# Anisotropic Delaunay Meshes of Surfaces

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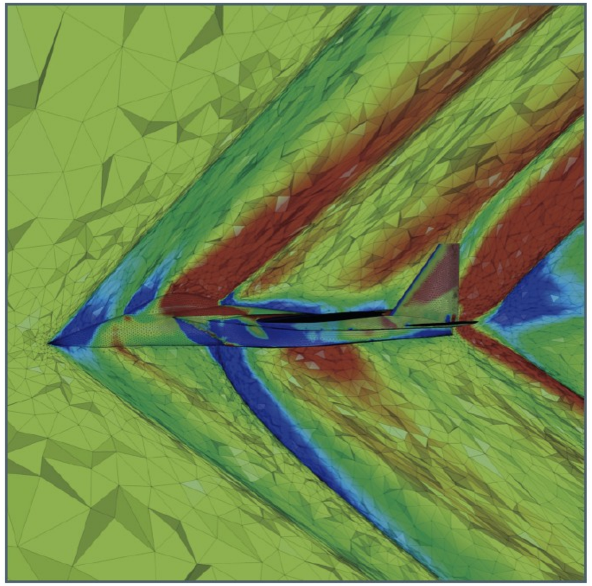
# Outline

- **Problem & Applications**
- State of the Art
- Algorithm
- Conclusions & Perspectives

# Problem : Anisotropic Mesh Generation



Courtesy A.Loiselle

Generate an anisotropic simplicial mesh

- with *simplicial elements* (triangles)
- elongated along *prescribed directions*

Anisotropic metric at each point

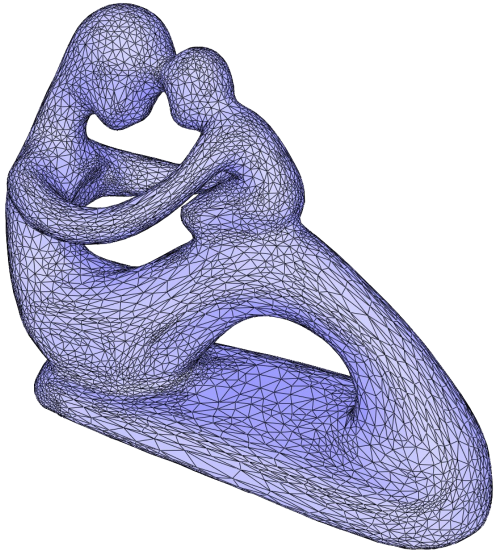
- everywhere in the domain Ω *metric field*

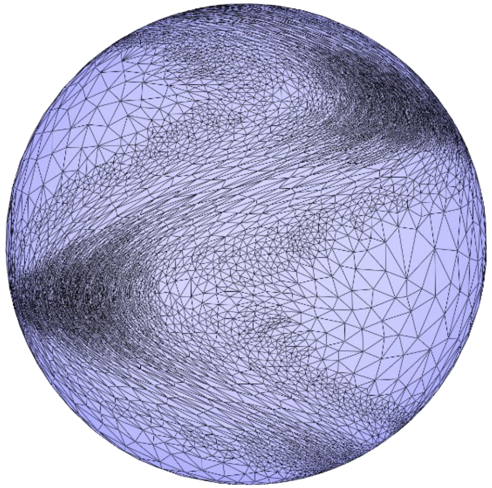
# Motivation

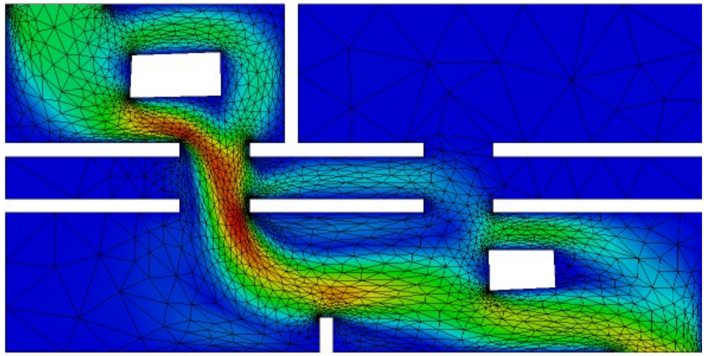
- Accuracy of surface **discretization**
  anisotropy ~ curvature tensor

- Reduction of **function interpolation error**
  anisotropy ~ Hessian

- **Adaptation**
  solving PDE’s for anisotropic phenomena







Courtesy C.Dobrzynski

# How to prescribe anisotropy

## Metric field $M_\Omega$

$\forall z \in \Omega \;\mapsto M(z)$ a symmetric positive definite matrix

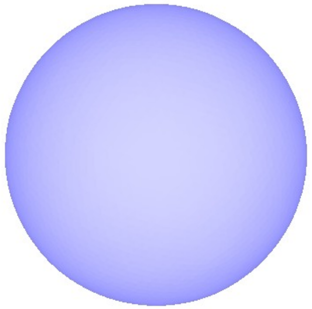
- **Approximation of surfaces** : second fundamental form
- **Fonction approximation** : Hessian of the function
- **Adaptive FEM for PDE** : error estimation on previous iteration solution, at each iteration

