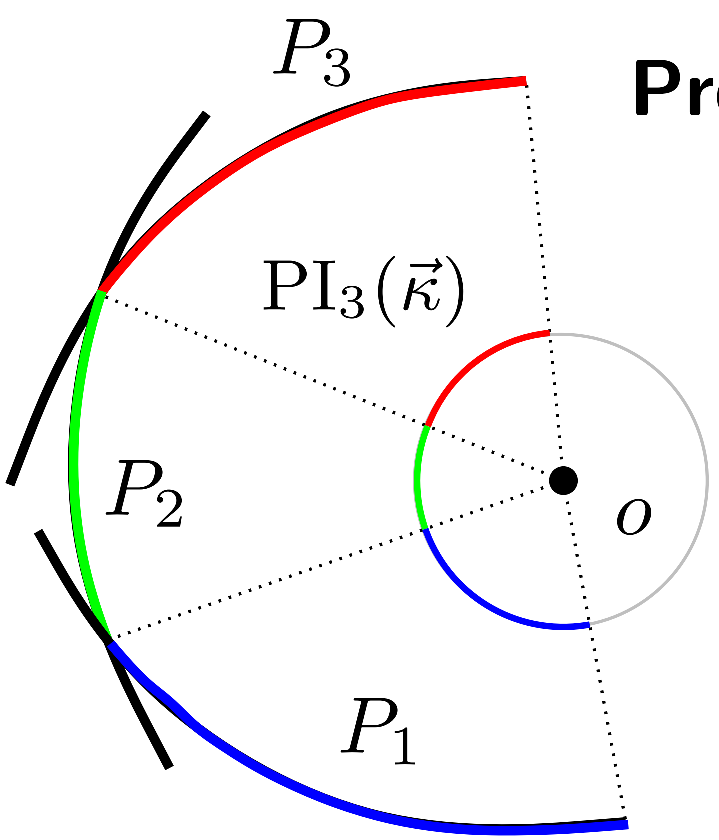

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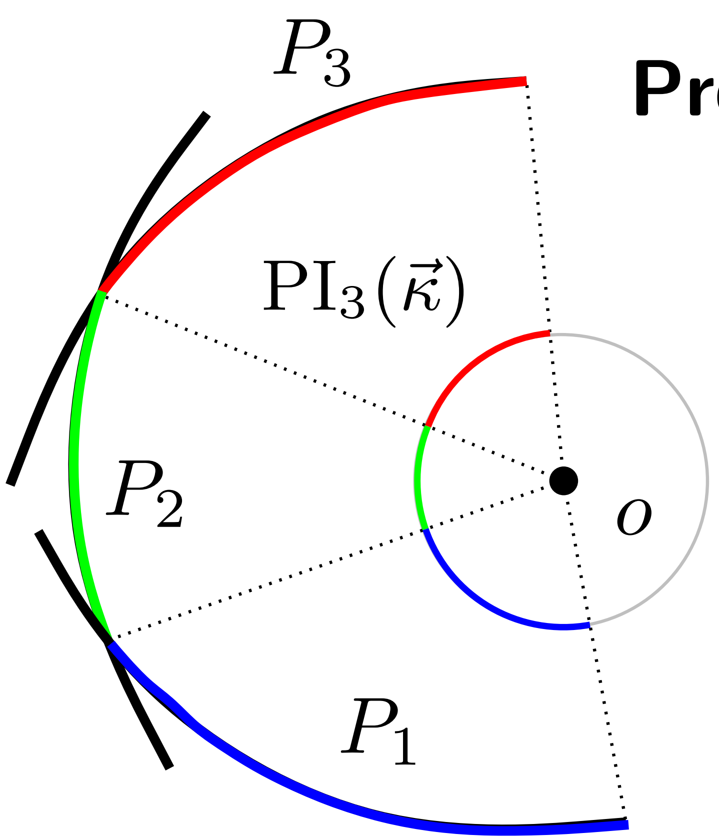

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$$\begin{aligned} x \in \mathrm{PI}_i(\vec{\kappa}) &\iff i \in \arg\min_j \frac{\kappa_j}{1-\langle x|y_j\rangle} \\ &\iff i \in \arg\min_j \log(\kappa_j) - \log(1 - \langle x|y_j\rangle) \\ &\iff i \in \arg\min_j \psi_j + c(x,y_j) \end{aligned}$$

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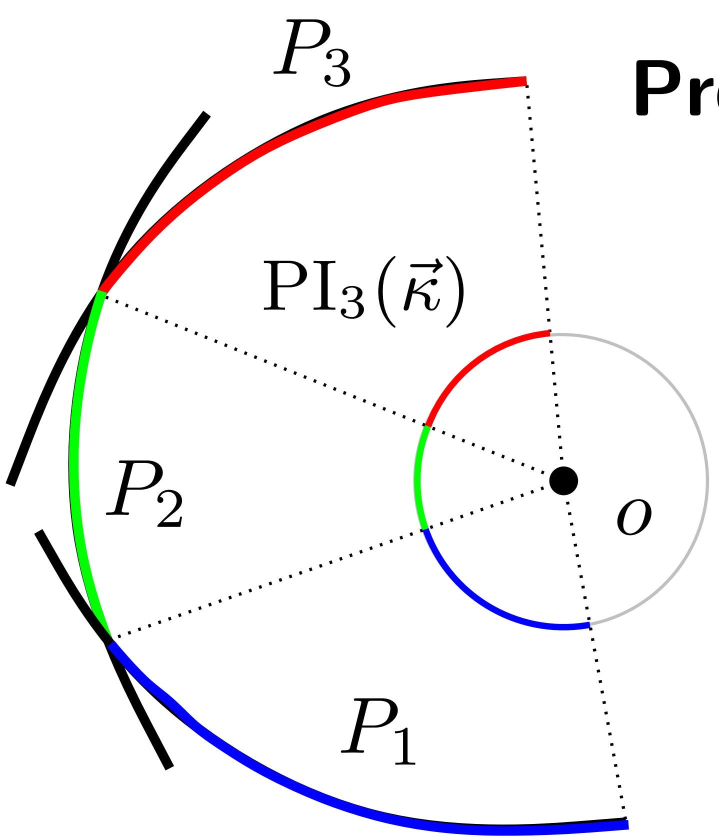

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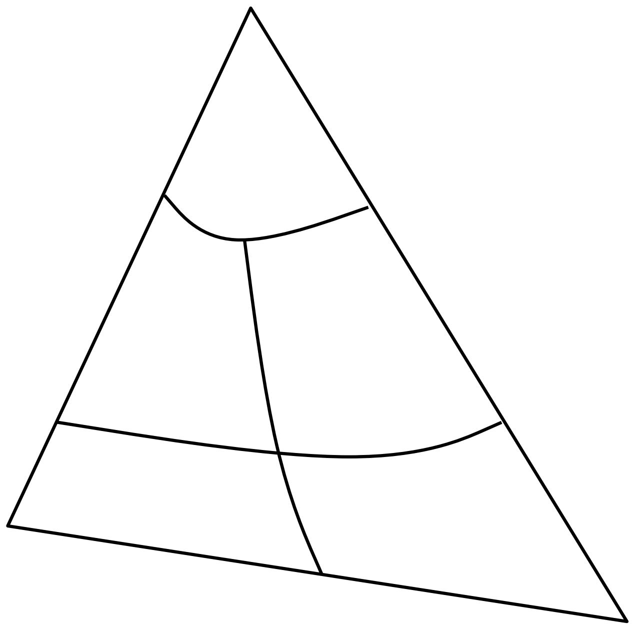
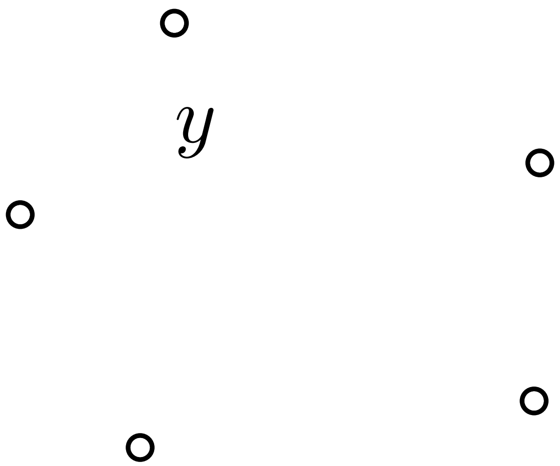
- An optimal transport problem Wang '04

Semi-discrete optimal transport

$\mu =$ probability measure on X with density, $X =$ manifold



$\nu =$ prob. measure on finite Y $= \sum_{y \in Y} \nu_y \delta_y$



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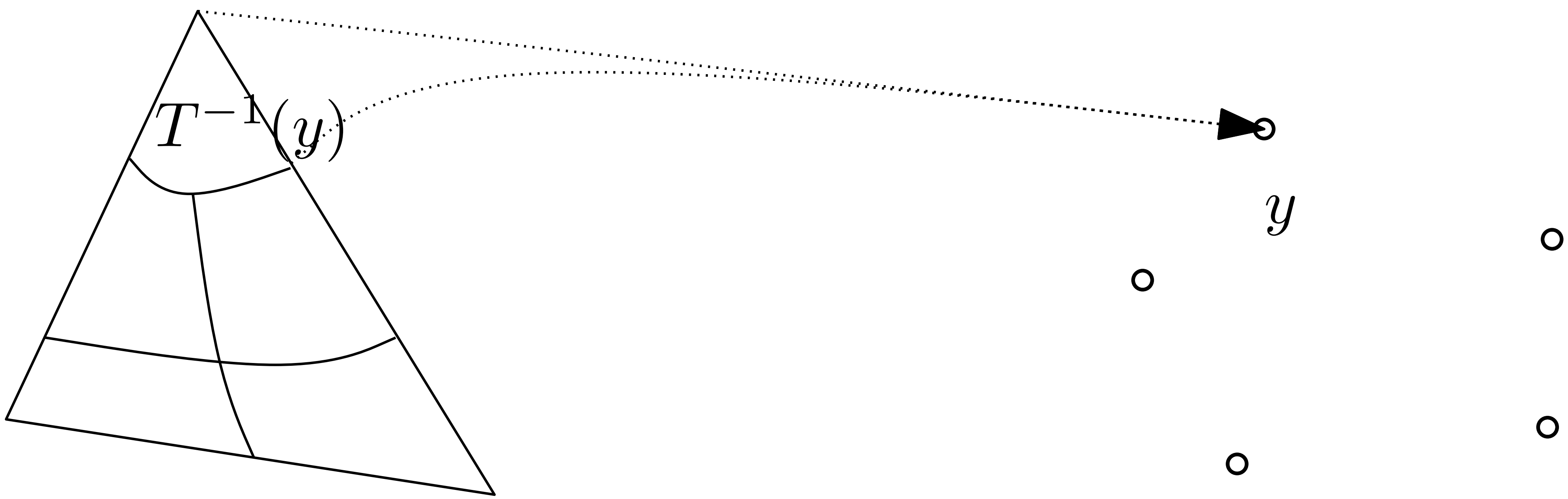

Transport map: $T : X \to Y$ s.t.

$$\forall y \in Y,\ \mu(T^{-1}(\{y\})) = \nu_y$$

in short: $T_{\#}\mu = \nu$.

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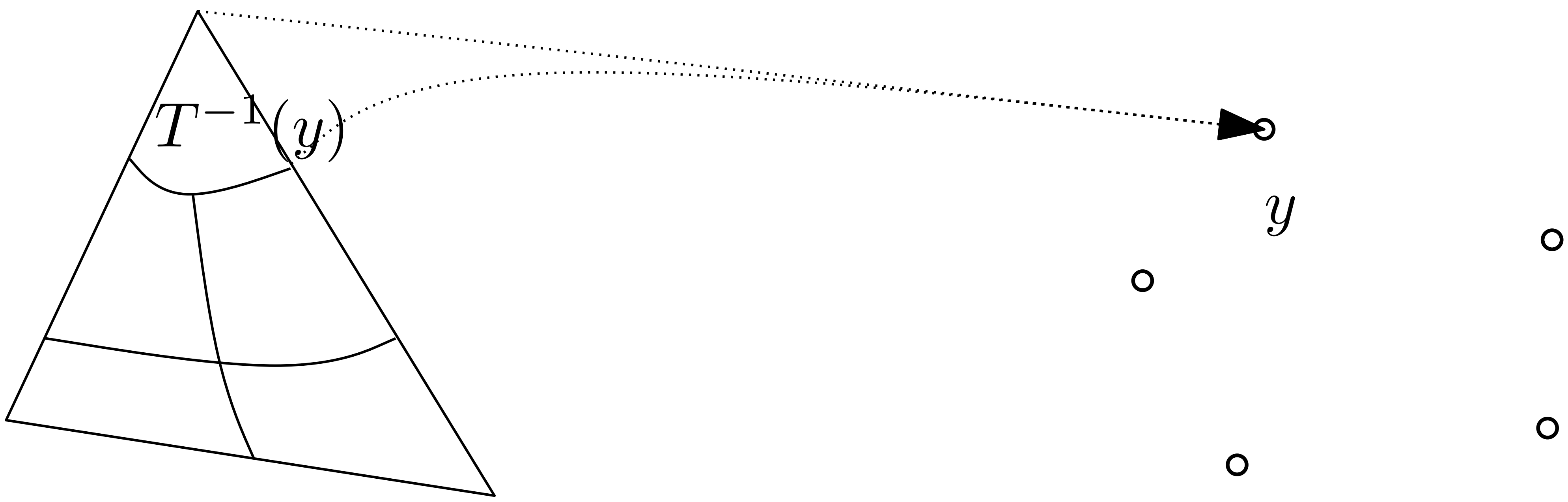

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$\mathcal{C}_c(T) = \int_X c(x, T(x))\, \mathrm{d}\, \mu(x)$

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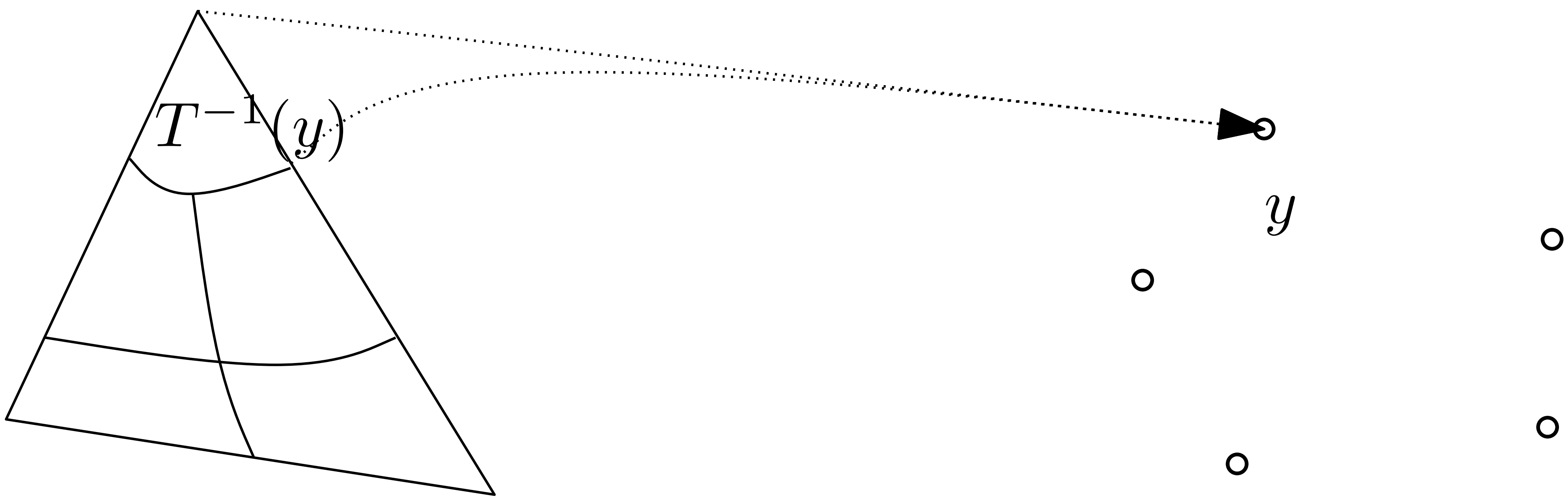

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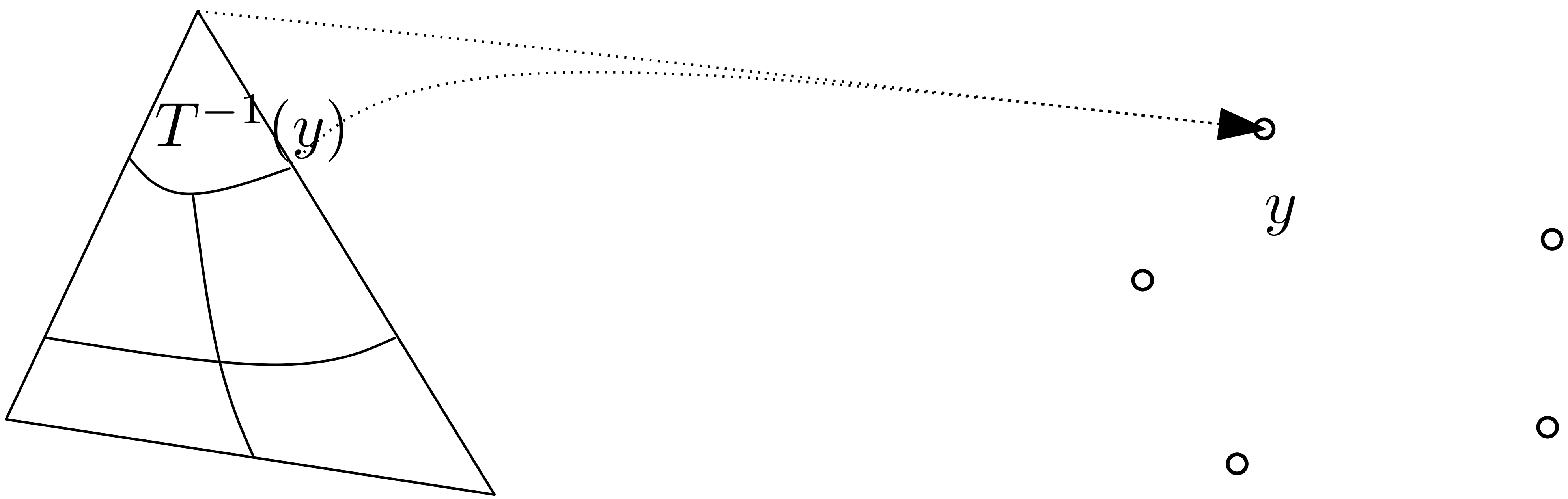

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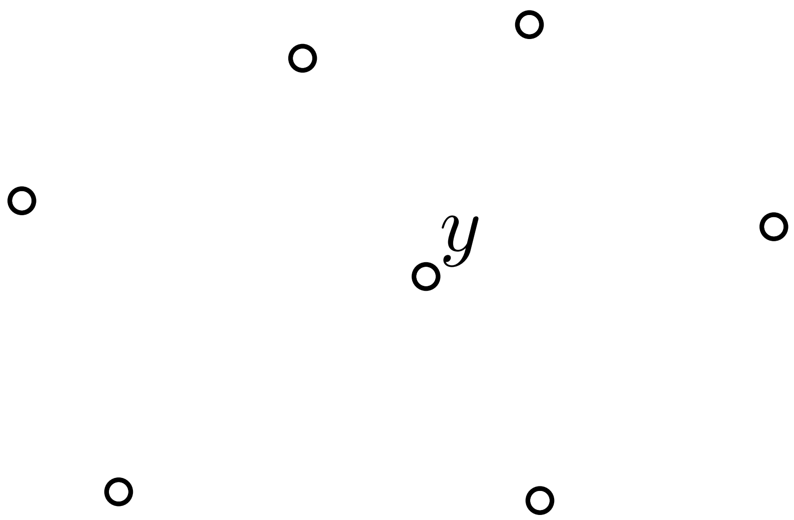
Monge problem: $\mathcal{T}_c(\mu, \nu) := \min\{\mathcal{C}_c(T); T_\# \mu = \nu\}$

Weighted Voronoi and Optimal Transport

Aurenhammer, Hoffman, Aronov '98 Merigot '2010



Y finite set, $\psi : Y \to \mathbb{R}$

We assume **(Twist)**, i.e. $c \in \mathcal{C}^\infty$ and $\forall x \in X$ the map $y \in Y \mapsto \nabla_x c(x, y)$ is injective.

