Numerical optimization of Dirichlet-Laplace eigenvalues on domains in surfaces

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Geometric problem

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Theoretical Question : Existence of a solution (λ, u)?
Spectral theorem: Yes!

There exist a sequence of real positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \cdots \nearrow \infty,$$

and a sequence of associated eigenfunctions $(u_k)_{k\geq 1}$ such that

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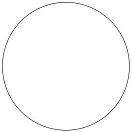
Moreover, the eigenfunctions (u_n) define a Hilbert basis of $H_0^1(\Omega)$.

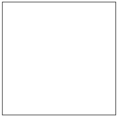
Motivations

- The Laplace-spectrum encodes informations about the underlying domain;
- Optimization w.r.t. the domain to understand the behaviour of the spectrum.

Theoretically known examples:



$\lambda_{1,\mathrm{Disc}_1} \simeq 18.168$



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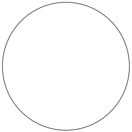

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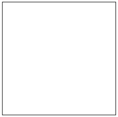

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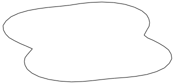

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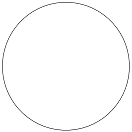

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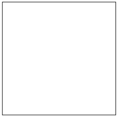

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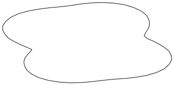

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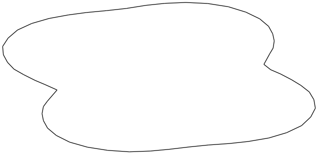

$\lambda_{1,h}(\sqrt{2}\Omega) \simeq 10.513$

Optimization problem

Question : What bounded domain $\Omega \subset \mathbb{R}^2$ minimizes $\lambda_{k,\Omega}$?

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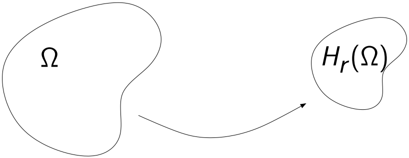
$\rightsquigarrow$ That is not a good question !



$$\lambda_{k,H_r(\Omega)} = \frac{1}{r^2}\lambda_{k,\Omega}$$

(the larger Ω is, the smaller the eigenvalue $\lambda_{k,\Omega}$ is)

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Or equivalently,

$$\min_{\substack{\mathrm{vol}(\Omega)=1,\\ \Omega \text{ bounded}}} \lambda_{k,\Omega} \quad \Leftrightarrow \quad \min_{\Omega \text{ bounded}} \mathrm{vol}(\Omega)\, \lambda_{k,\Omega}$$

Few known results:

Theorem (Faber-Krahn, 1923)

Let B be the ball of volume 1. Then,

$$\lambda_{1,B} = \min\left\{ \lambda_{1,\Omega} \,\middle|\, \Omega \subset \mathbb{R}^2, \mathrm{vol}(\Omega) = 1 \right\}.$$


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Theorem (Krahn-Szegö, 1926)

Let B_2 be the union of two identical balls, $\mathrm{vol}(B_2) = 1$. Then,

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B_2

- These theorems also hold in $\mathbb{R}^n$, $n \geq 3$;

Another result:

Theorem (Bucur 2012 & Mazzoleni, Pratelli 2013)

There exists a minimizer for $\lambda_{k,\Omega}$, $k \geq 3$, among all quasi-open sets Ω of given volume. Moreover, it is bounded and has finite perimeter.

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Theorem (Bucur 2012 & Mazzoleni, Pratelli 2013)

There exists a minimizer for $\lambda_{k,\Omega}$, $k \geq 3$, among all quasi-open sets Ω of given volume. Moreover, it is bounded and has finite perimeter.

However, it does not provide the shape of the minimizing domain!

Open problem

For $k \geq 3$, what is the bounded domain of volume 1 in $\mathbb{R}^2$ which minimizes $\lambda_{k,\Omega}$?

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$\rightsquigarrow$ numerics !

Numerical processing

Weak formulation of problem $(\mathcal{P})$:

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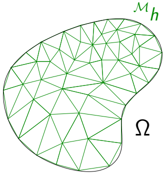
$\rightsquigarrow$ A discrete framework is required.