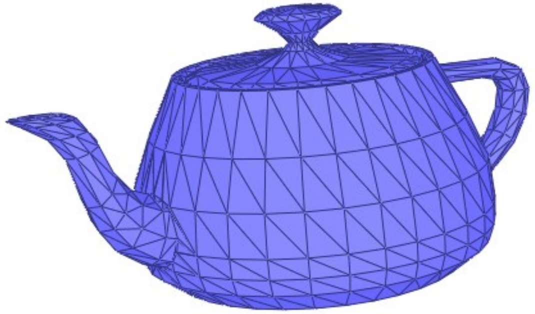
# Contributions

## Anisotropic surface mesh generator

**Large variety of inputs**

- Geometries
- Metric fields





**Quality of output**

(shape, size, approximation, complexity)

**Delaunay framework**

- Restricted Delaunay triangulation
- Delaunay refinement

→ guarantees

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# Previous work (1/3)

## Heuristics for 2D or 3D meshes

- **Ellipses packing** [Li et al.99], [Yamakawa,Shimada.03]
- **Anisotropic Delaunay refinement** [Borouchaki et al.99], [Frey,Alauzet.04], [Dobrzynski,Frey.08]
- **Continuous mesh** [Loseille,Alauzet.09]
- **Anisotropic mesh optimization** [Heckbert.96], [Li et al.05]



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# Previous work (2/3)

## Heuristics for surface meshes

- **Quad meshing** based on principal curvature lines [Alliez et al. 03]
- **Anisotropic adaptation** [Labelle,Shewchuk'03], [Boissonnat et al.'05], [Du,Wang'05],[Cheng et al.'06]
- **Locally uniform anisotropic Delaunay meshes** [Boissonnat et al.'10]

# Previous work (3/3)

## Voronoi approaches

- **Voronoi diagram on Riemannian manifold** [Leibon,Letscher'00], [Bougleux et al.'08]
- **Anisotropic Voronoi diagram**[Jiao et al.06]
- **Grid based** [Azernikov, Fischer.05]

# Outline

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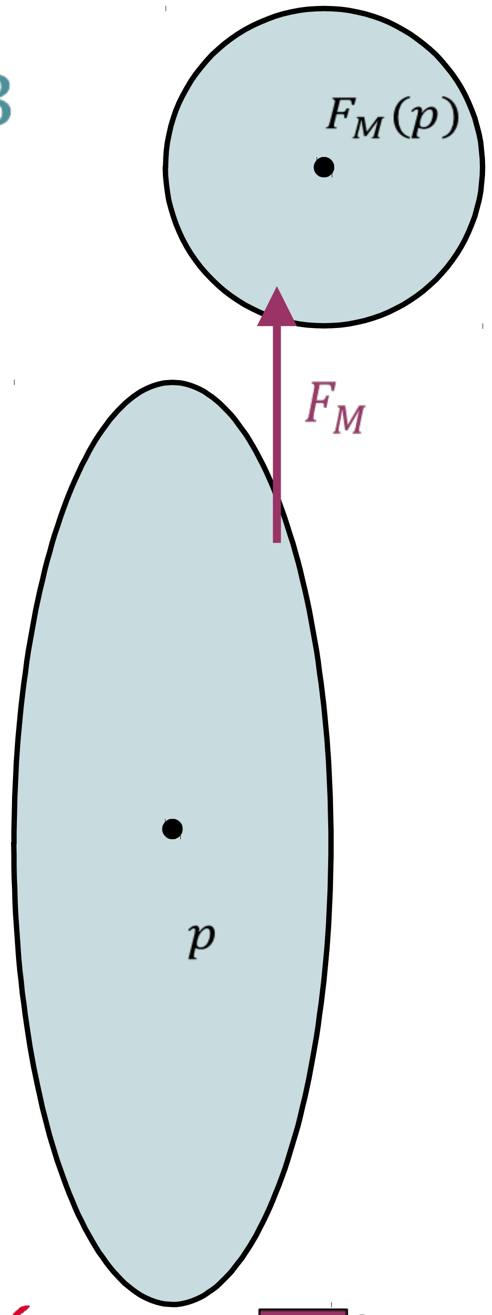
# Definitions – Anisotropic Metric in $\mathbb{R}^3$

- symmetric positive definite $3 \times 3$ matrix $M$
- distance : $d_M(a,b) = \sqrt{(a-b)^t M (a-b)}$
- $\exists$ matrix $F_M$ s.t. $\det(F_M) > 0$ and $F_M^t F_M = M$
  $\Longrightarrow d_M(a,b) = \| F_M(a-b) \|$