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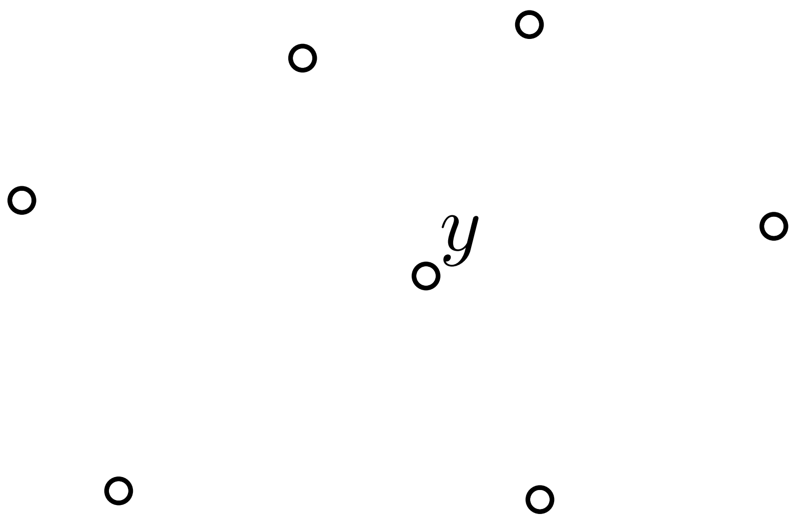

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$$T_c^\psi(x) = \arg\min_{y \in Y} c(x,y) + \psi(y)$$

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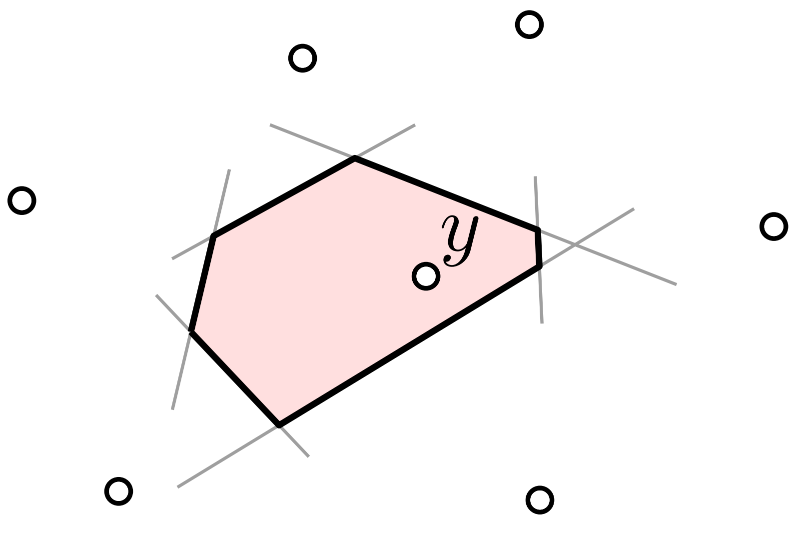
Aurenhammer, Hoffman, Aronov '98 Merigot '2010



Y finite set, $\psi : Y \to \mathbb{R}$

We assume **(Twist)**, i.e. $c \in \mathcal{C}^\infty$ and $\forall x \in X$ the map $y \in Y \mapsto \nabla_x c(x, y)$ is injective.

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$$\mathrm{Vor}_c^\psi(y) = \{x \in \mathbb{R}^d ;\, T_c^\psi(x) = y\}$$

$=$ generalized weighted Voronoi cell

Weighted Voronoi and Optimal Transport

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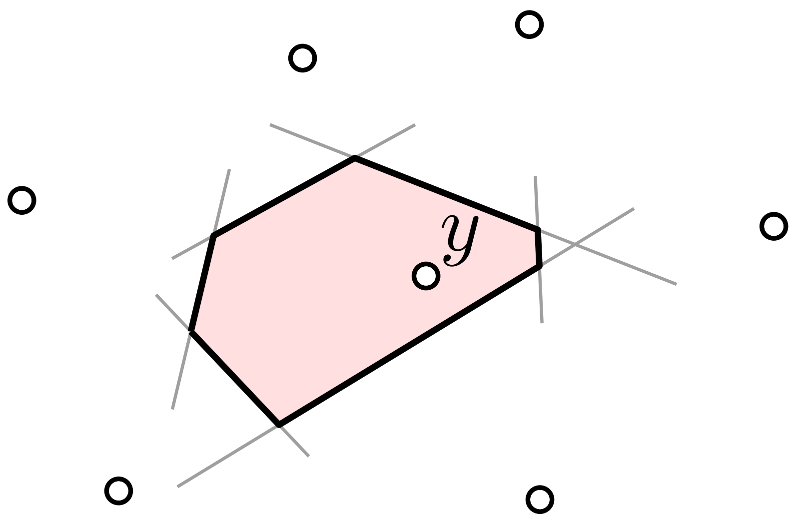

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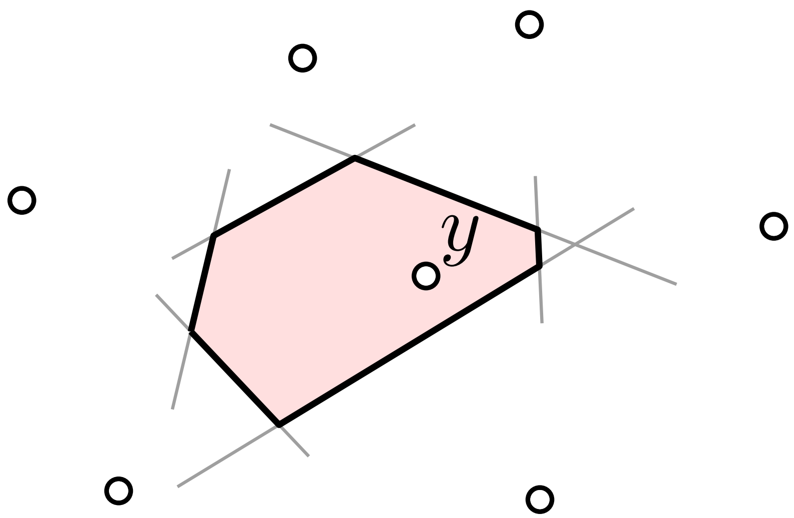

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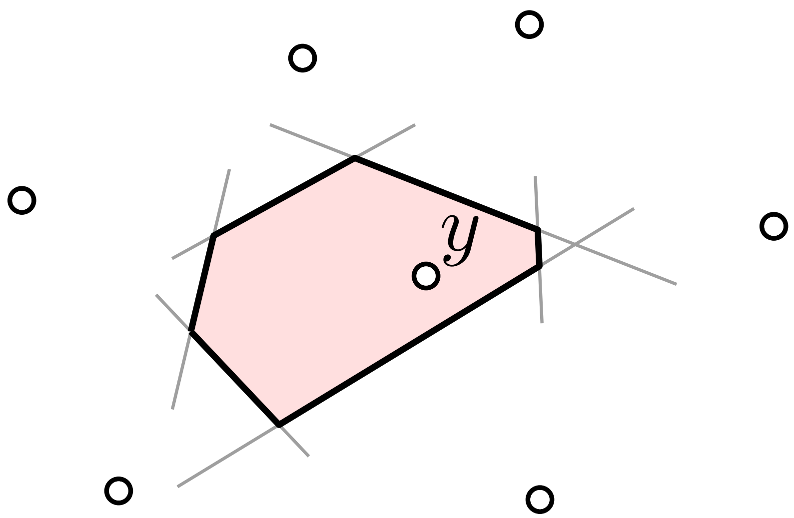

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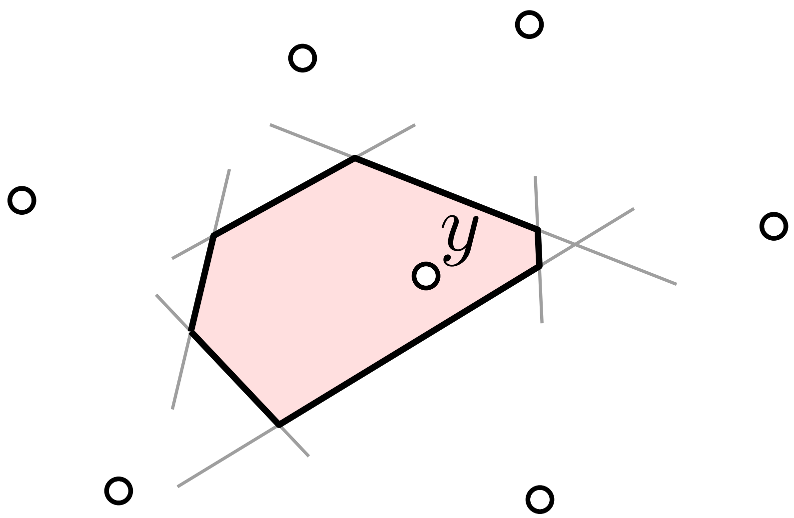

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- Converse ?

Back to the Reflector Antenna Problem

Lemma: With $c(x, y) = -\log(1 - \langle x|y \rangle)$, and $\psi_i := \log(\kappa_i)$,

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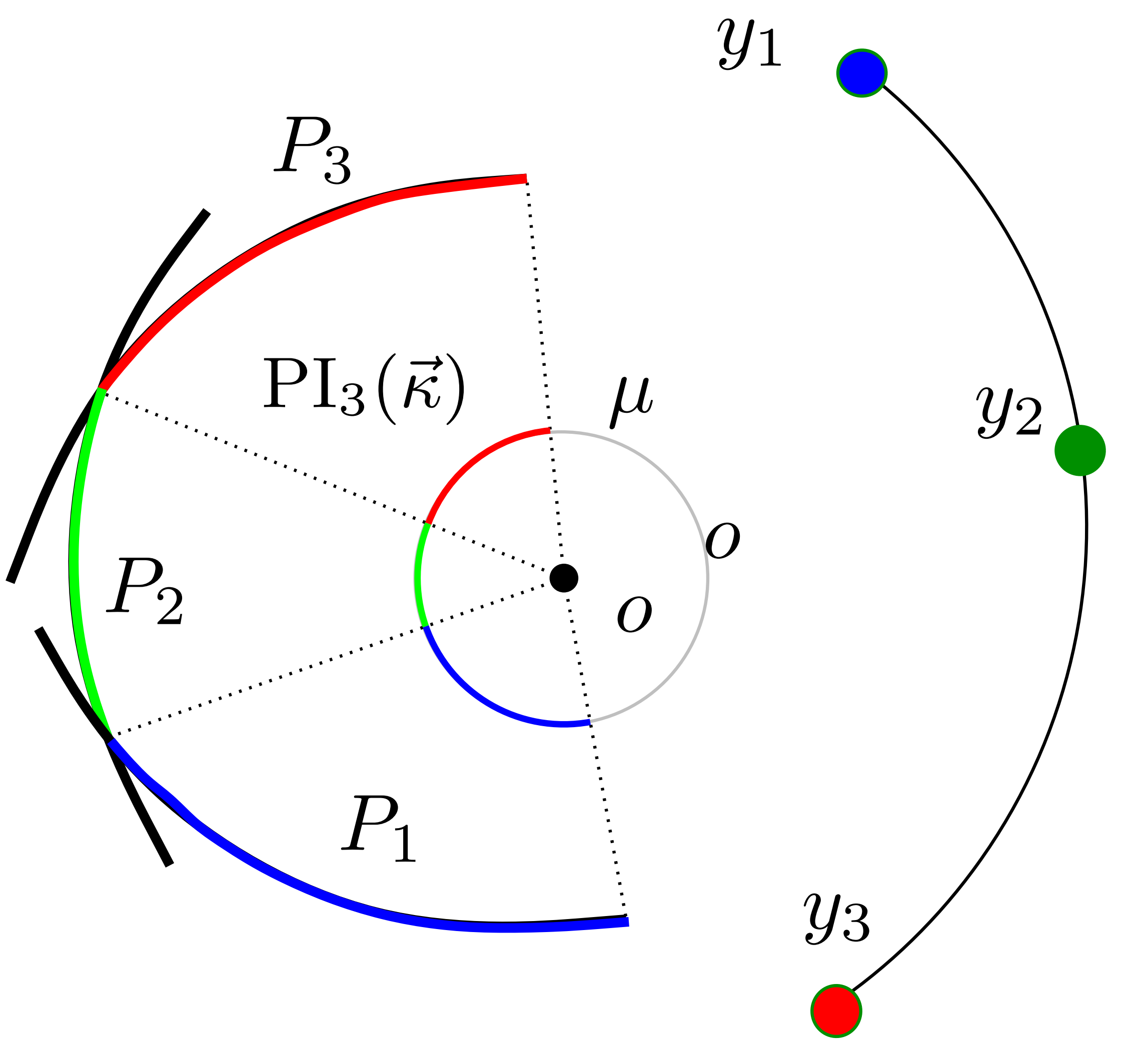

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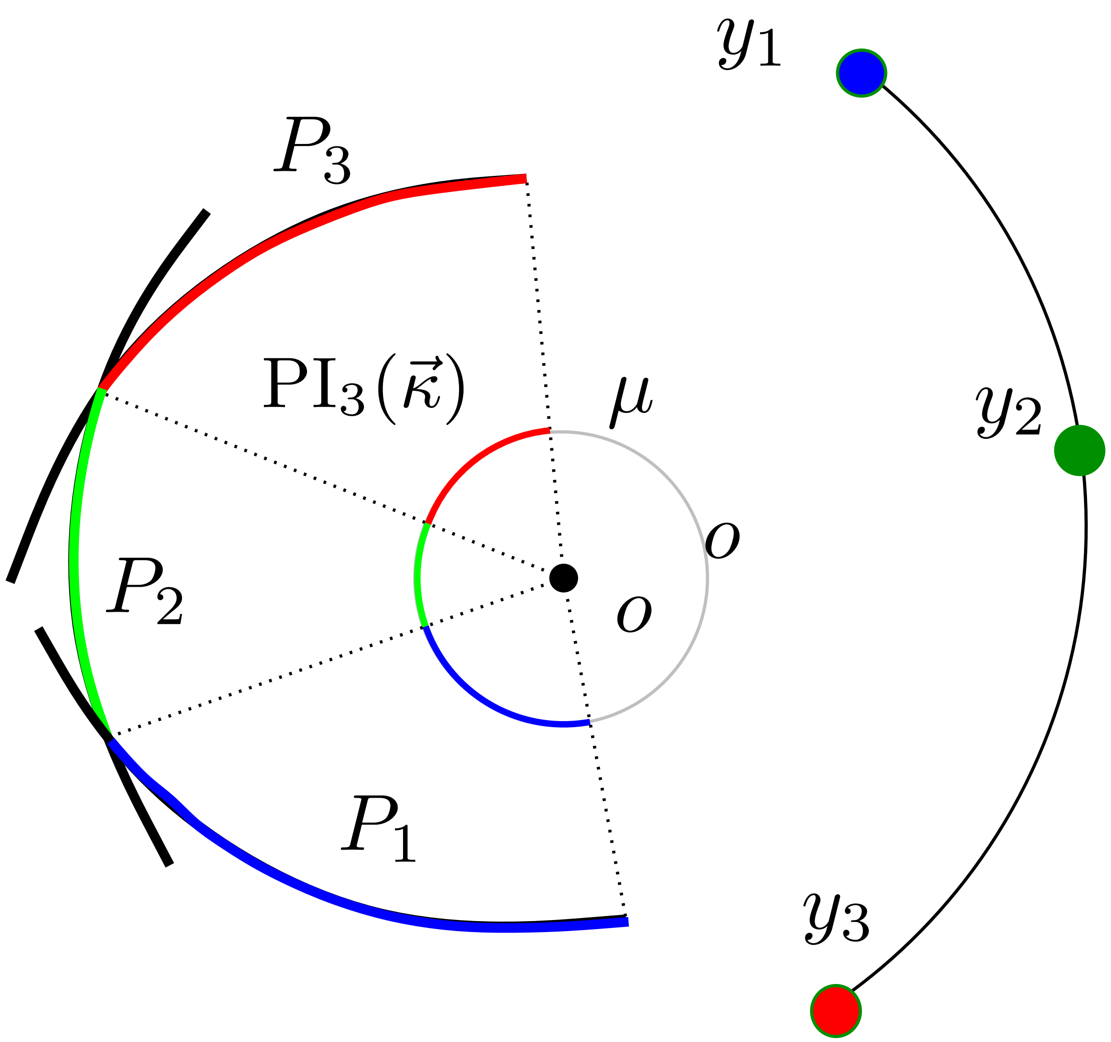

Optimal transport formulation

- $\mathrm{PI}_i(\vec{\kappa}) = \mathrm{Vor}_c^{\psi}(y_i)$.
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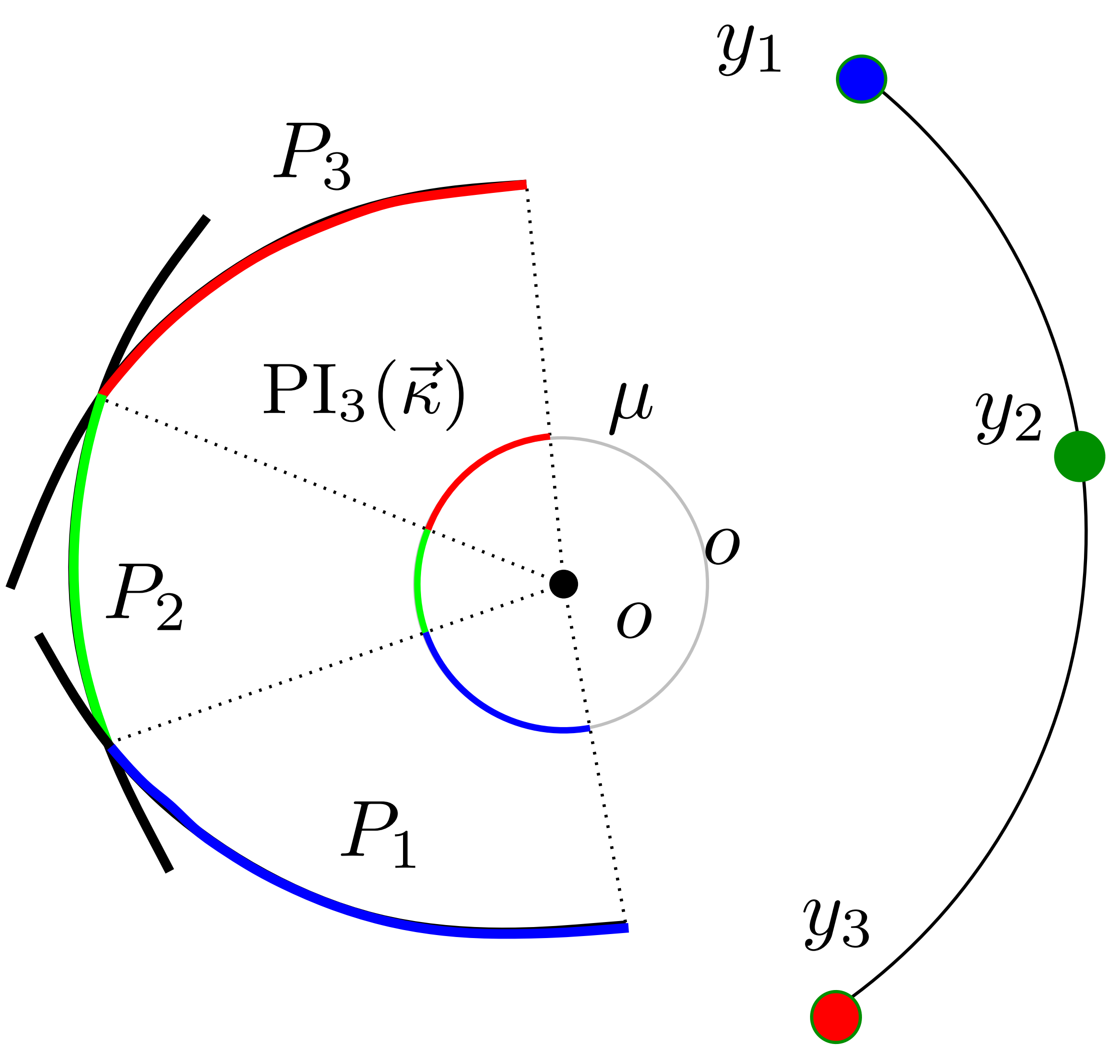

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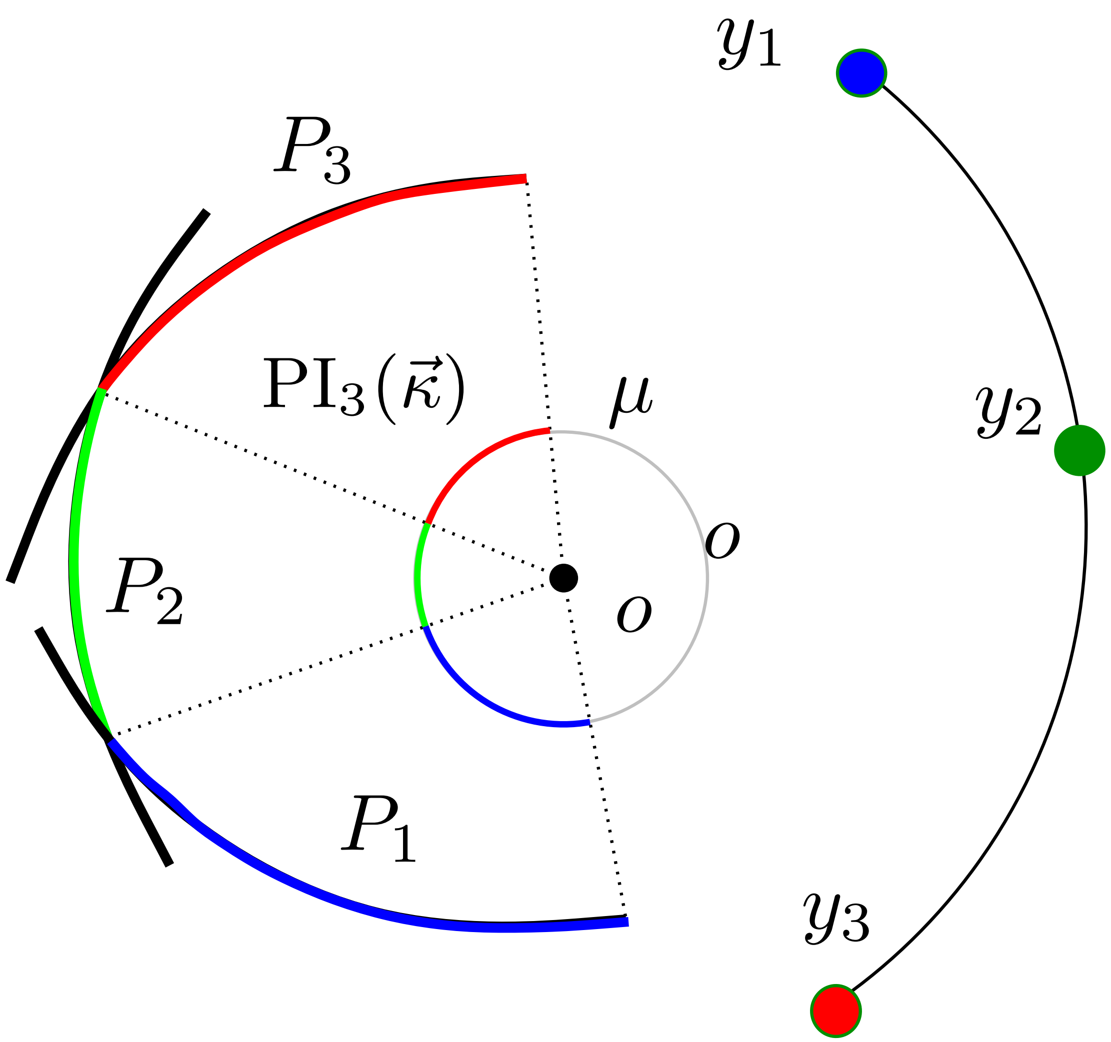

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The map T_c^{ψ} is a c-optimal transport between μ and $T_{c\#}^{\psi}\mu$.