Galerkin approximation

Discretization of Ω into triangles K of type $\mathcal{P}_1 \rightsquigarrow$ we get a mesh $\mathcal{M}_h$ with N nodes inside Ω ;



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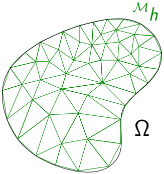

Instead of $H_0^1(\Omega)$ in $(\mathcal{WP})$, consider the finite dimensional space

$$V_h := \left\{ \varphi \in \mathcal{C}^0(\overline{\Omega}) \,|\, \varphi_{|\partial\Omega} = 0,\ \varphi_{|K} \text{ linear } \forall K \in \mathcal{M} \right\} \ ;$$

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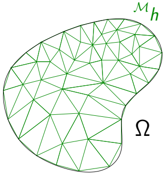

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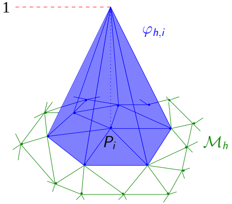
Basis $\{\varphi_{h,i}\}_{i=1}^N$ of V_h :

$$\varphi_{h,i}(P_j) = \delta_{ij},$$



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$\rightsquigarrow$ Lanczos algorithm to solve $(\mathcal{WP}_h)$.

Shape optimization

The idea is to use a descent algorithm to minimize the *cost functional* $J(\Omega) = \lambda_k(\Omega)\,\mathrm{vol}(\Omega)$.

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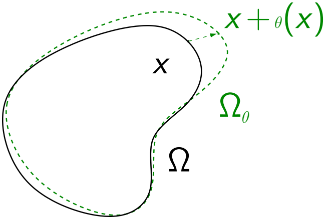
Given an initial domain Ω, we allow deformations of the form

$$\Omega_\theta = (\mathrm{id}+\theta)(\Omega),\ \theta \in W^{1,\infty}(\Omega).$$



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Then P_i is moved onto

$$P_i' := P_i - d_i \vec{n}, \text{ with } d_i = J'(\Omega_0)(\theta_i).$$

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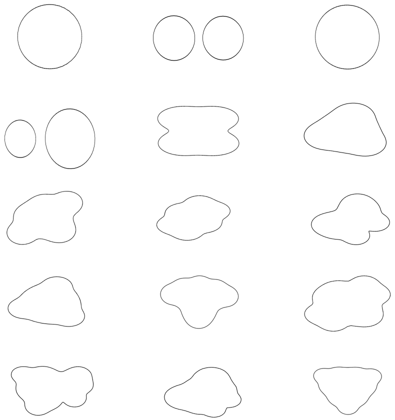
Then, we obtain a new domain, we can mesh it, compute the associated eigenvalues and eigenfunctions, move the new boundary, and so on…

15 first candidates to be minimizing domains of volume 1 in $\mathbb{R}^2$.

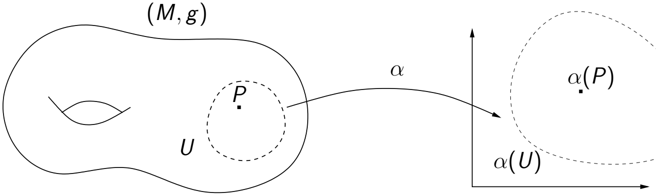


k	λ_k
1	18.17
2	36.39
3	46.30
4	64.78
5	78.53
6	89.05
7	106.51
8	120.01
9	134.06
10	144.82
11	160.55
12	174.37
13	188.84
14	202.22
15	211.16

Previously found by Oudet ('04, partly) and Antunes-Freitas ('12)

Generalization to surfaces

Let (M, g) be a Riemannian manifold of dimension 2



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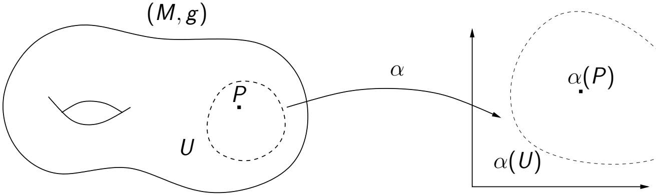

Mesh $\alpha(U)$ in order to consider manifold non embeddable in $\mathbb{R}^3$.
$\rightsquigarrow$ use the expression of the Laplacian in local coordinates:

$$\Delta f = \frac{1}{\sqrt{\det(G)}} \sum_{j,k=1}^{2} \partial x_j \left(G^{jk} \sqrt{\det(G)} \partial x_k f \right).$$

Generalization to surfaces

It implies several modifications. For instance,

- for the computation:

$$(\mathcal{WP}_h) \begin{cases} \text{find } u_h \in V_h, u_h \not\equiv 0, \text{ and } \lambda > 0 \text{ such that} \\ \int\limits_\Omega \nabla u_h^t G^{-1} \nabla \varphi_{h,i} \sqrt{\det G} = \lambda \int\limits_\Omega u_h \varphi_{h,i} \sqrt{\det G}, \\ \text{for all } i = 1, \dots, N. \end{cases}$$

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- for the optimization:
  There is no homothety any more! The volume constraint has to be taken into consideration. $\rightsquigarrow$ Lagrange multiplier.