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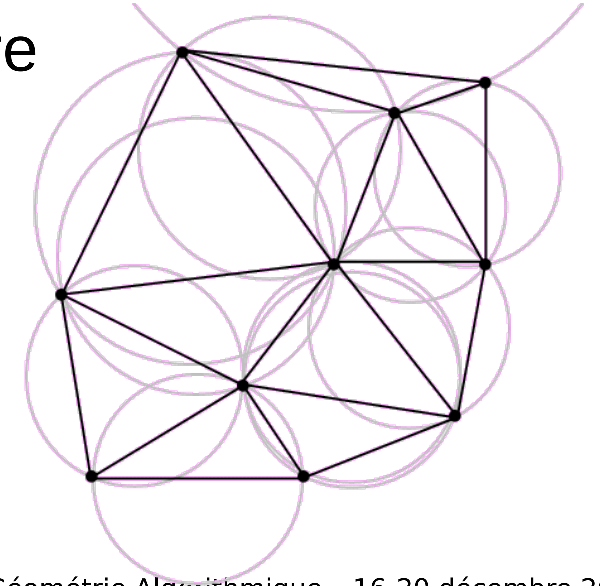
# Definitions – Delaunay triangulation in a uniform anisotropic metric

## Notations

- *M-distance* $d_M(a,b) = \sqrt{(a-b)^t M(a-b)} = ||F_M(a-b)||$
- *M-sphere* $S_M(c,r) = \{x : \ d_M(c,x) = r\}$
- *M-Delaunay*

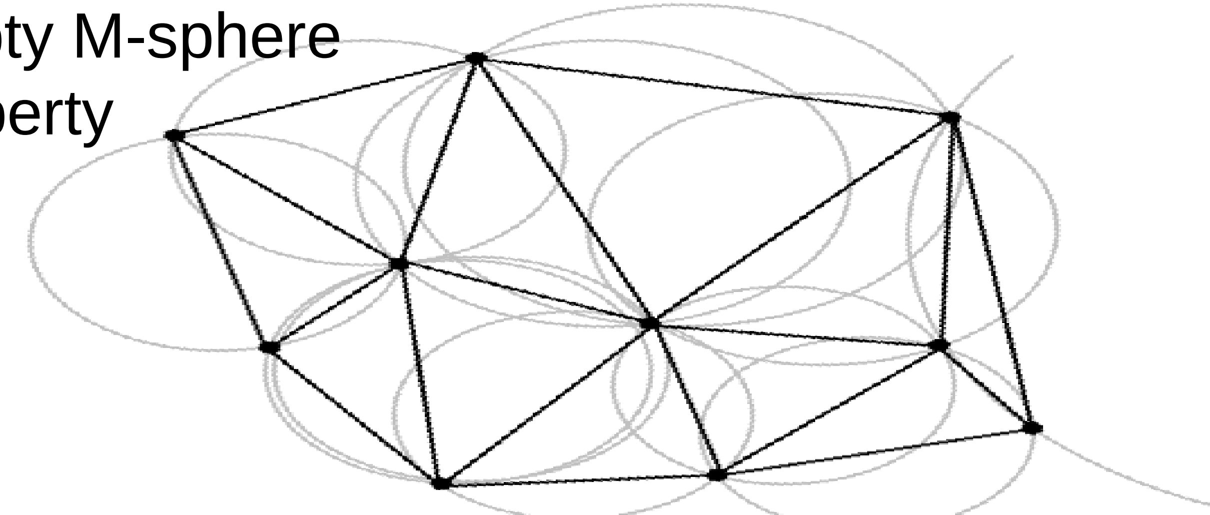
### Delaunay triangulation $DT(V)$

empty sphere property



### Delaunay triangulation $DT_M(V)$

empty M-sphere property





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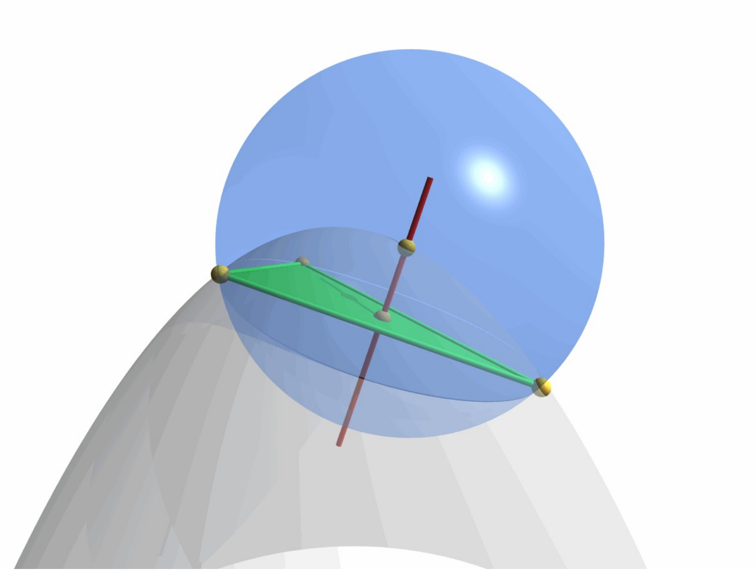
# Definitions – Delaunay triangulation in a uniform anisotropic metric