Problem (FF): Find $\psi_1, \ldots, \psi_N$ such that $T_{c\#}^{\psi}\mu = \nu$.

Supporting paraboloids algorithm' 99

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While $\exists y \neq y_0$ such that $\mu(\mathrm{Vor}_c^{\psi}(y)) \leq \nu_y - \delta$, **do:**

decrease $\psi(y)$ s.t. $\mu(\mathrm{Vor}_c^{\psi}(y)) \in [\nu_y, \nu_y + \delta]$,

Result: ψ s.t. for all y, $|\mu(\mathrm{Vor}_c^{\psi}(y)) - \nu_y| \leq \varepsilon$.

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Concave maximization

Theorem: $\vec{\kappa}$ solves **(FF)** iff $\vec{\psi} = \log(\vec{\kappa})$ maximizes

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- A consequence of Kantorovich duality.

Proof of concave maximization thm

Supdifferentials. Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ and $\psi \in \mathbb{R}^d$.

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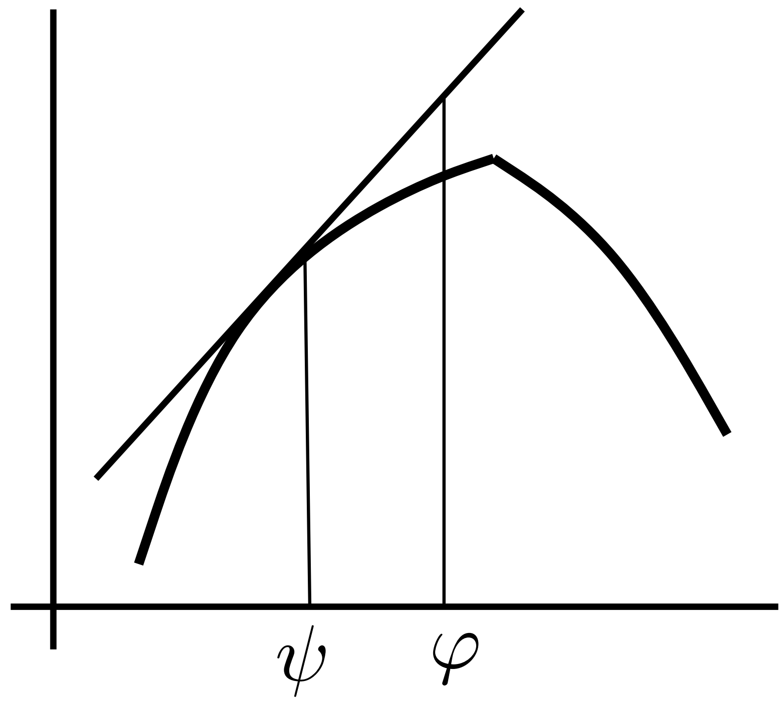

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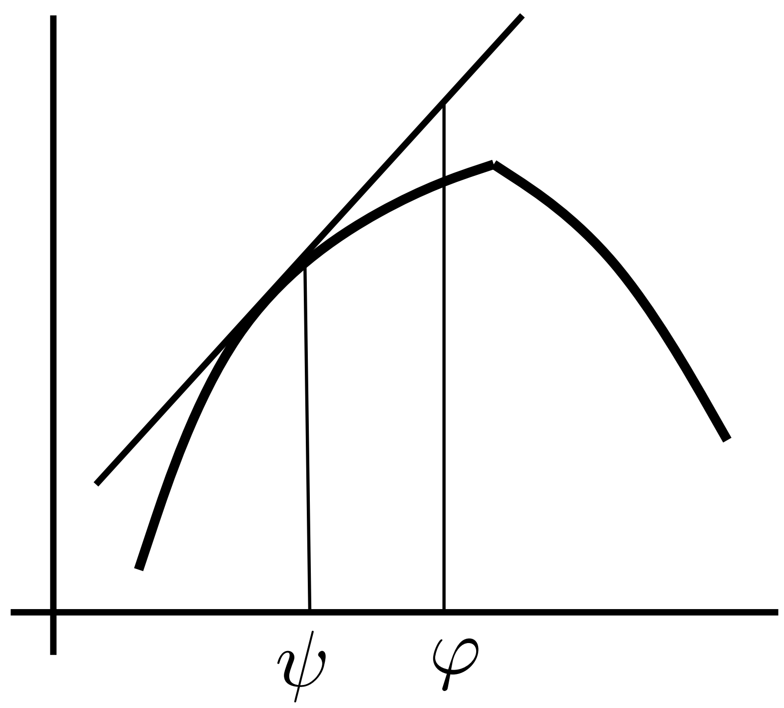

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$$= \langle D\Phi(\psi) | \varphi - \psi \rangle$$

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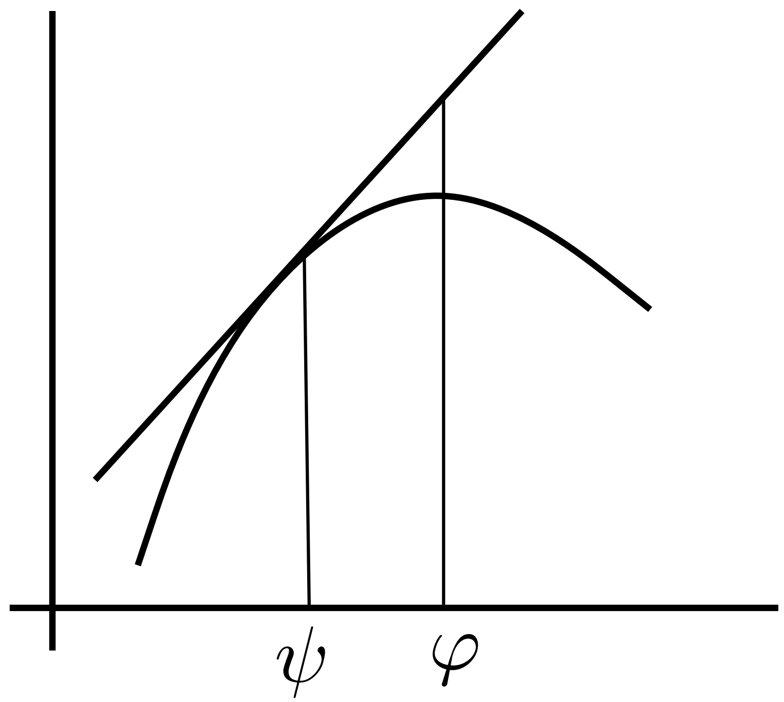

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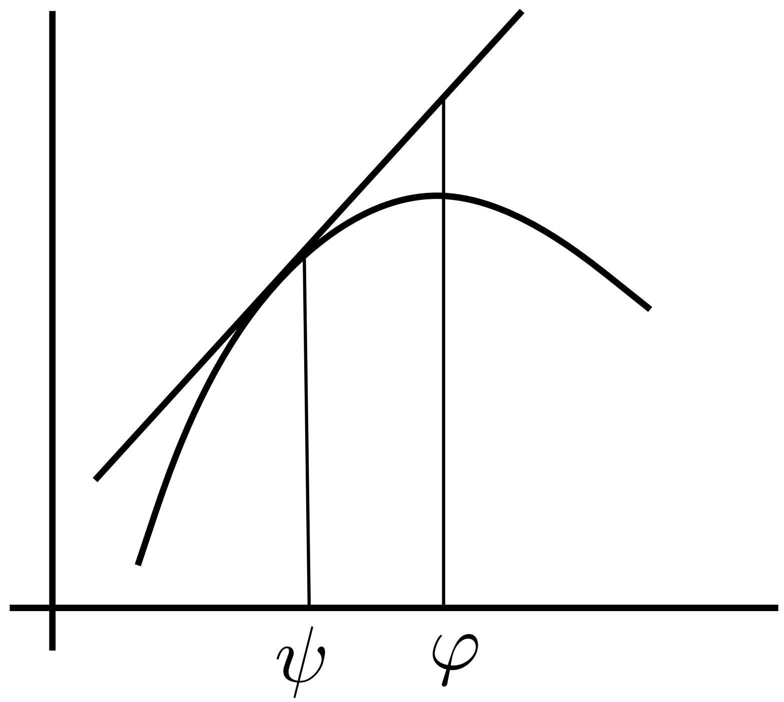

- $D\Phi(\psi) \in \partial^+\Phi(\psi) \Rightarrow \Phi$ concave.
- $D\Phi(\psi)$ depends continuously on $\psi \Rightarrow \Phi$ of class C^1.
- ψ maximum of $\Phi \Leftrightarrow \mu(\mathrm{Vor}_c^{\psi}(y_i)) = \nu_i \ \forall i$

2. Implementation

Implementation of Convex Programming ($-\Phi$)

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Main difficulty: computation of $\mathrm{Vor}_c^\psi(y)$

Computation of the generalized Voronoi cells

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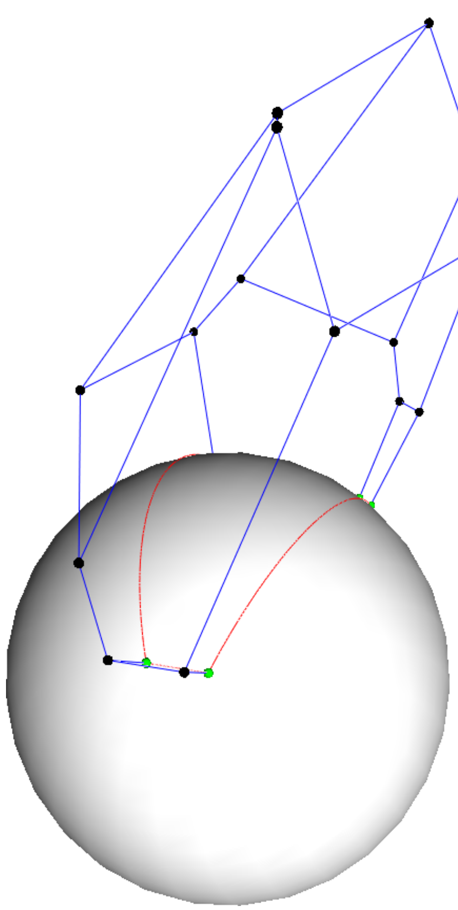
Lemma: With $\vec{\psi} = \log(\vec{\kappa})$, $p_i := -\frac{y_j}{2\kappa_j}$ and $\omega_i := -\|\frac{y_j}{2\kappa_j}\|^2 - \frac{1}{\kappa_j}$,

$$\mathrm{Vor}_c^\psi(y_i) = \mathrm{Pow}_P^\omega(p_i) \cap \mathcal{S}^2$$



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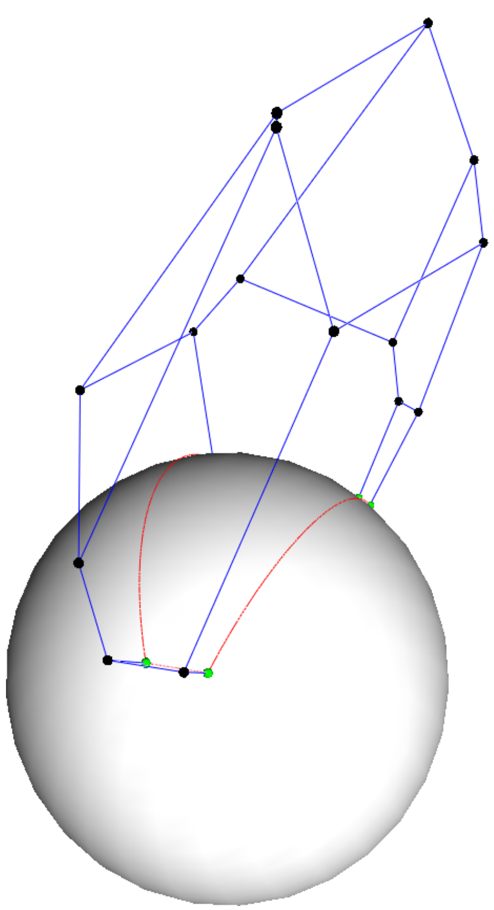

Proof: $x \in \mathrm{Vor}^{\psi}_{c}(y_i) \subseteq \mathcal{S}^2_o$

$$\iff i \in \arg\min_j \frac{\kappa_j}{1 - \langle x | y_j \rangle}$$

$$\iff i \in \arg\min_j \langle x | \frac{y_j}{\kappa_j} \rangle - \frac{1}{\kappa_j}$$

$$\iff i \in \arg\min_j \|x + \underbrace{\boxed{\frac{y_j}{2\kappa_j}}}_{-p_j}\|^2 \underbrace{\boxed{- \|\frac{y_j}{2\kappa_j}\|^2 - \frac{1}{\kappa_j}}}_{\omega_j}$$

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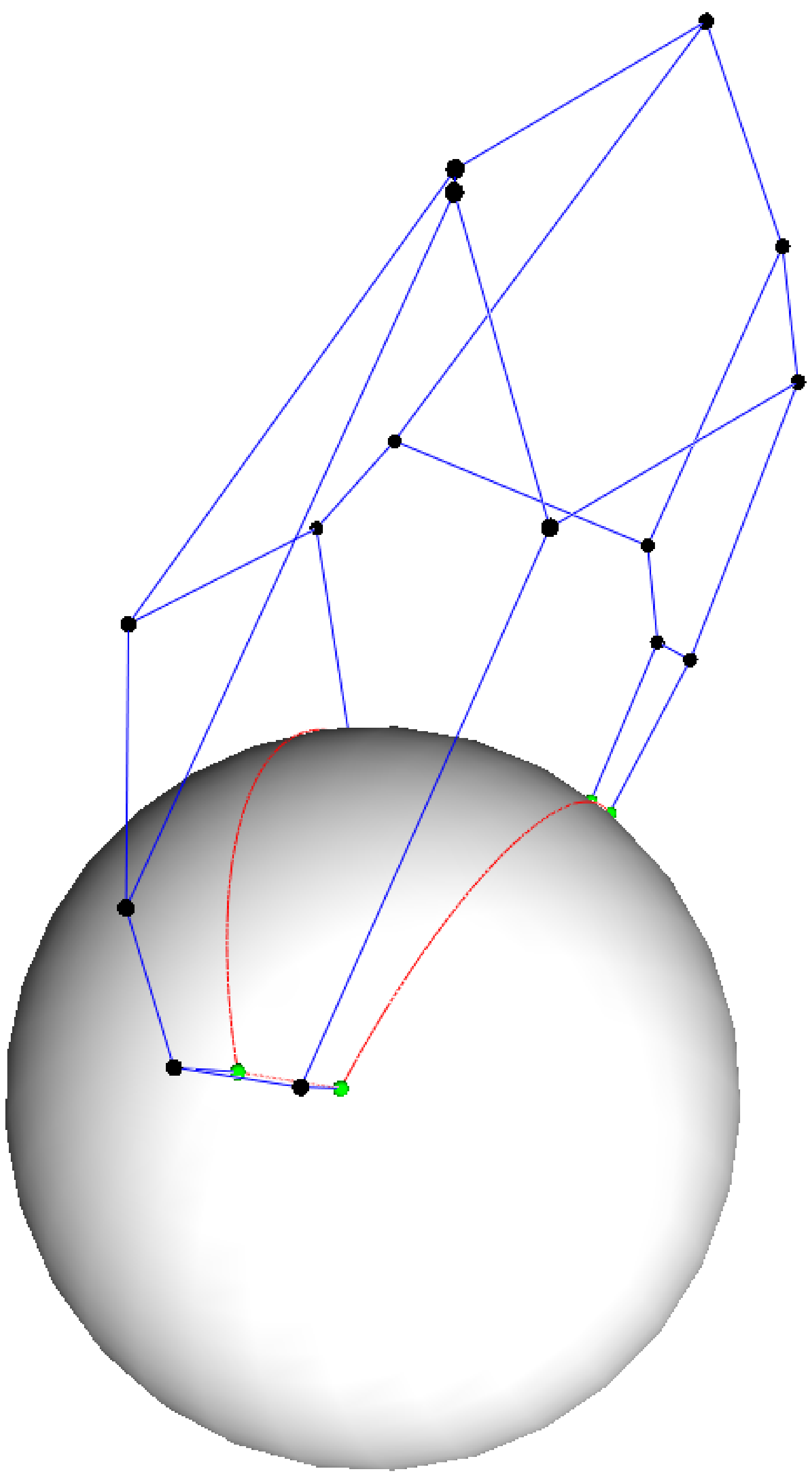

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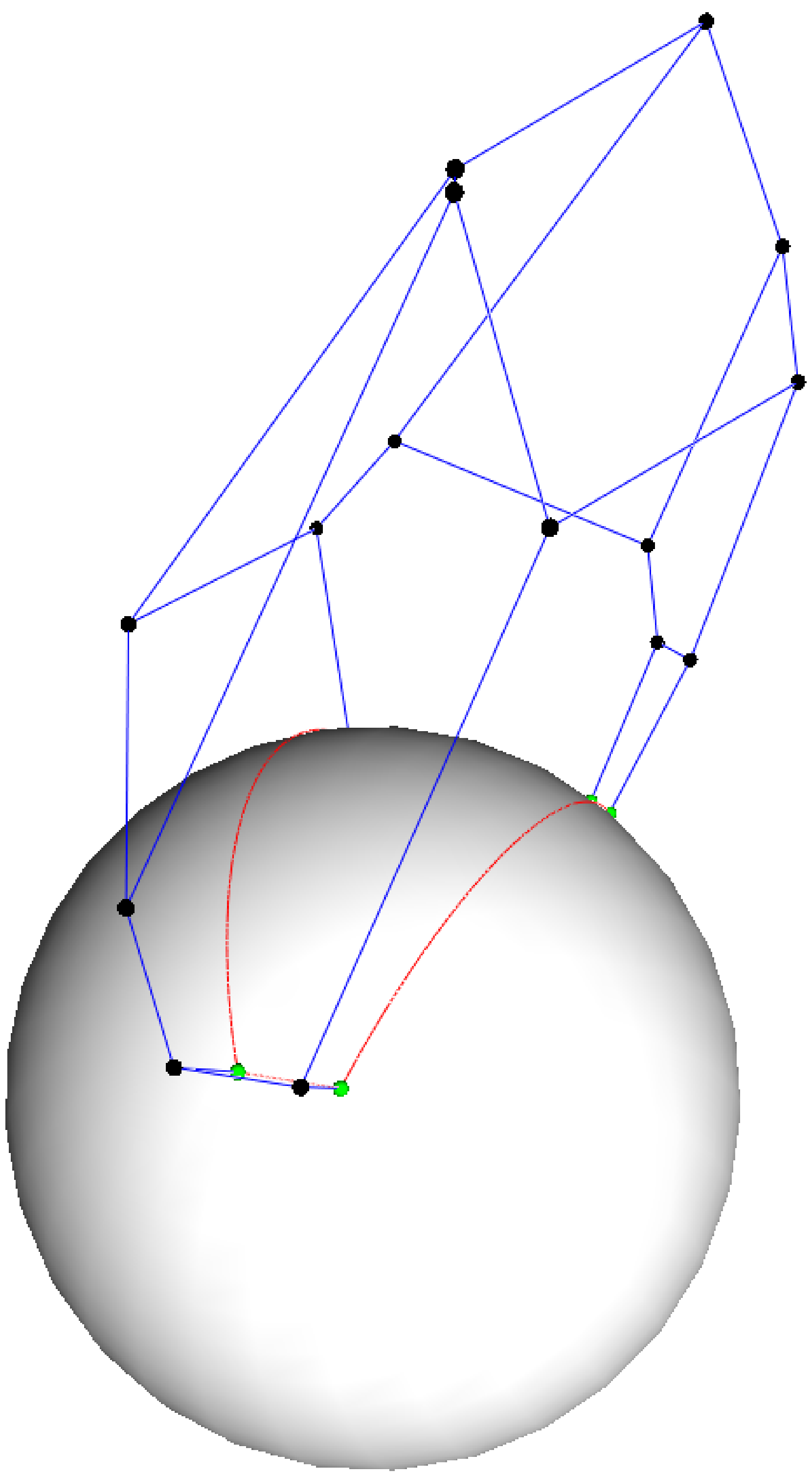