We look for a saddle point of the functional

$$J(\mu, \Omega) = \lambda_k(\Omega) + \mu(\mathrm{vol}(\Omega) - V_0),$$

where V_0 is the volume of the initial domain Ω_0.

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$\rightsquigarrow$ We get a similar formula for the shape optimization.

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- in the Poincaré disc $\mathbb{D}^2$ (curvature = -1);

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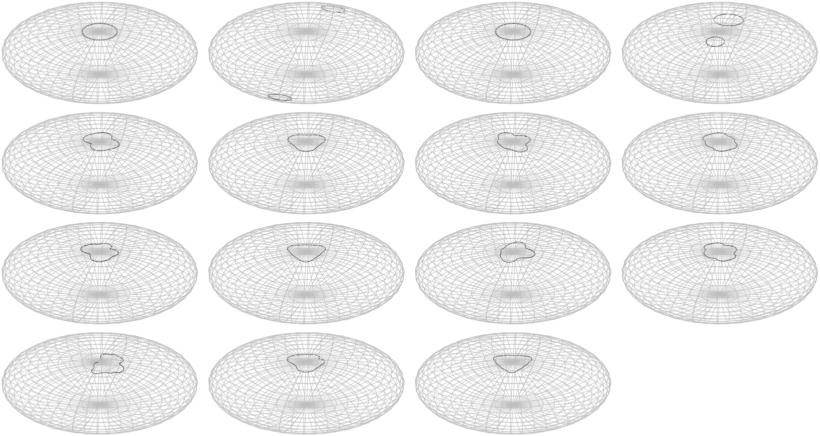
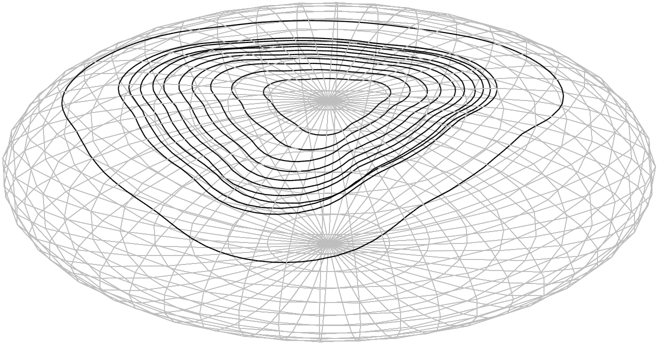
For small domains in surfaces, the results are *similar*

- in the sphere $\mathbb{S}^2$ (curvature = + 1);
- in the Poincaré disc $\mathbb{D}^2$ (curvature = -1);
- in a hyperboloid H (curvature between 0 and 1)

$$H = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0, x^2 + y^2 - z^2 = -1\}.$$



Optimal domains $\Omega_k^* \subset \mathbb{S}^2$ (volume 0.1) minimizing each of the first k eigenvalues.



Plot of the optimizers for $\lambda_{10}(\Omega^*_{10,\mathbb{S}^2})$ and $\text{vol}(\Omega^*_{10,\mathbb{S}^2}) = 0.1, 0.2, \ldots, 0.9, 1$ and 2.

Comparison between manifolds

Comparison of the first eigenvalues for optimal domains of volume 0.01 in $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{D}^2$ and the hyperboloid H.

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$$\begin{array}{ccccccc} \lambda_1(B_{\mathbb{S}^2,0.01}) & < & \lambda_1(B_{\mathbb{R}^2,0.01}) & < & \lambda_1(B_{\mathbb{D}^2,0.01}) & < & \lambda_1(B_{H,0.01}) \\ \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr \\ 1816.57 & & 1816.80 & & 1817.67 & & 1819.10 \end{array}$$

Thank you for your attention !