## Notations

- *restricted Delaunay*
- *M-restricted Delaunay*
- ***M-surface Delaunay ball and center $c_M$***



- compute $s = dual_M(restricted\ facet)$
- $c_M = s \cap \text{Surface}$

# Our Approach : Locally Uniform Anisotropic Delaunay Meshes

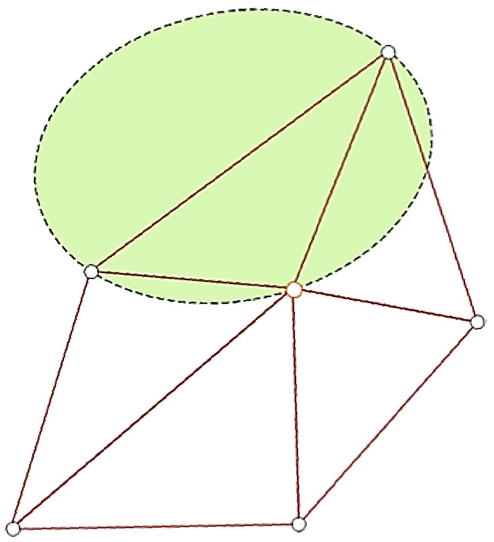
Notations :

$V$ a set of vertices, $v \in V$ a vertex

$M_v$ metric at $v$,

$Del_v(V)$ the Delaunay triangulation of $V$ computed in $M_v$

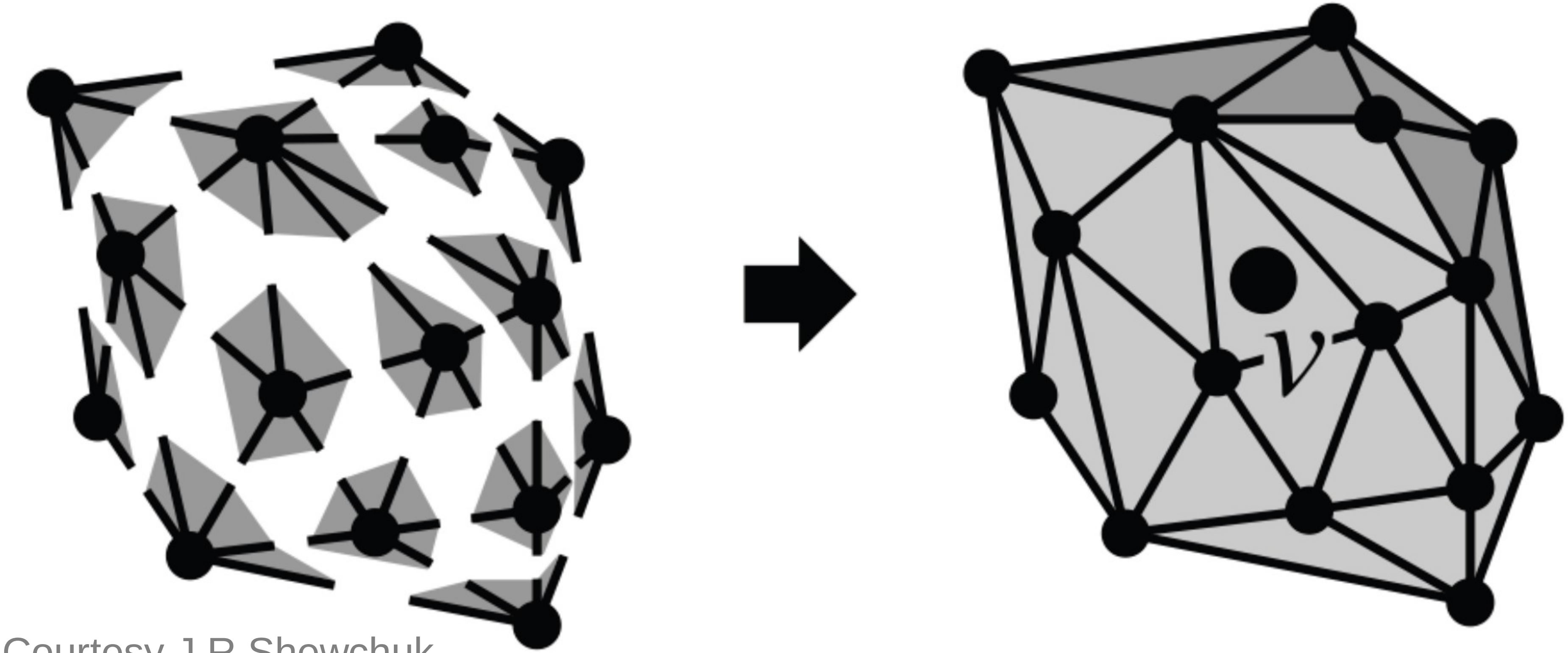
$T_v$ the 3D star of $v$ in $Del_v(V)$



Build a mesh s.t. **the star of each vertex is Delaunay** wrt the metric at that vertex