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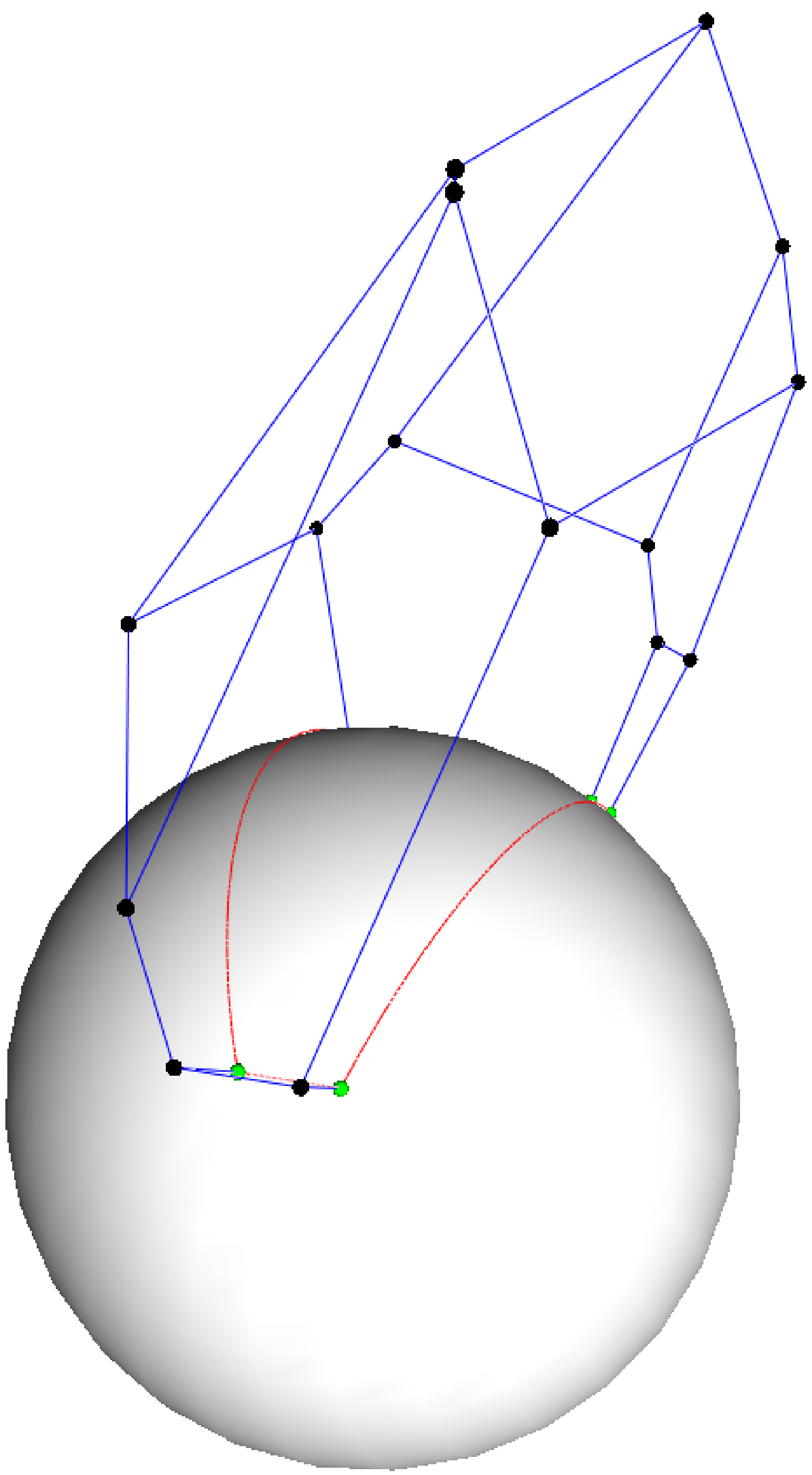

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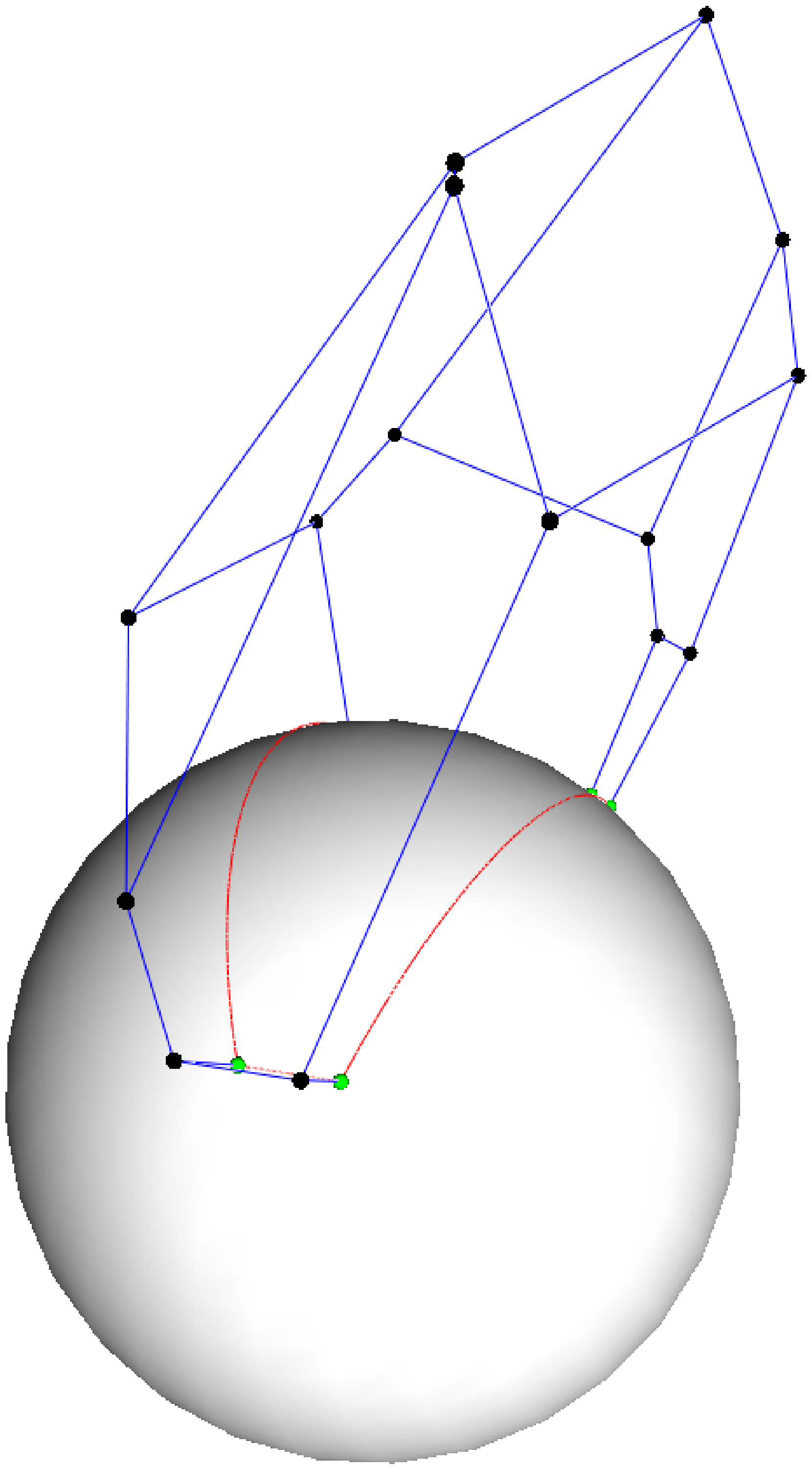

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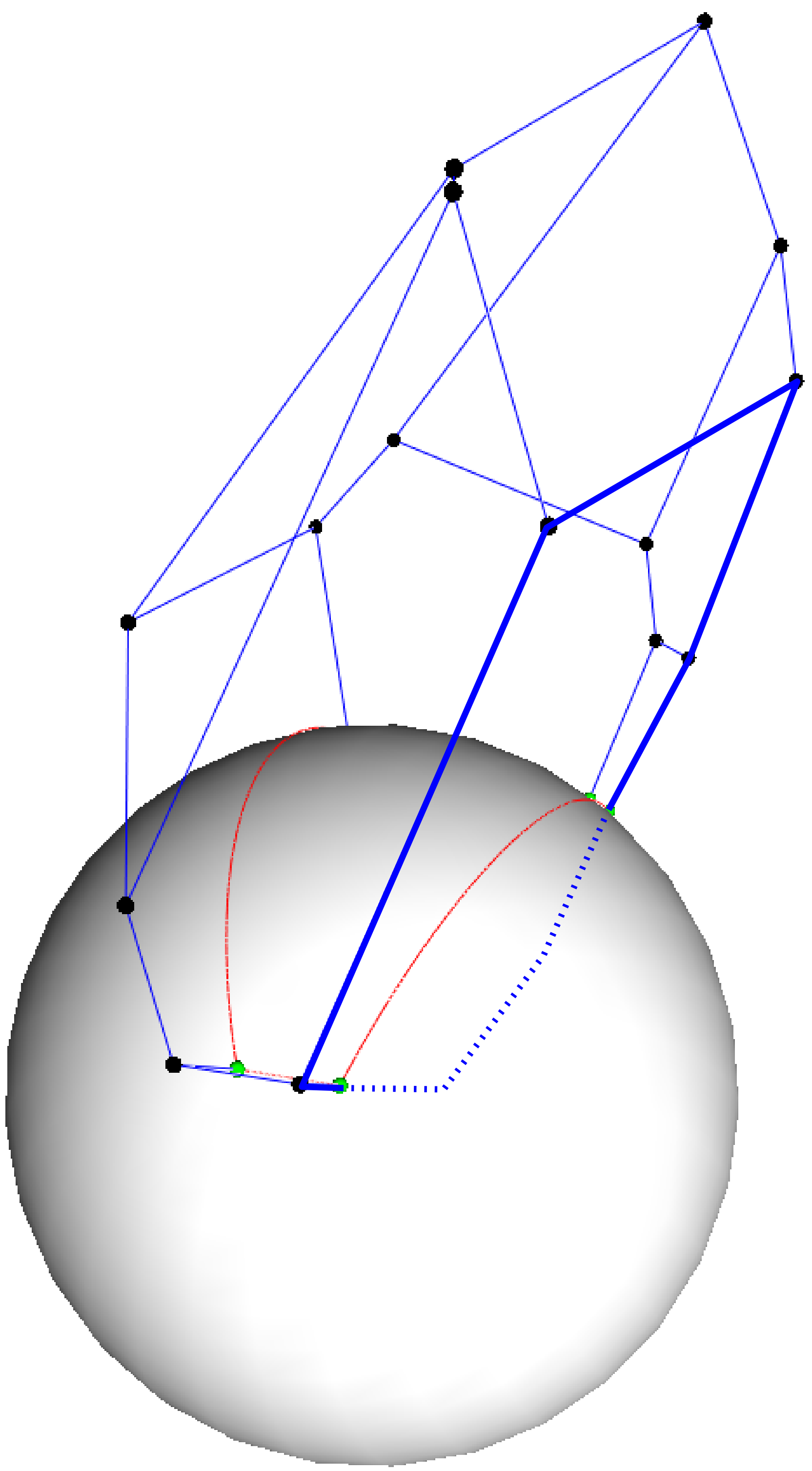

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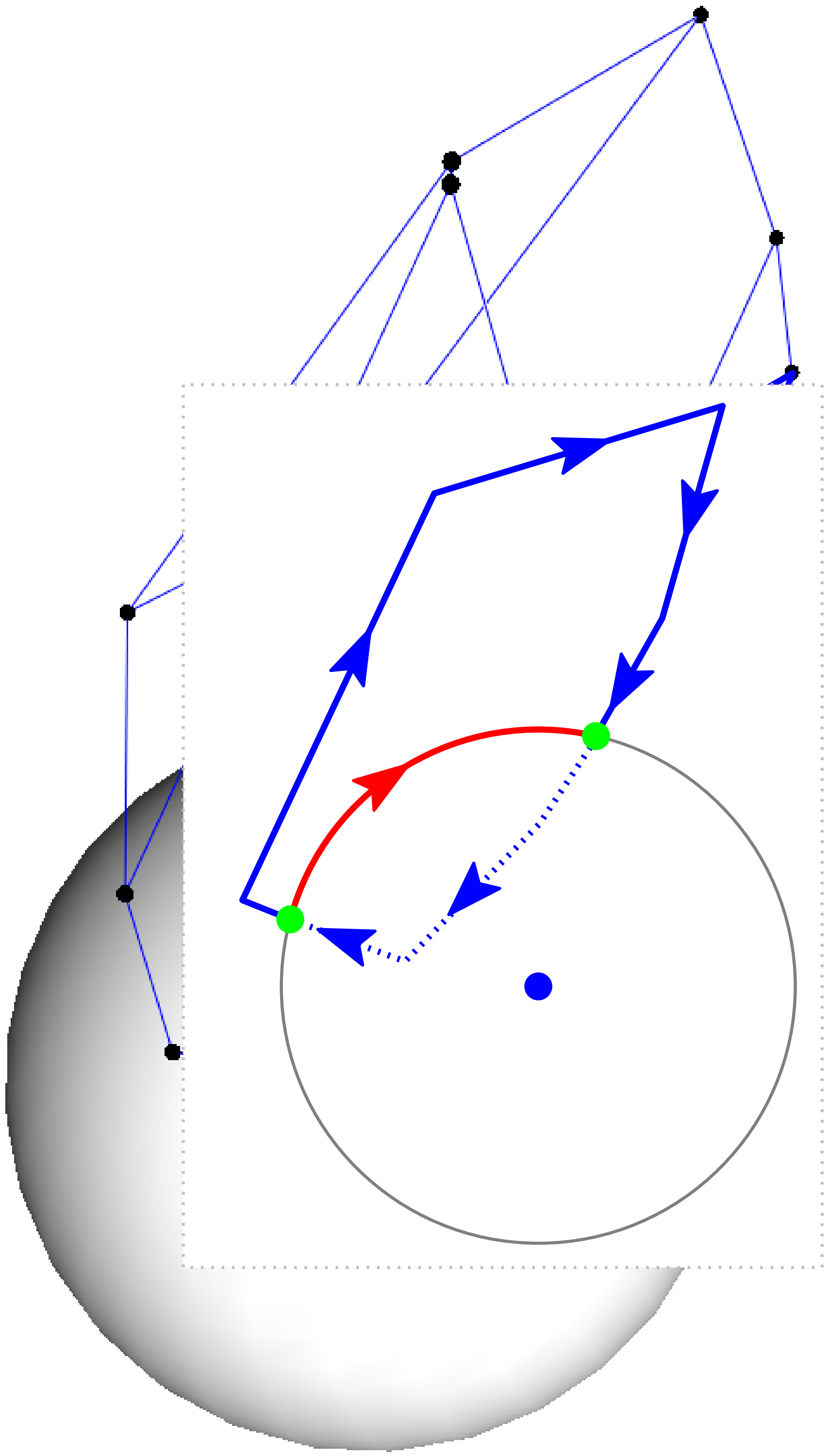

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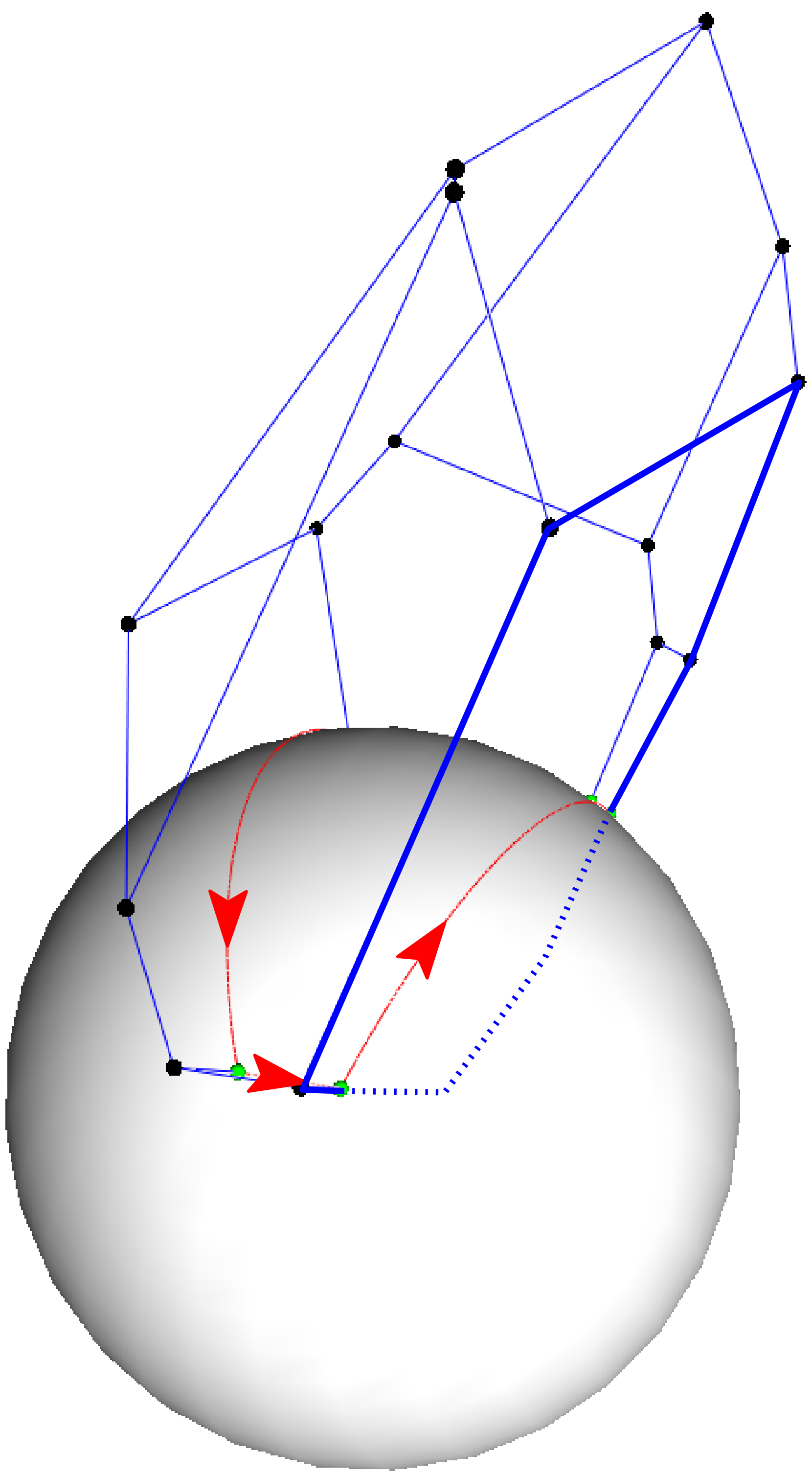

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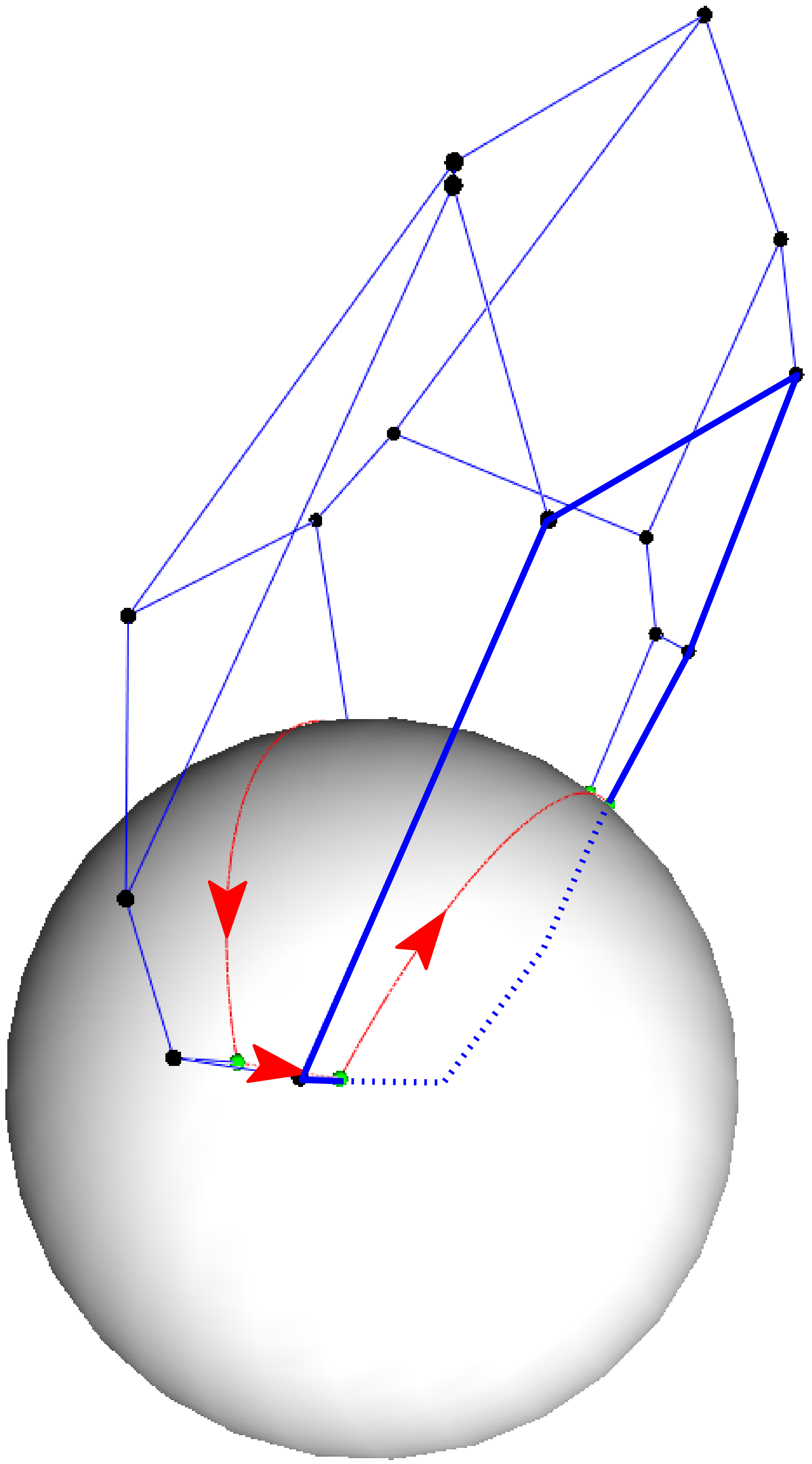