# Star set

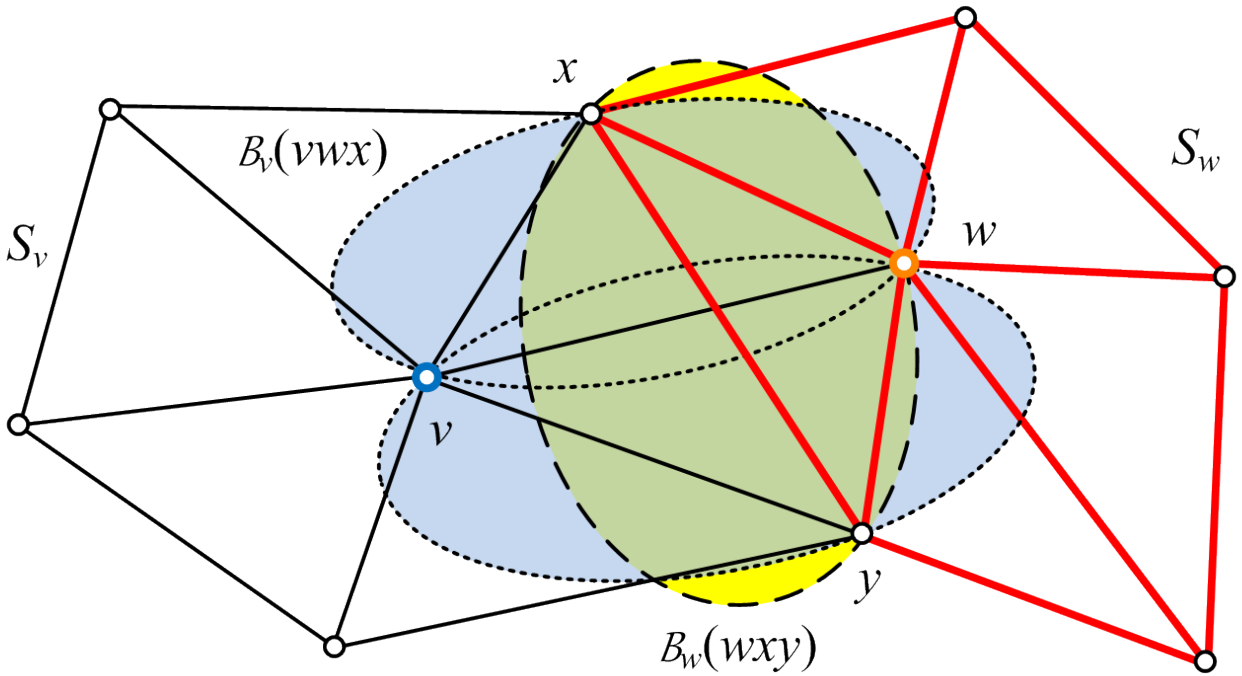
- $\mathbb{T}(V)$ the star set $\{T_v : v \in V\}$
- $\mathbb{S}(V)$ the surface star set $\{S_v : v \in V\}$




Courtesy J.R.Shewchuk

# Star set - Inconsistencies

$S(V)$ the surface star set $\{S_v : v \in V\}$




Inconsistency :

simplex $\tau$
with vertices $\{v, w, \ldots\}$
is in $S_v$ but not in $S_w$

# Meshing Algorithm

**Input :**

- Domain $\Omega$
- Metric field defined over $\Omega$
- Criteria :
  - consistency
  - quality of simplices

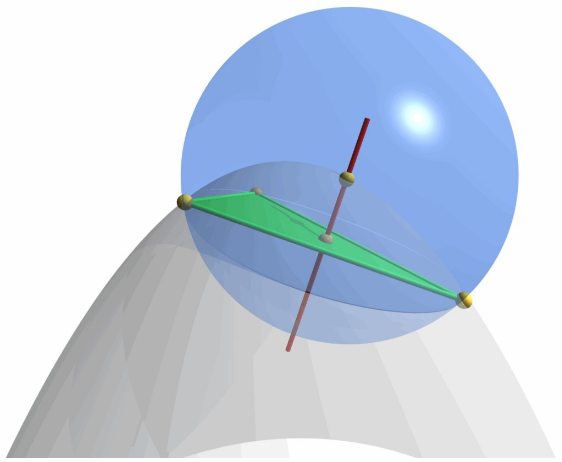
# Definitions - Simplex Quality in Anisotropic metric

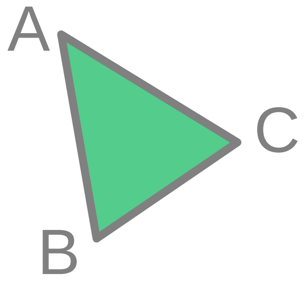
A triangle $\tau$ in a metric $M$.