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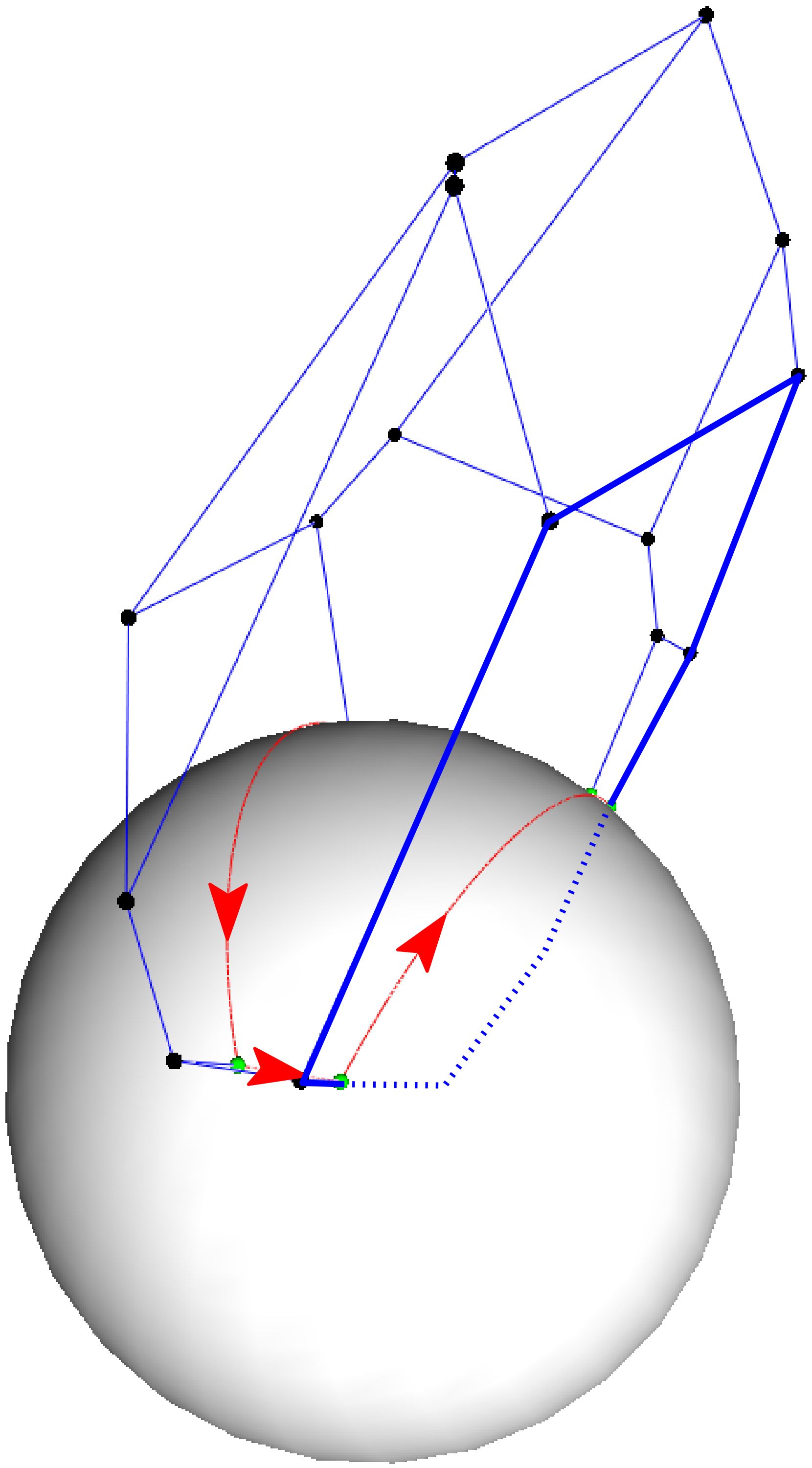

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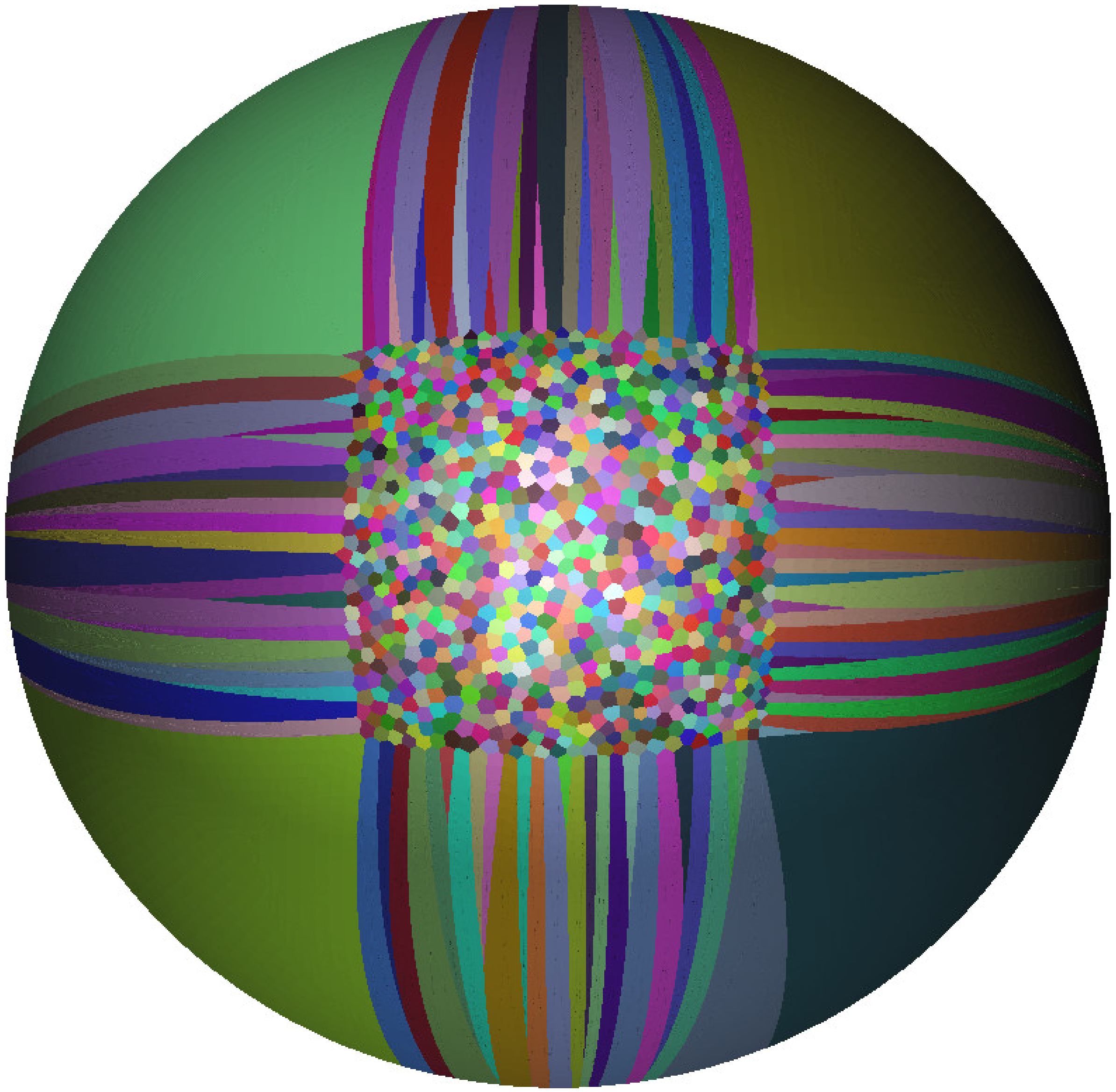
Complexity: $\mathrm{O}(N \log N + C)$ where $C =$ complexity of the Power diagram.

Numerical results (1)

$\nu = \sum_{i=1}^{N} \nu_i \delta_{x_i}$ obtained by discretizing a picture of G. Monge.

$\mu =$ uniform measure on half-sphere $\mathcal{S}^2_+$ $N = 1000$



drawing of $(\mathrm{Vor}_c^{\psi}(y_i))$ (on $\mathcal{S}^2_+$) for $\psi = 0$

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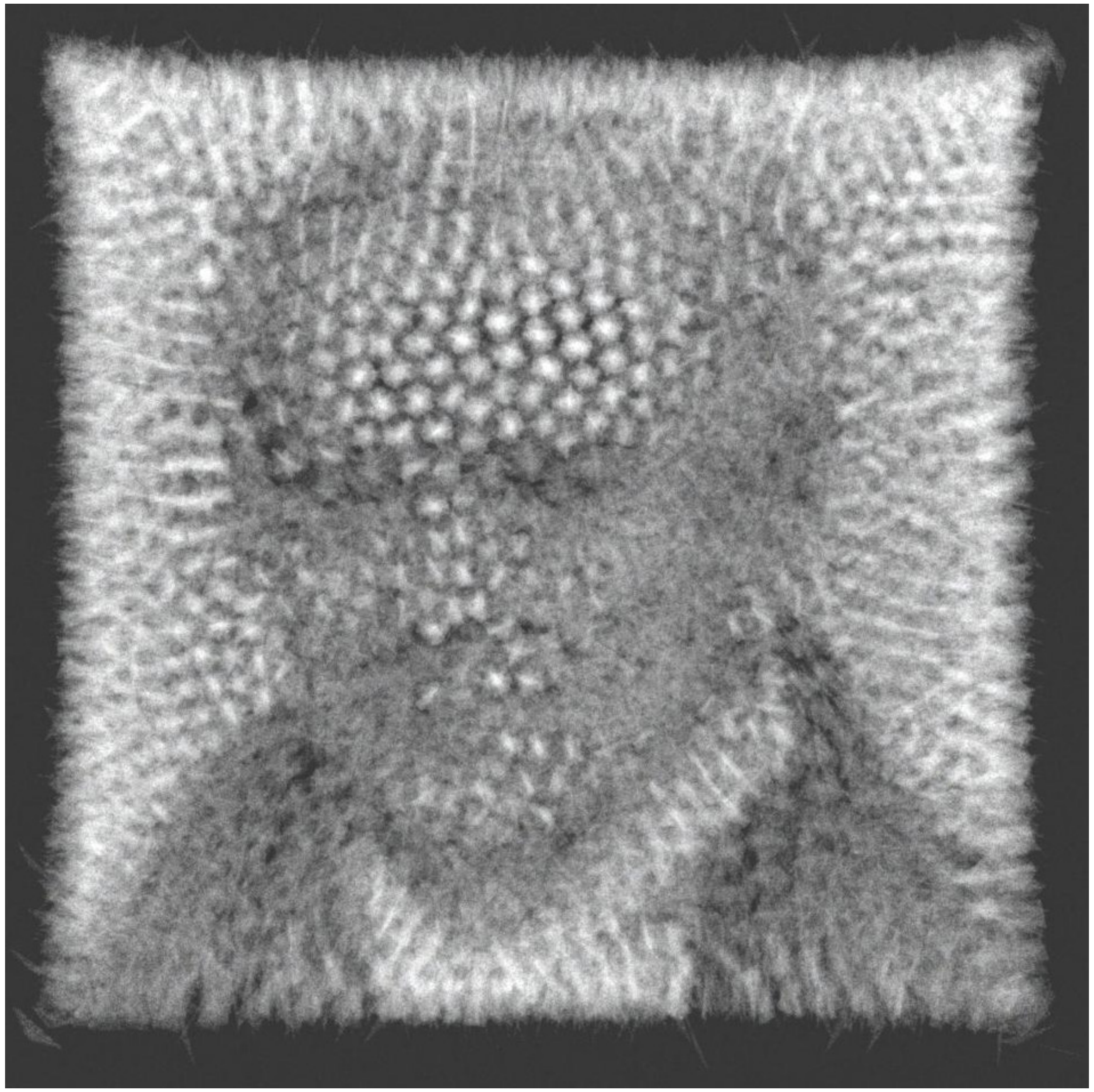

rendering of the image reflected at infinity (using LuxRender)

Numerical results (2)

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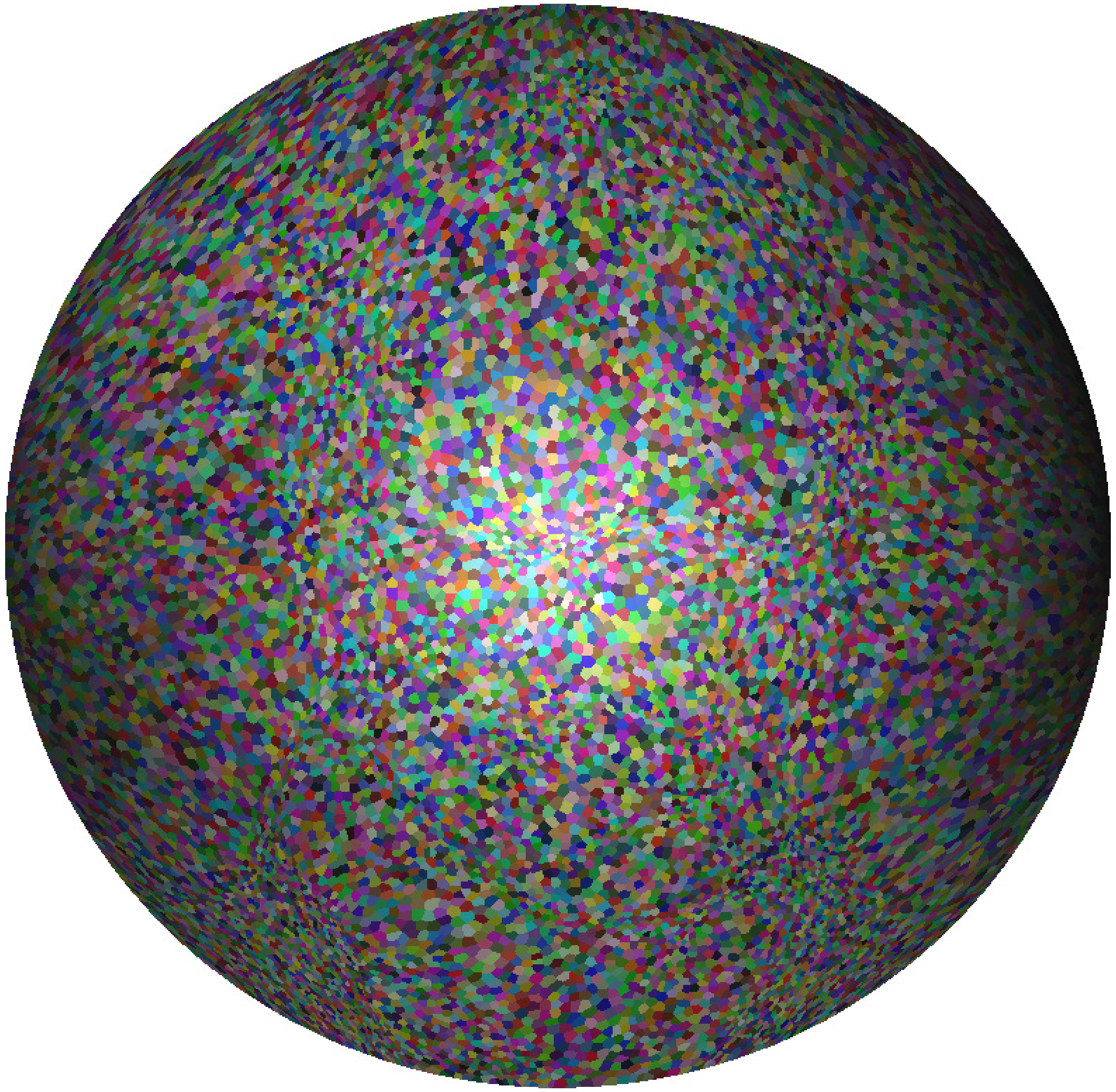

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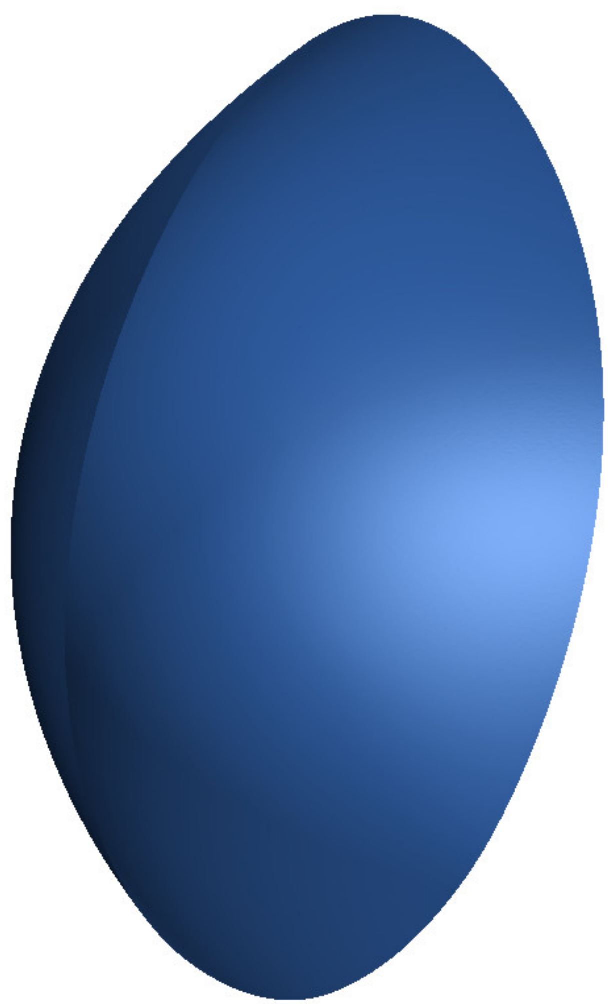

solution to the far-field reflector problem: $R(\kappa_{\text{sol}})$

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3. Complexity of paraboloid intersection

Complexity of the paraboloid intersection (PI)

Theorem: For N paraboloids, the complexity of the diagram $(\mathrm{PI}_i(\vec{\kappa}))_{1 \leq i \leq N}$ is $O(N)$.

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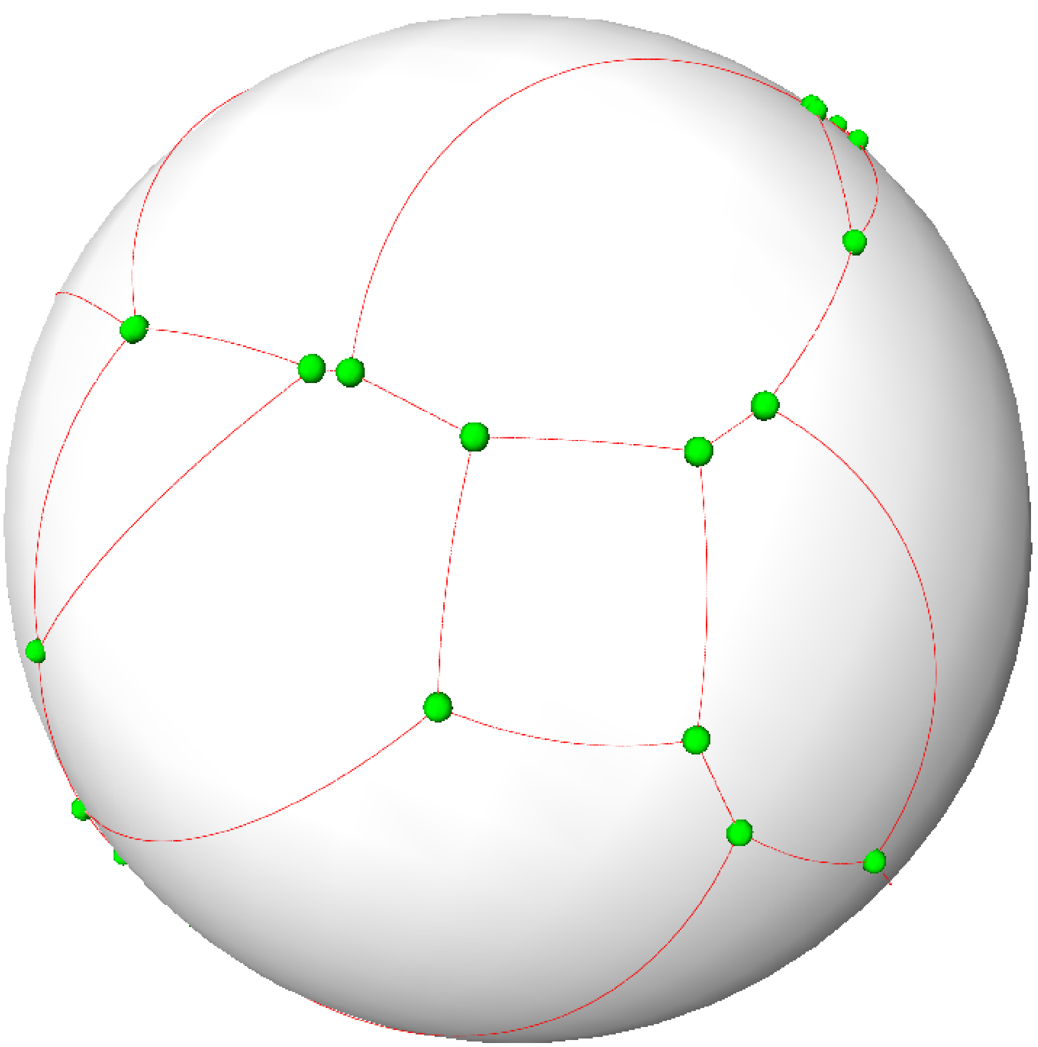
Complexity: $E + F + V$, where



$E = \#$ edges

$V = \#$ vertices

$F =$ total $\#$ of connected components

Complexity of the paraboloid intersection (PI)

Theorem: For N paraboloids, the complexity of the diagram $(\mathrm{PI}_i(\vec{\kappa}))_{1\leq i\leq N}$ is $O(N)$.

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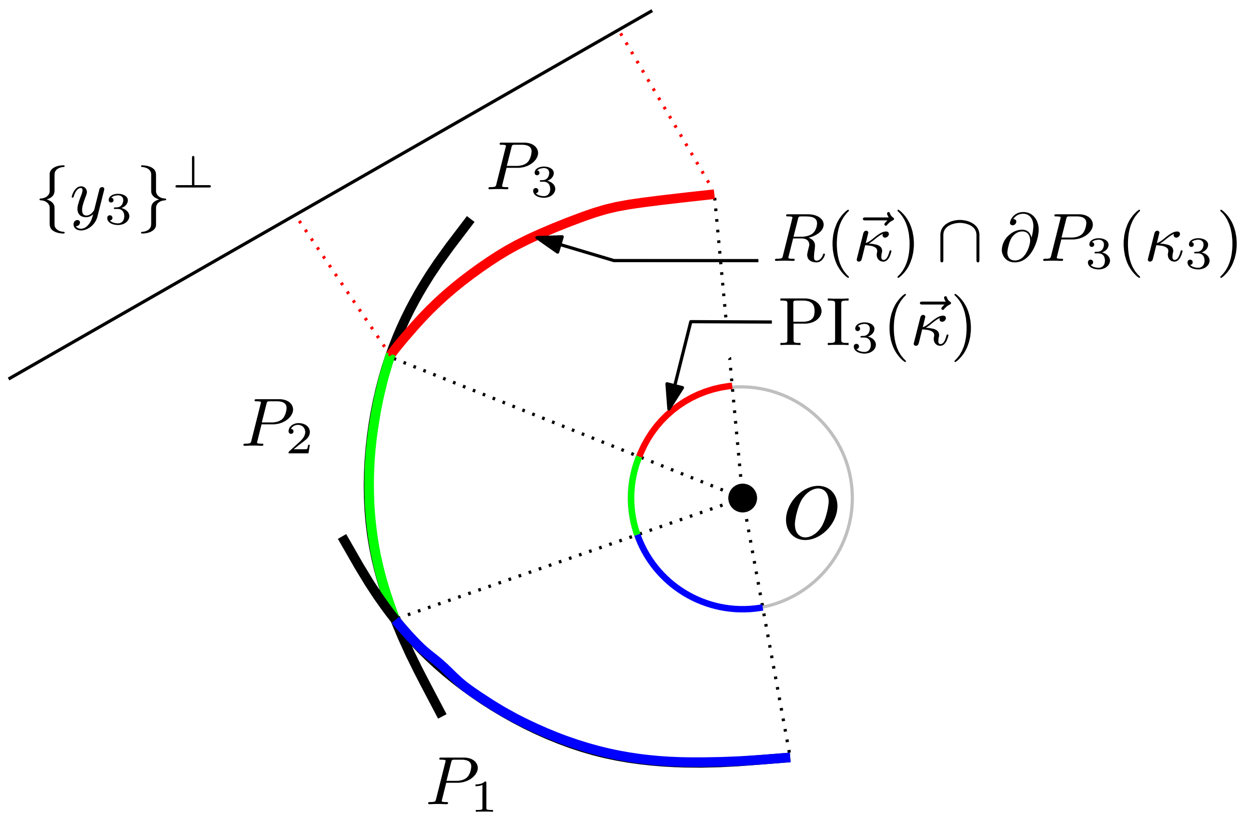
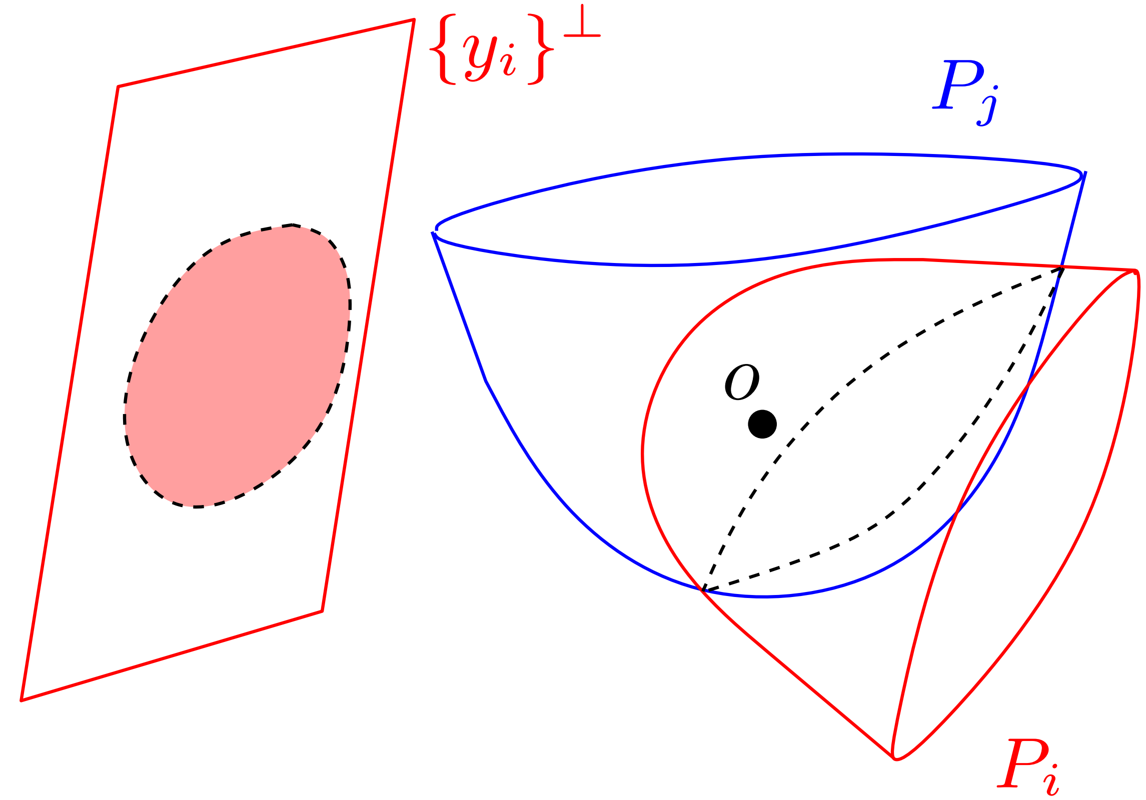


Lemma: The projection of $\partial P_i \cap P_j$ onto the plane $\{y_i^{\perp}\}$ is a disc.



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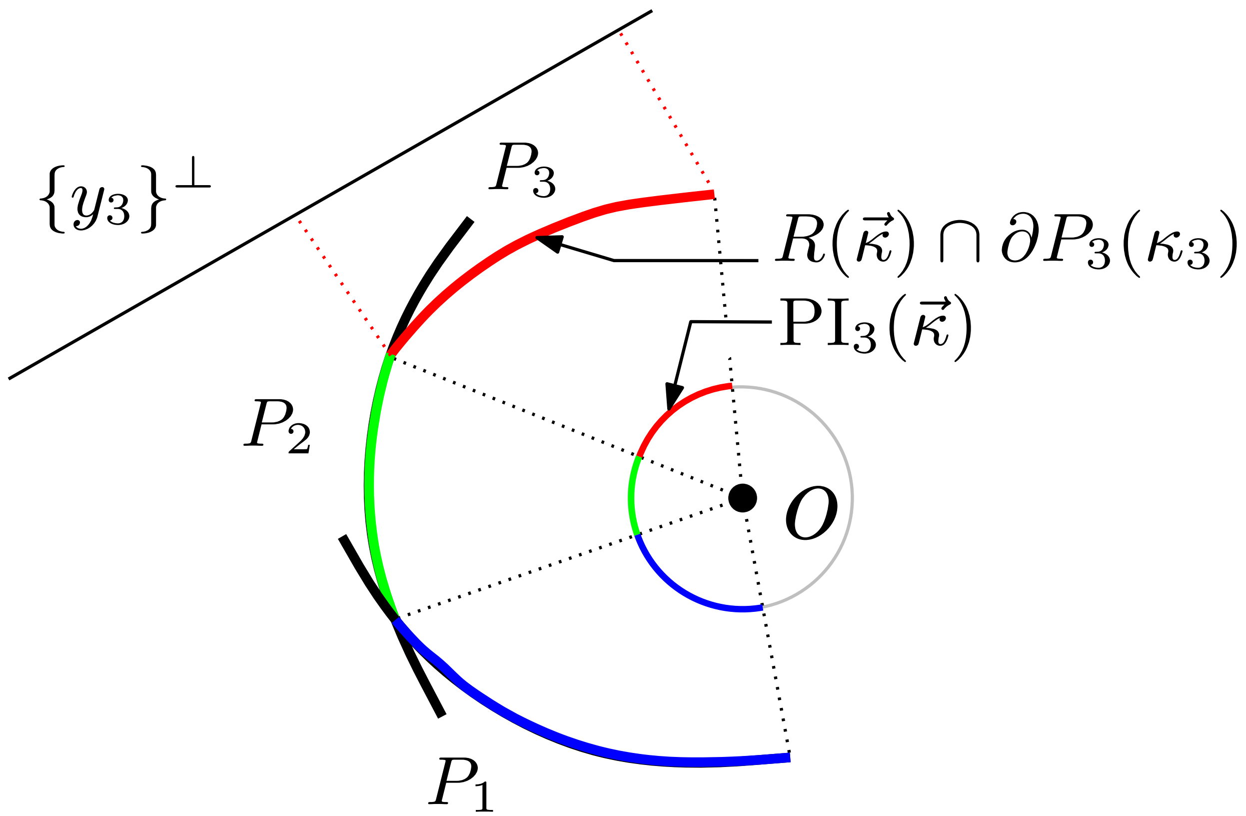
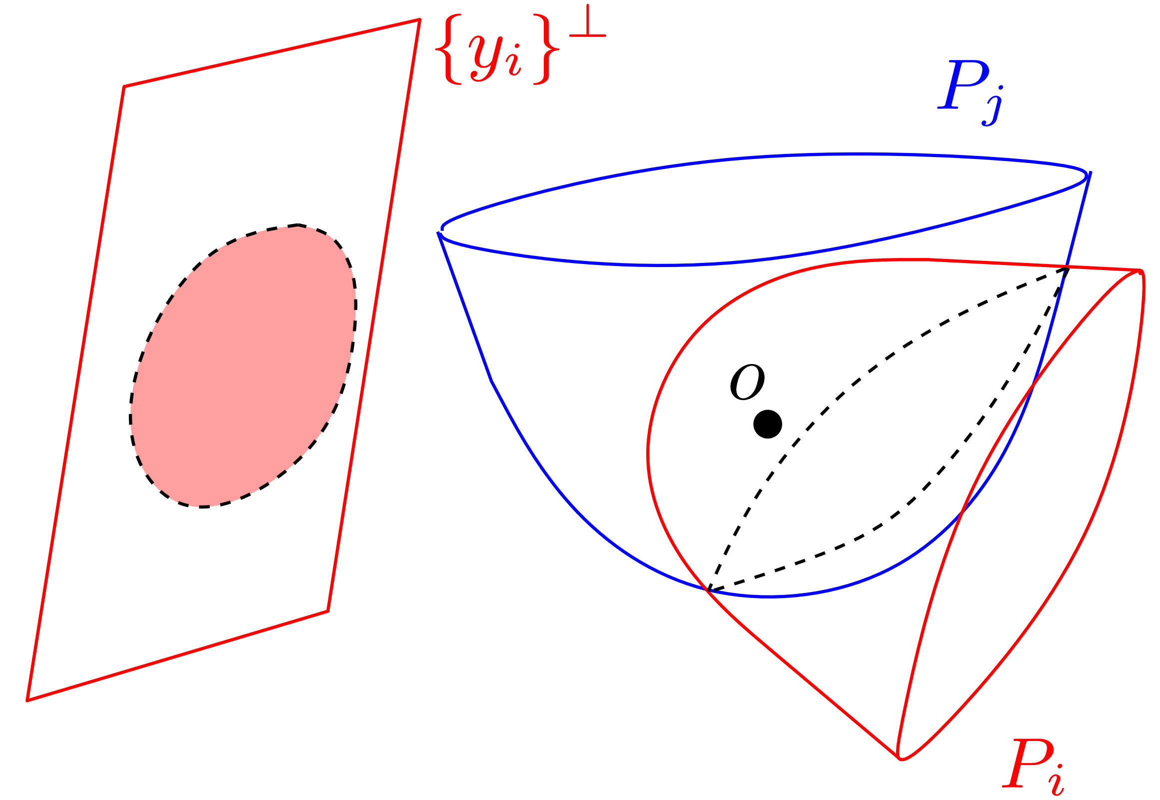


Lemma: The projection of $\partial P_i \cap P_j$ onto the plane $\{y_i^\perp\}$ is a disc.



$\Longrightarrow$ the projection of $R(\vec{\kappa}) \cap \partial P_i$ on $\{y_i\}^\perp$ is convex

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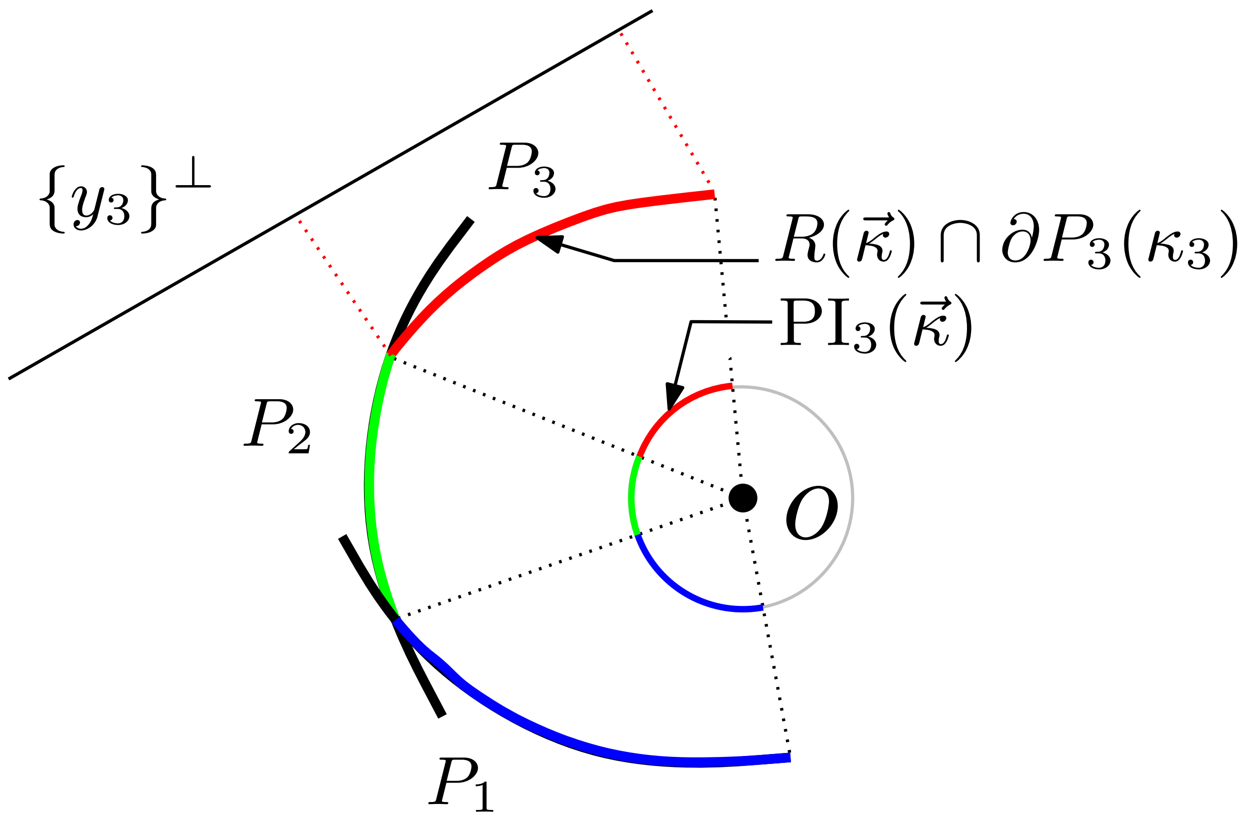

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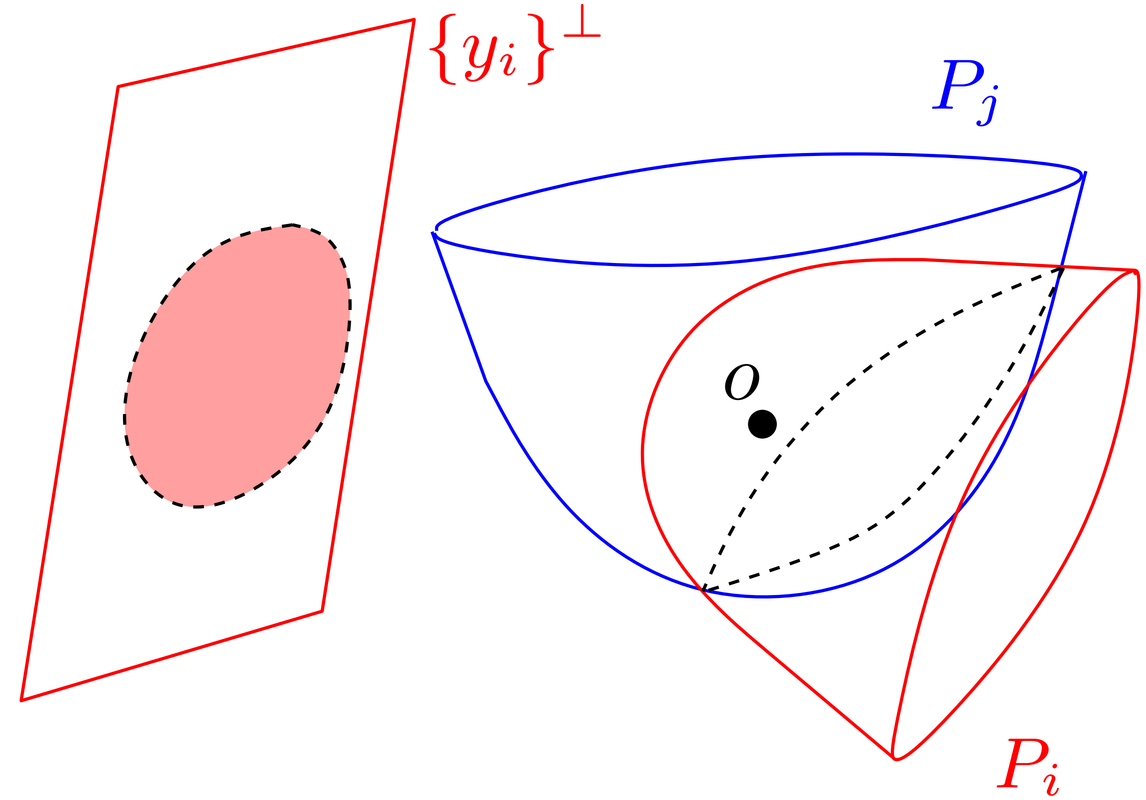

$\Longrightarrow$ the projection of $R(\vec{\kappa}) \cap \partial P_i$ on $\{y_i\}^\perp$ is convex

$\Longrightarrow$ $\mathrm{PI}_i(\vec{\kappa})$ is connected.

Complexity of the paraboloid intersection (PI)

Theorem: For N paraboloids, the complexity of the diagram $(\mathrm{PI}_i(\vec{\kappa}))_{1\leq i\leq N}$ is $O(N)$.

Proof:

- $F \leq N$
- Every vertex has 3 edges $\Rightarrow 3V \leq 2E$.