Quality evaluation :

- Size – radius $r_M(\tau) \leq r_0$
- Approximation – Haussdorf distance $d(c(\tau), p_S(c(\tau)) \leq h_0$
- Shape – radius-edge ratio $\rho_M(\tau) = \frac{r_M(\tau)}{e_M(\tau)} \leq \rho_0$
- Distortion – $\gamma(\tau) = \max\big(\gamma(A,B), \gamma(B,C), \gamma(C,A)\big) \leq \gamma_0$

  with $\gamma(A,B) = \gamma(M_A, M_B) = \max(||F_A F_B^{-1}||, ||F_B F_A^{-1}||)$
- **Consistency**





# Meshing Algorithm

Apply following refinement rules with priority order :

1. Sizing field

   While $\exists \tau \in S_v$ s.t. $r_v(\tau) \geq \mathrm{sf}(c_v(\tau))$, refine $\tau$

2. Radius-edge ratio

   While $\exists \tau \in S_v$ s.t. $\rho_v(\tau) \geq \rho_0$, refine $\tau$

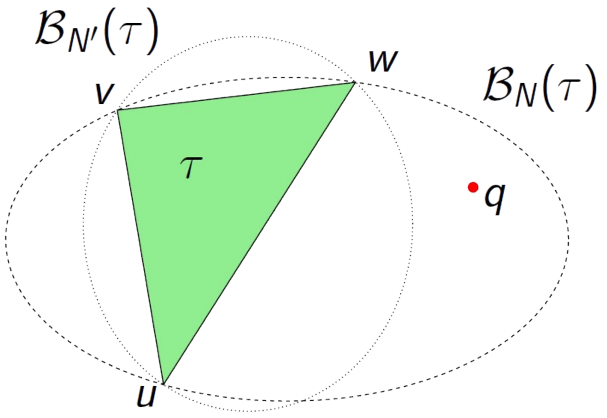
3. Inconsistencies

   While $\exists$ an inconsistent simplex $\tau \in S_v$, refine $\tau$

M-surface Delaunay ball center

pick valid point

# Quasi-cospherical configurations




A subset of sites $\{u, v, w, x\}$ is a $(\gamma_0, M)$-cospherical configuration if there exist two metrics $N$ and $N'$ s.t. :

- $\gamma(M, N) \leq \gamma_0$, $\gamma(M, N') \leq \gamma_0$, and $\gamma(N, N') \leq \gamma_0$
- $Del_N(U) \neq Del_{N'}(U)$

« quasi-cospherical » for short

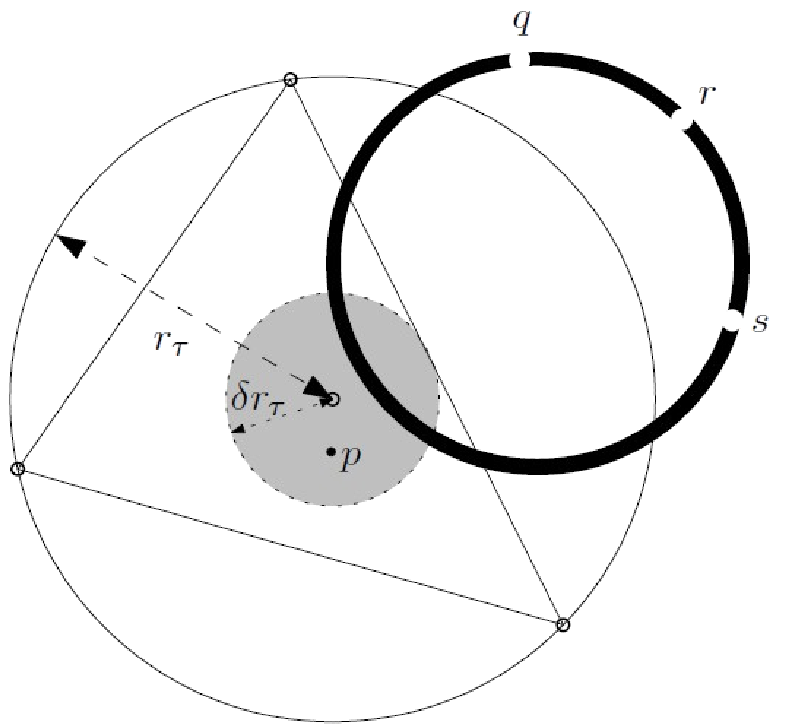
In (isotropic) Delaunay refinement :

refinement points are circumcenters

Here :