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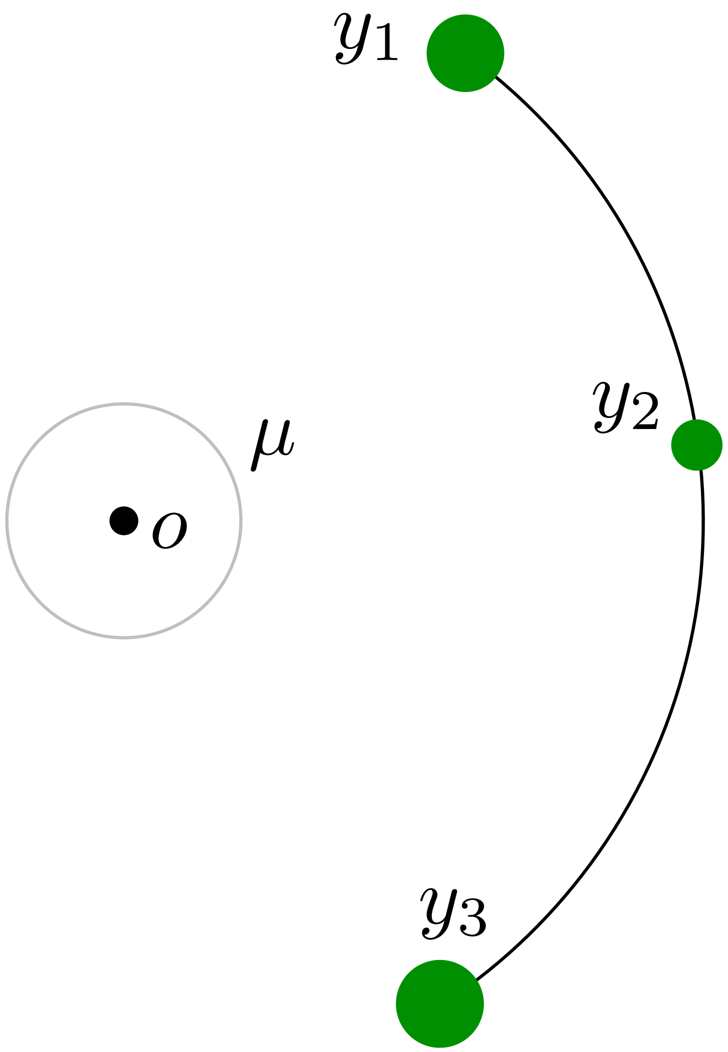
- $F \leq N$
- Every vertex has 3 edges $\Rightarrow 3V \leq 2E$.
- Euler's formula $V - E + F = 2$ implies

$$V \leq 2F - 4 \text{ and } E \leq 3F - 6.$$

4. Other types of reflectors

Other type : paraboloid union (PU)



Punctual light at origin o, μ measure on $\mathcal{S}^2_o$

Prescribed far-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $\mathcal{S}^2_\infty$

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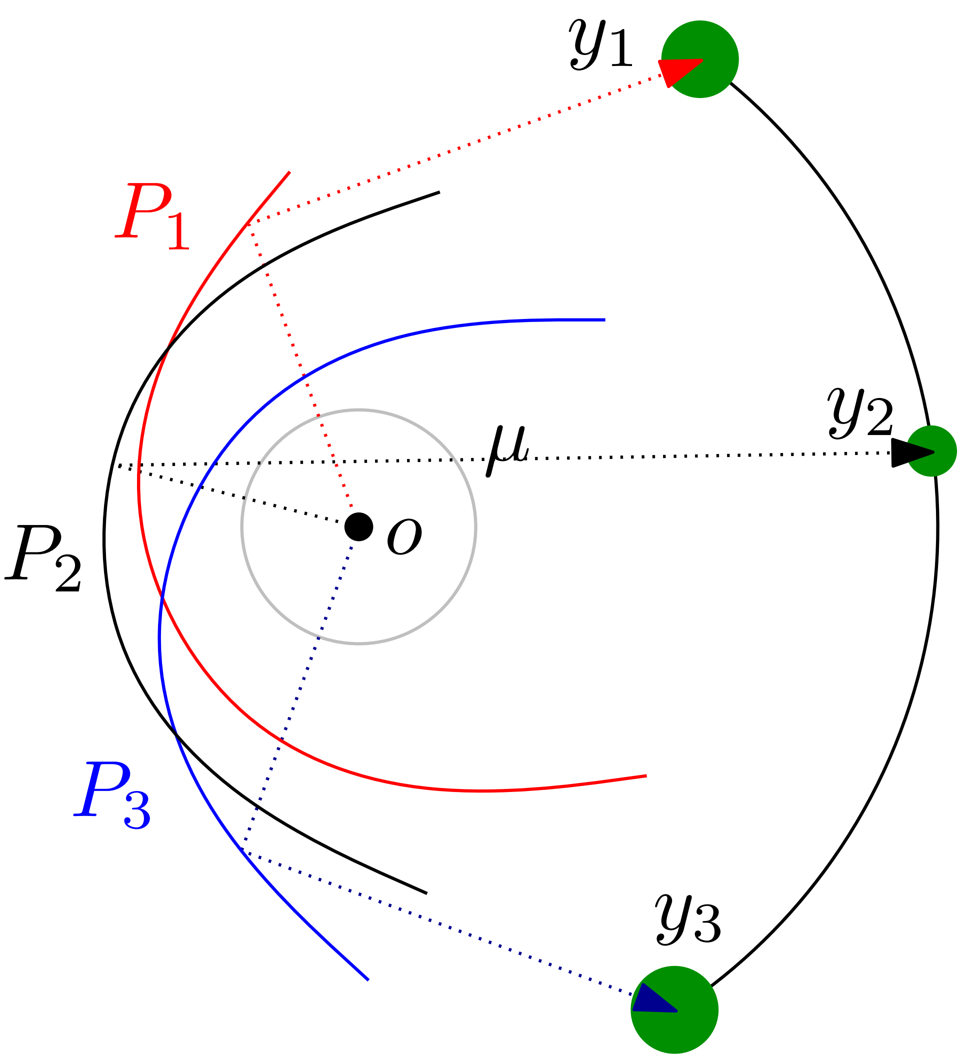

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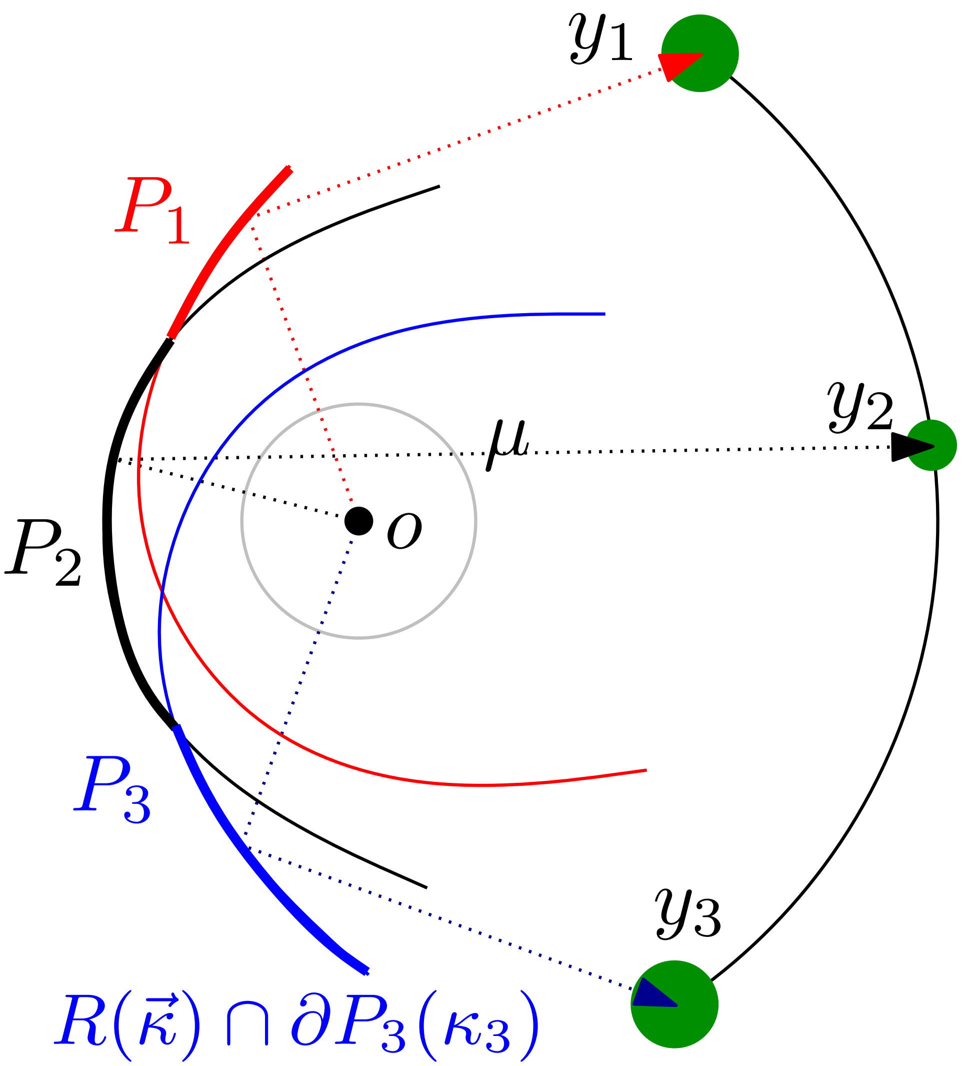

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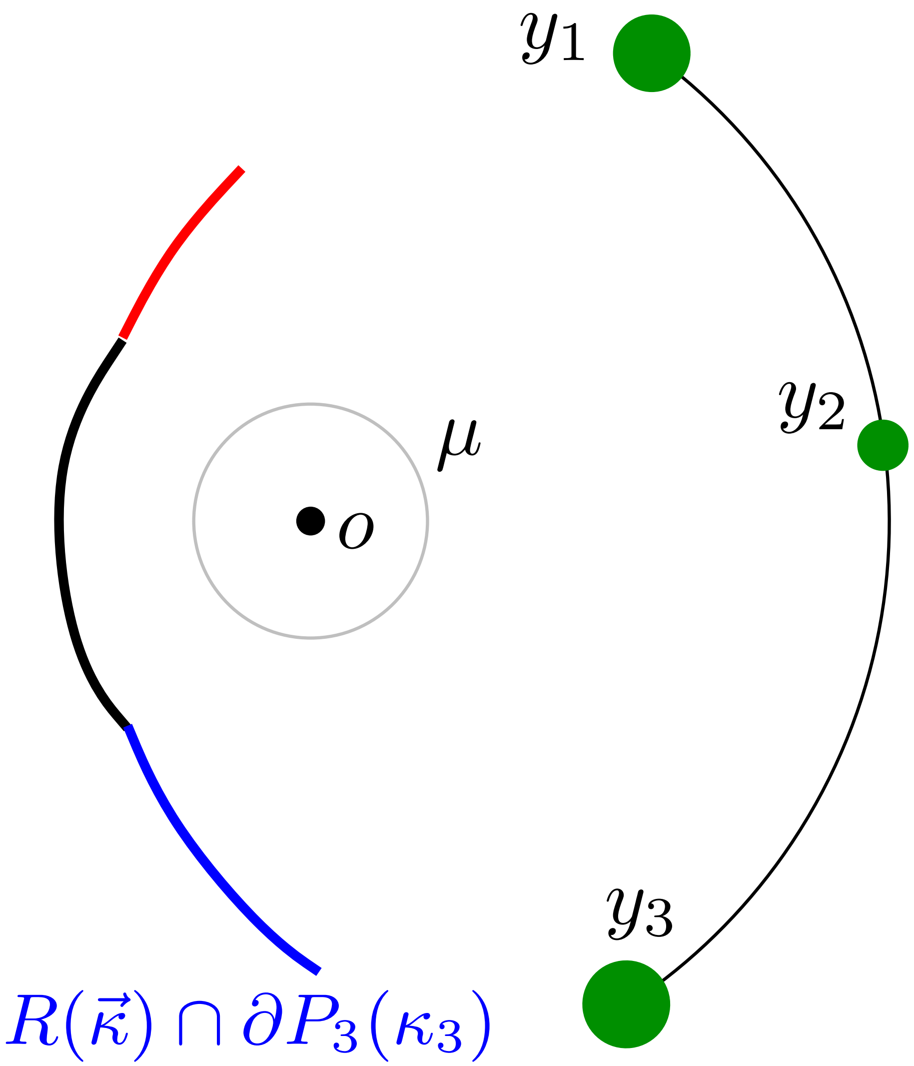

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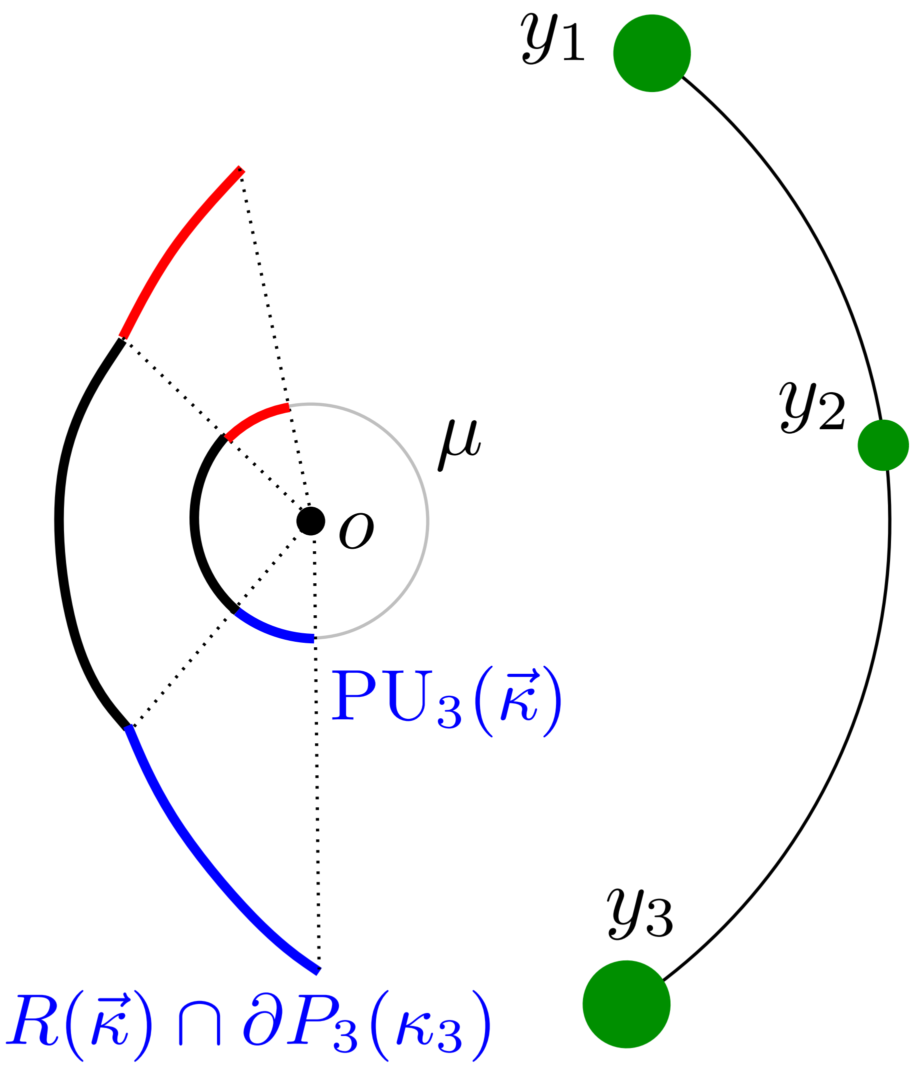

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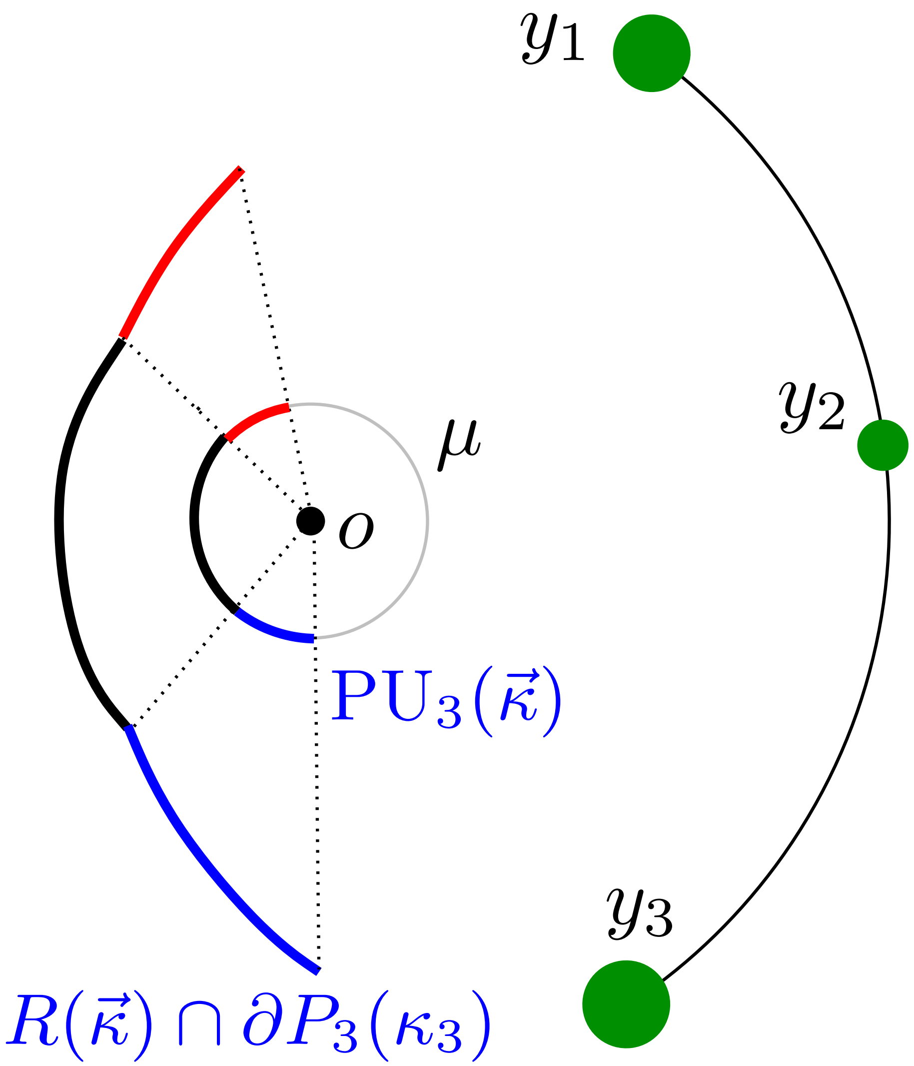

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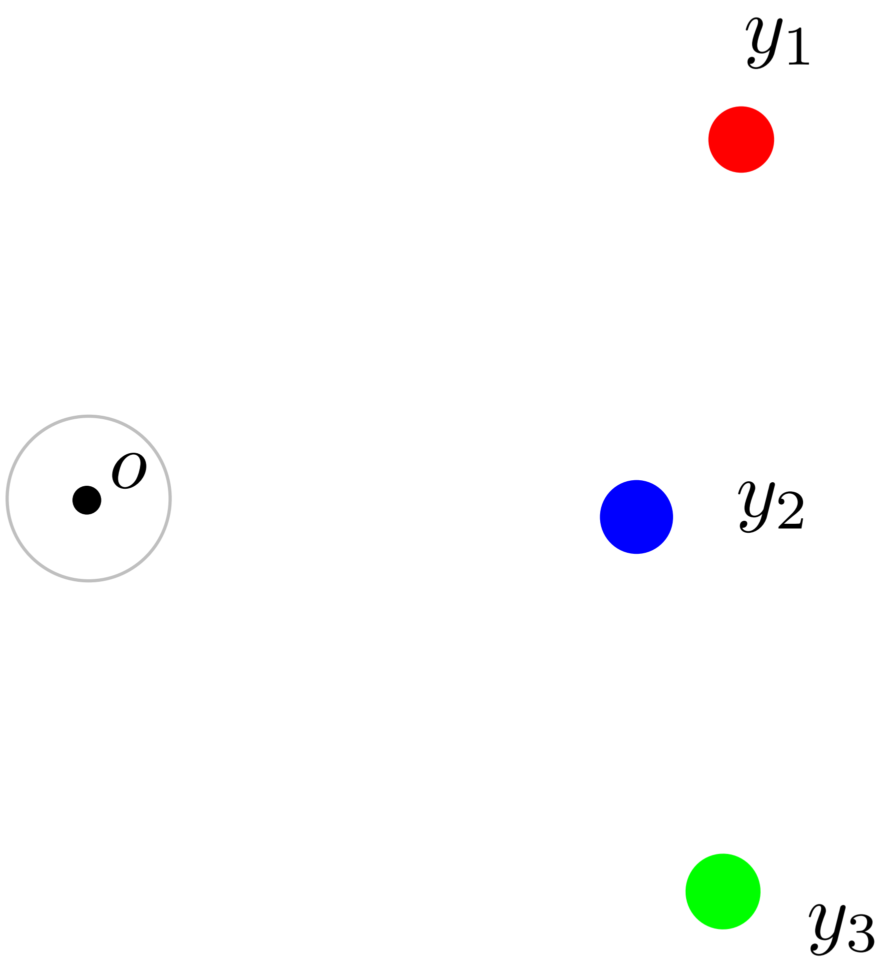
Far-field reflector antenna problem:

Problem (FF'): Find $\kappa_1, \ldots, \kappa_N$ such that for every i, $\mu(\mathrm{PU}_i(\vec{\kappa})) = \nu_i$.

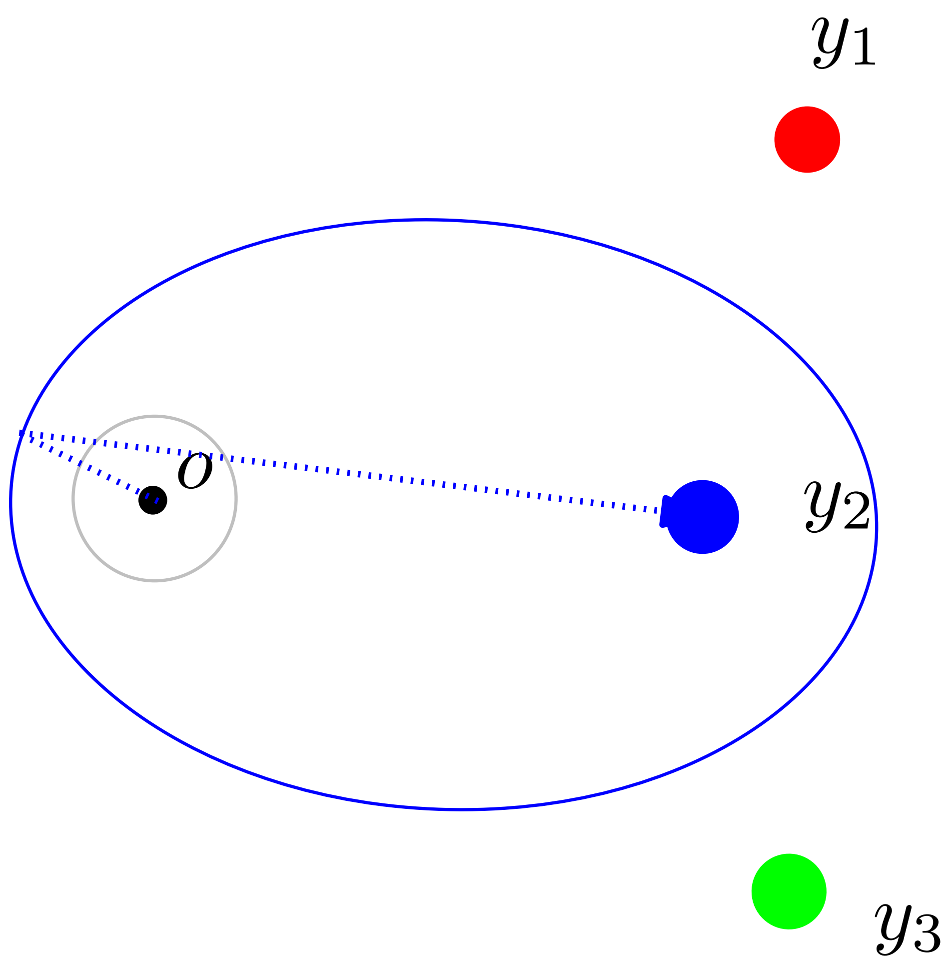
Near-Field Reflector Antenna Problem



Punctual light at origin o, μ measure on $\mathcal{S}_o^2$

Prescribed near-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $\mathbb{R}^3$

Near-Field Reflector Antenna Problem



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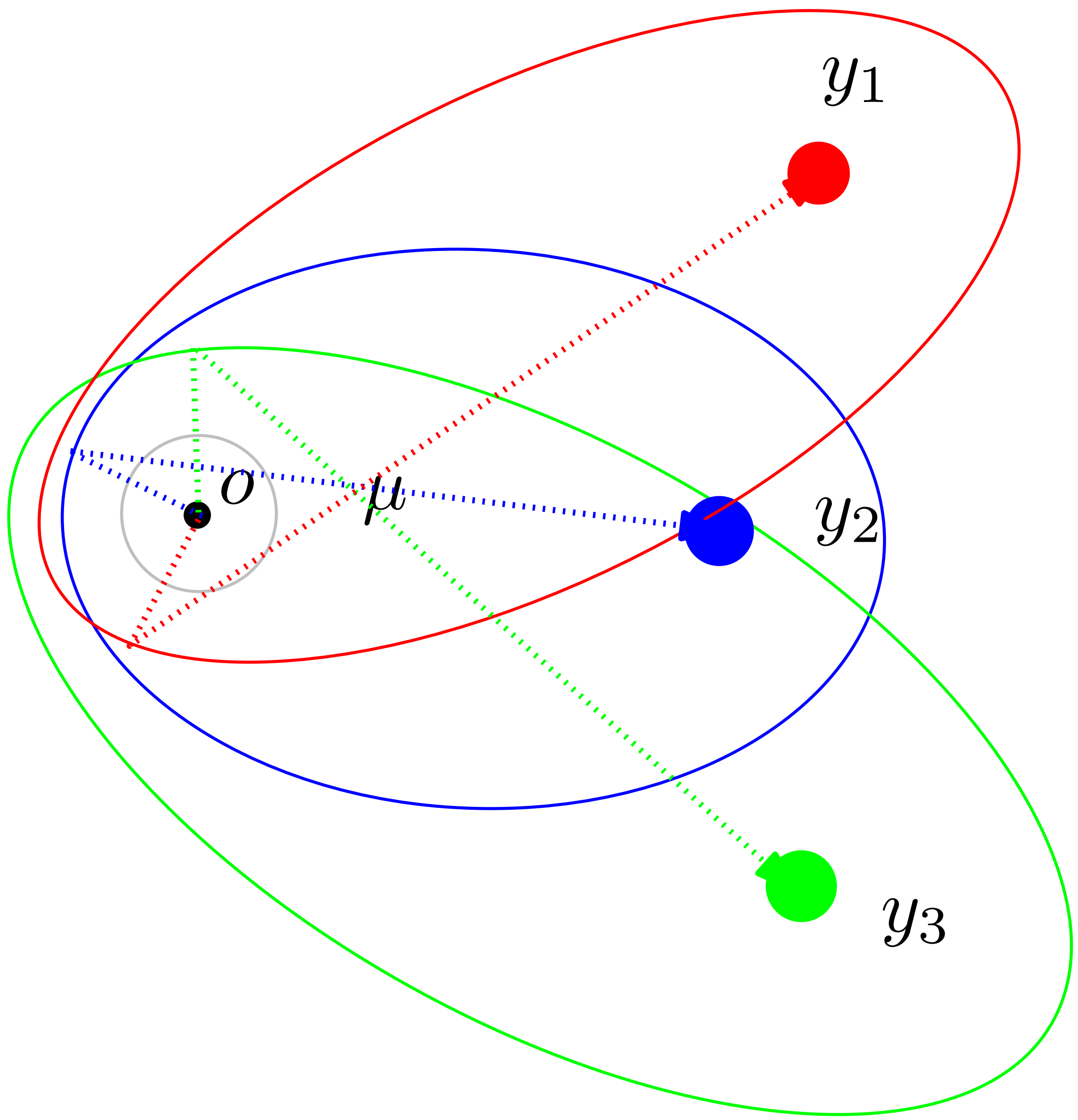

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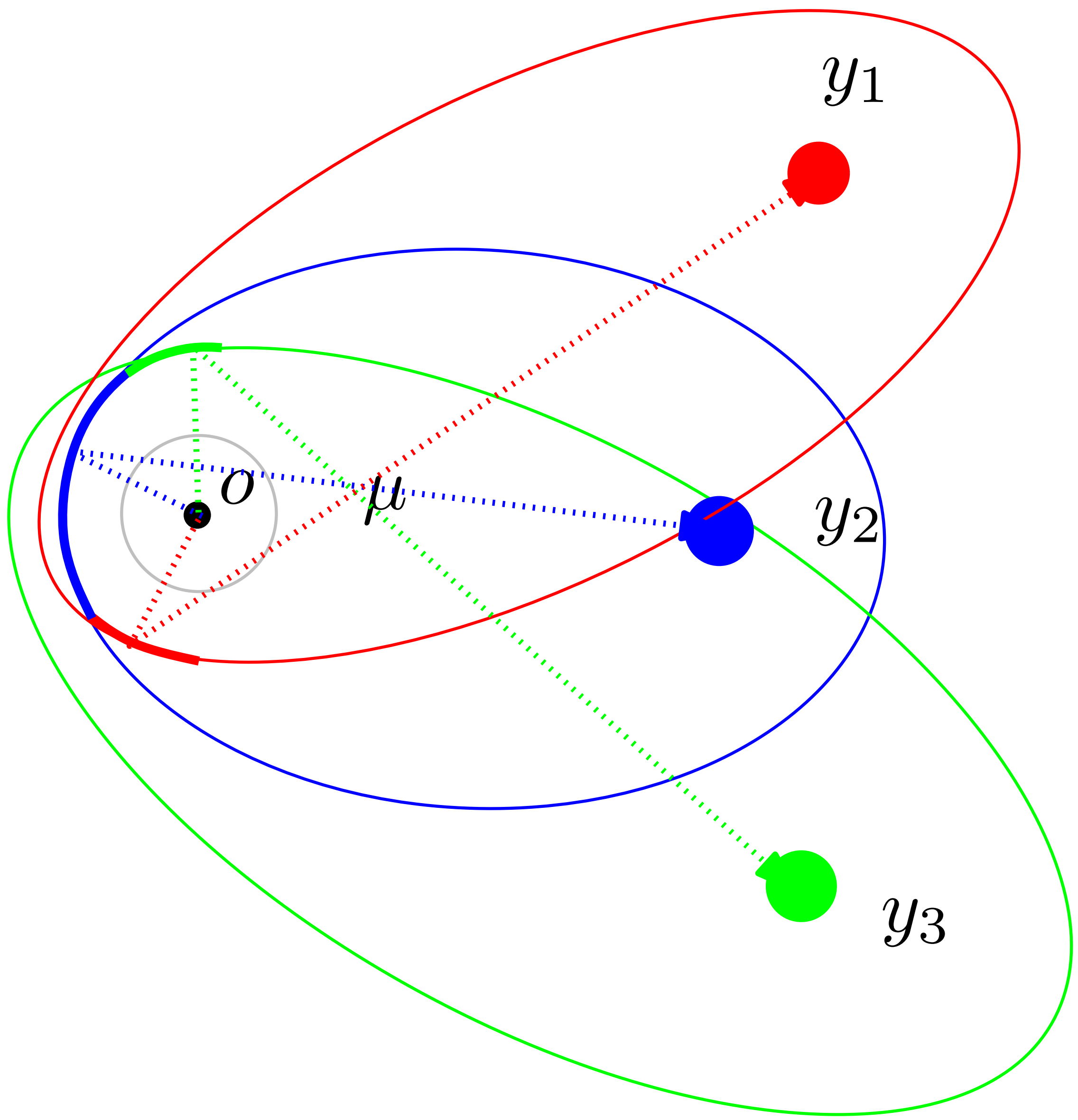
Near-Field Reflector Antenna Problem



Punctual light at origin o, μ measure on $\mathcal{S}_o^2$

Prescribed near-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $\mathbb{R}^3$

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$$R(\vec{e}) = \partial \left(\cap_{i=1}^{N} E_i(e_i) \right)$$

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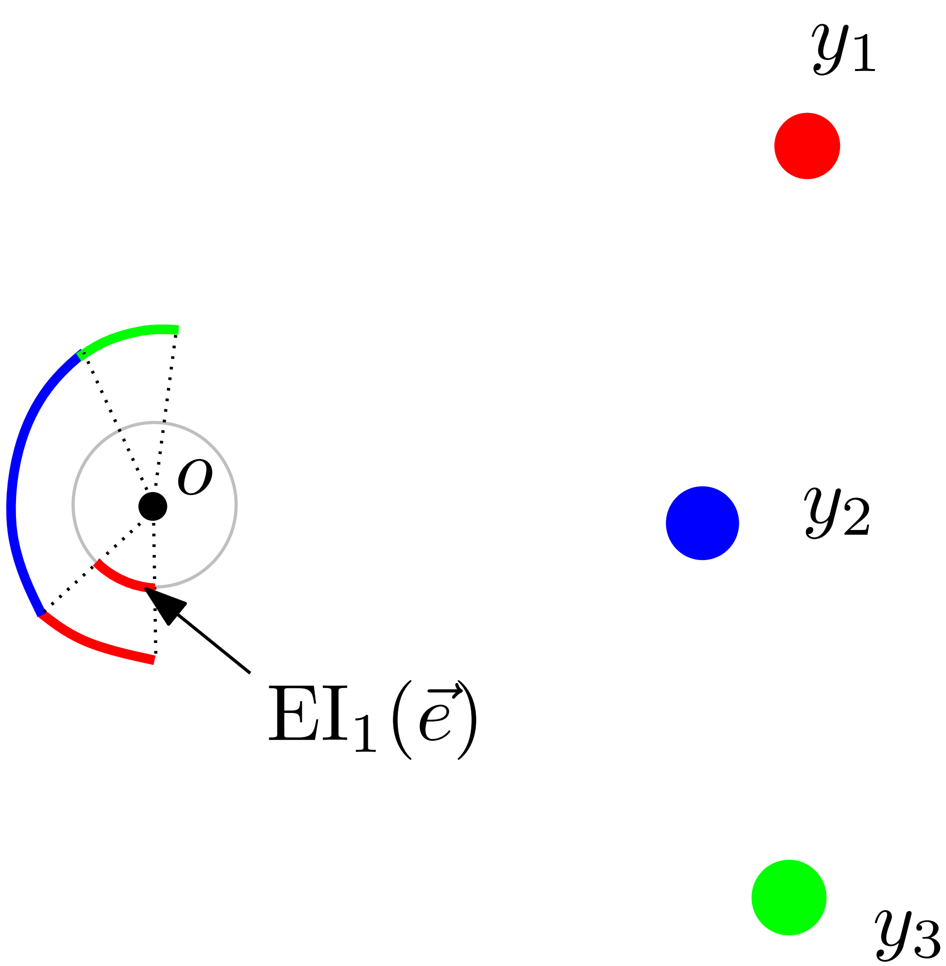

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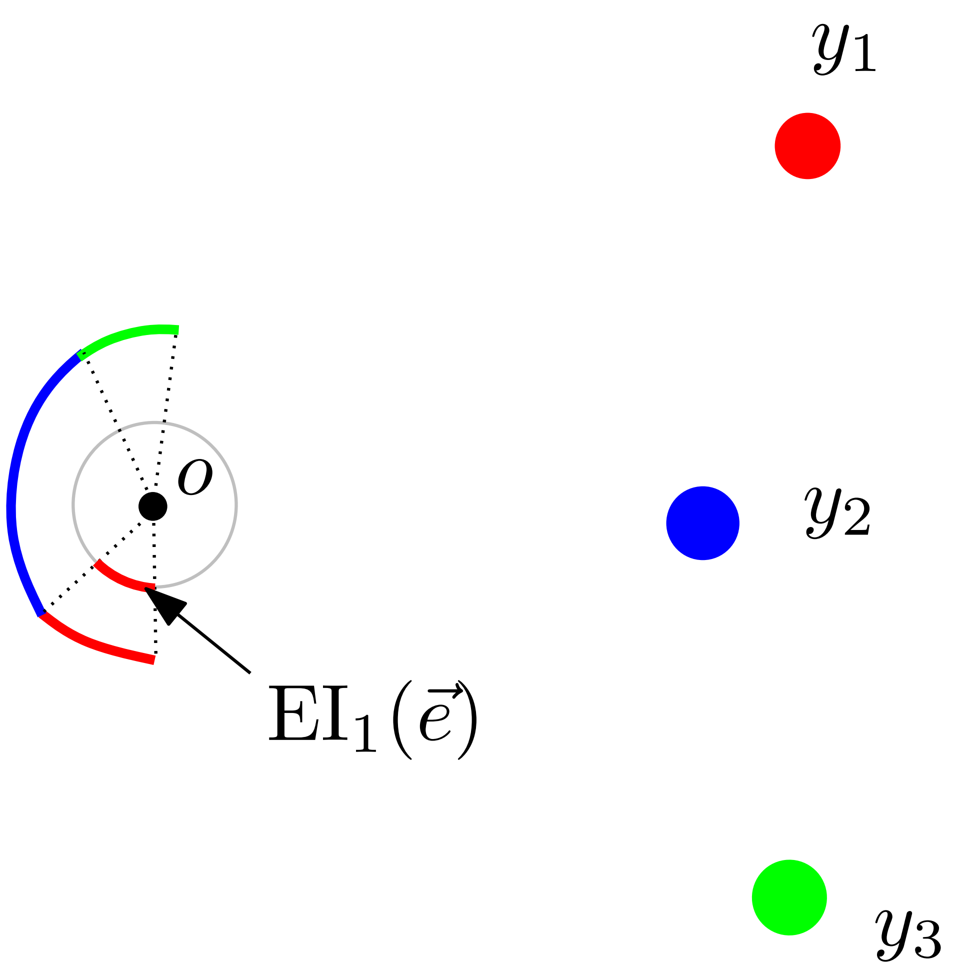

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Near-field reflector antenna problem:

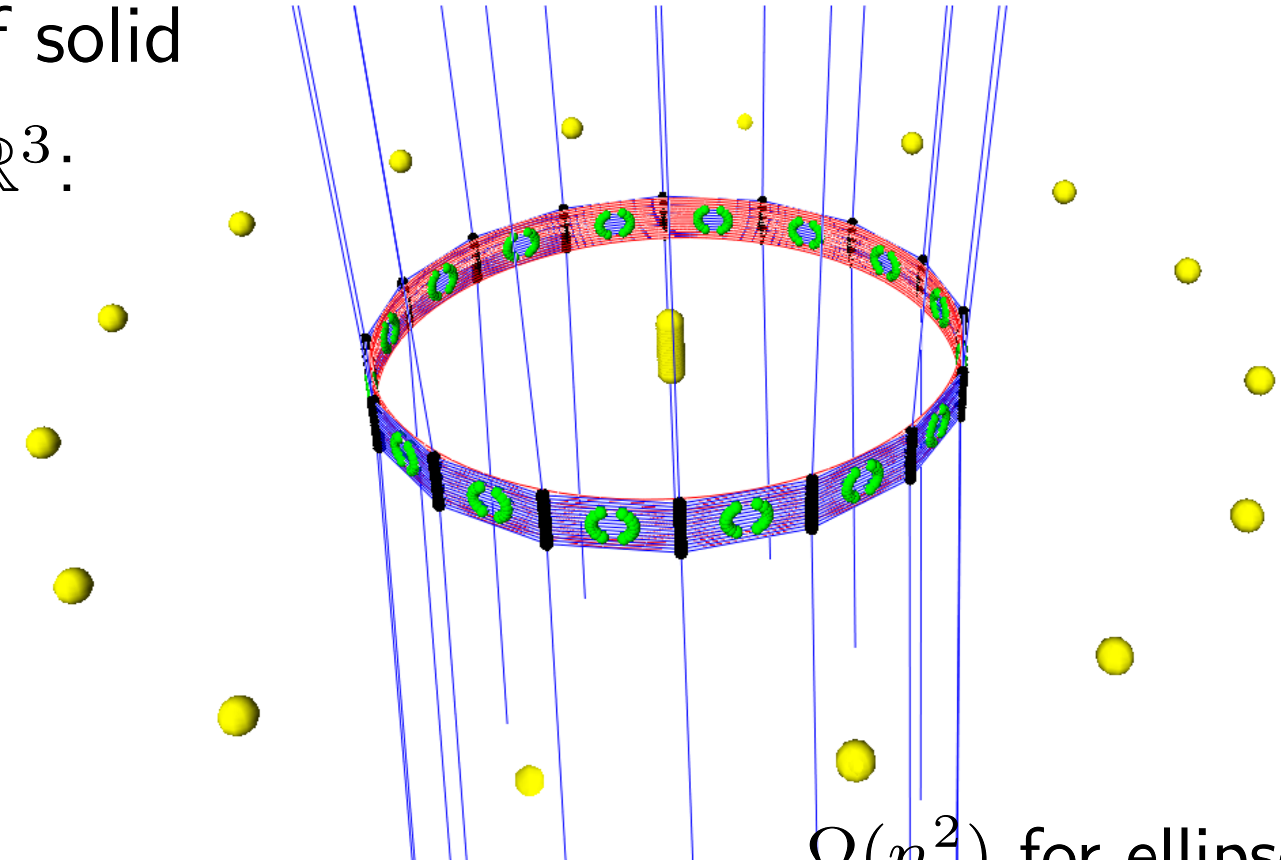
Oliker '04

Problem (NF): Find $e_1, \ldots, e_N$ such that for every i, $\mu(\mathrm{EI}_i(\vec{e})) = \nu_i$.

amount of light reflected to the point y_i.

Complexity of a single iteration

Complexity of union/intersection of solid confocal quadric of revolutions in $\mathbb{R}^3$:



$\Omega(n^2)$ for ellipsoids

	Combinatorial complexity	Computational c.
Paraboloid intersection	$\Theta(n)$	$\Theta(n \log n)$
Paraboloid union	$\Omega(n)$	$O(n^2)$
Ellipsoid intersection	$\Theta(n^2)$	$\Theta(n^2)$
Ellipsoid union	$\Theta(n^2)$	$\Theta(n^2)$

↑ $\#$ faces $+$ points $+$ edges

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A simple quasi-Newton scheme can be used to solve rather large (15k points) geometric instances of optimal transport.

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- quantitative stability results ?

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Thank you!