may lead to cascading quasi-cospherical configurations

# Pick a valid refinement point




## Picking region [Sliver prevention, Li'00]

τ a bad simplex in star $S_v$

$P_v(\tau) = B_v(c_v(\tau), \delta r_v(\tau))$

## Picking lemma (short. see [Boissonnat et al'2011])

For any $\delta$, it is possible to find a valid refinement point in $P_v(\tau)$

↓

creates no new (and small) quasi-cospherical configurations

# Implementation details

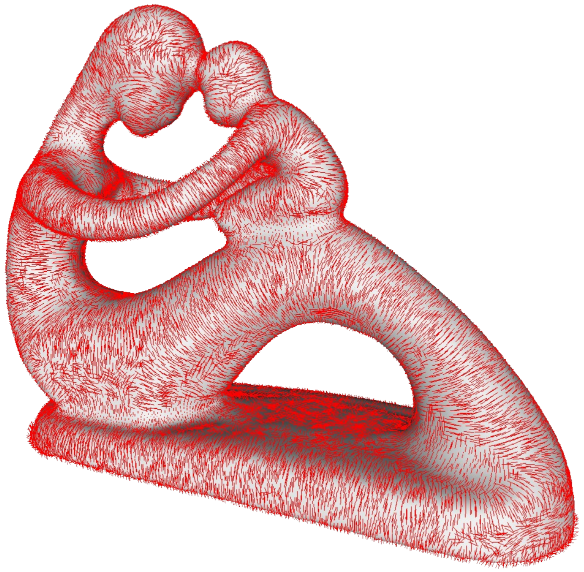
- implemented with CGAL
- store a « small triangulation » per vertex : its *star*
- to avoid degeneracies (e.g. planar regions)
  Insert points on the medial axis [Amenta'98]

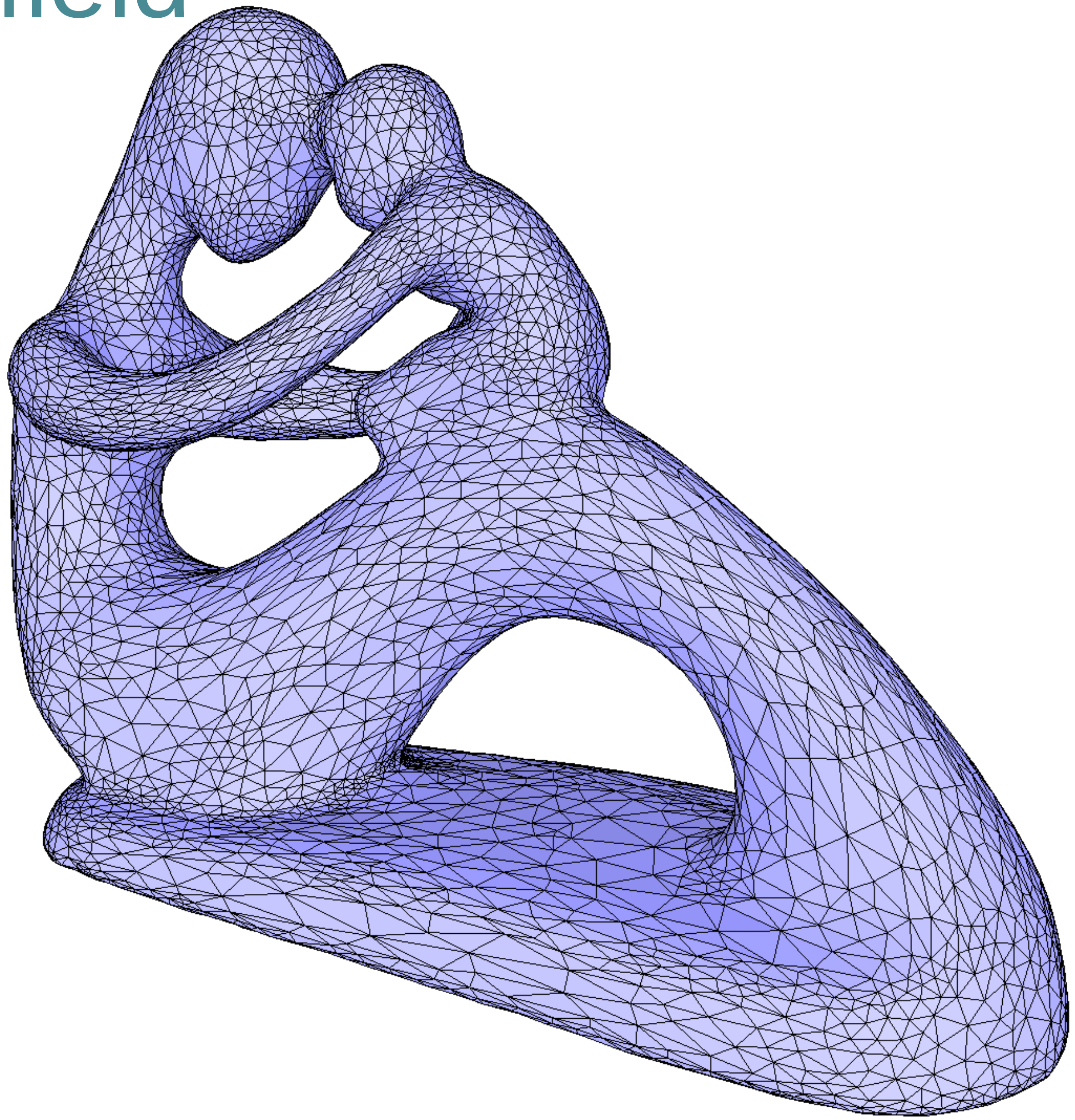
# Curvature-driven metric field

**$M_v$ has :**

eigenvectors = {principal directions}

eigenvalues = {sqrt(curvatures)}





# Results – Stretched ellipsoids

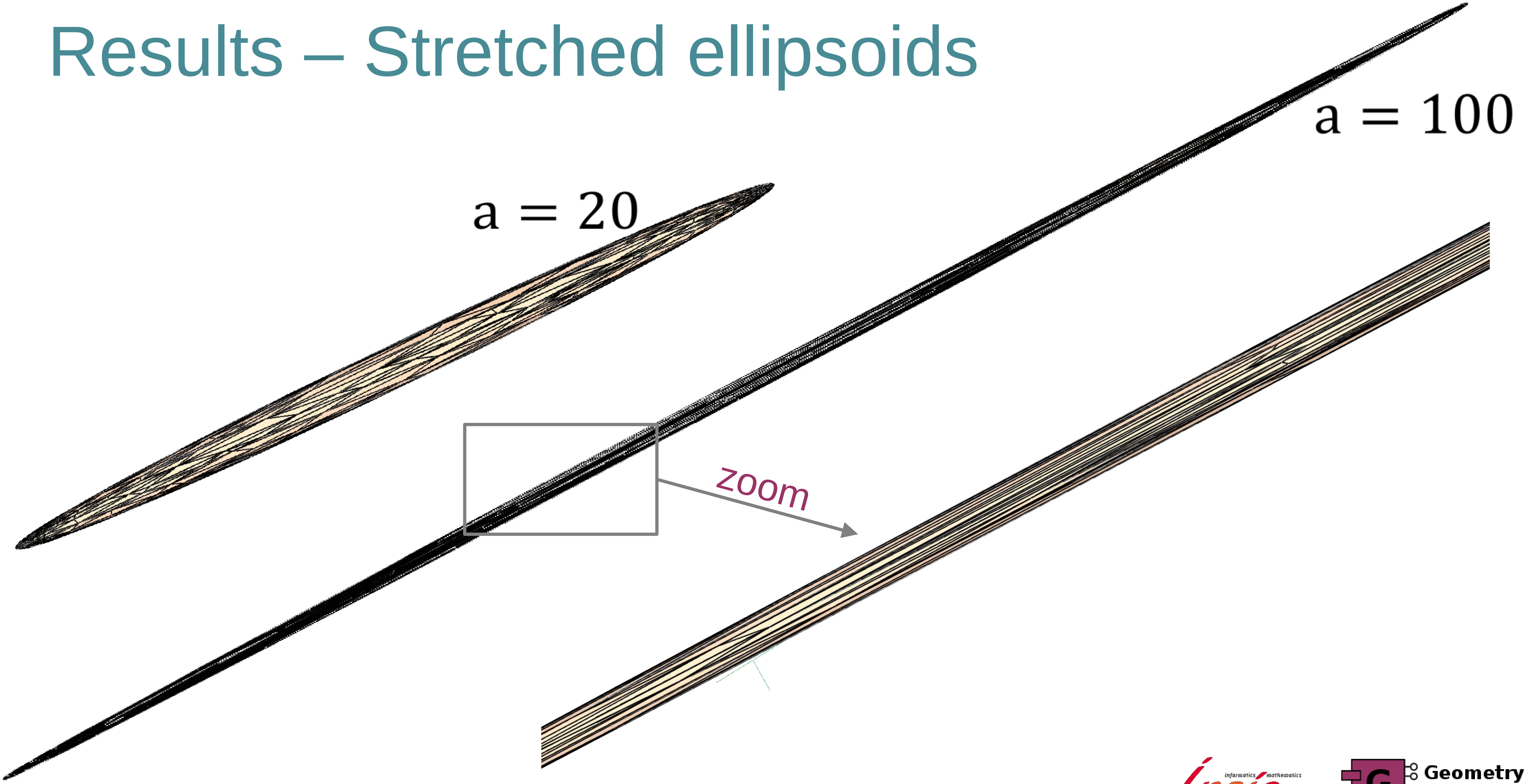
Ellipsoid $\frac{x^2}{a^2} + y^2 + z^2 - 1 = 0$

Curvature tensor fixed approximation error

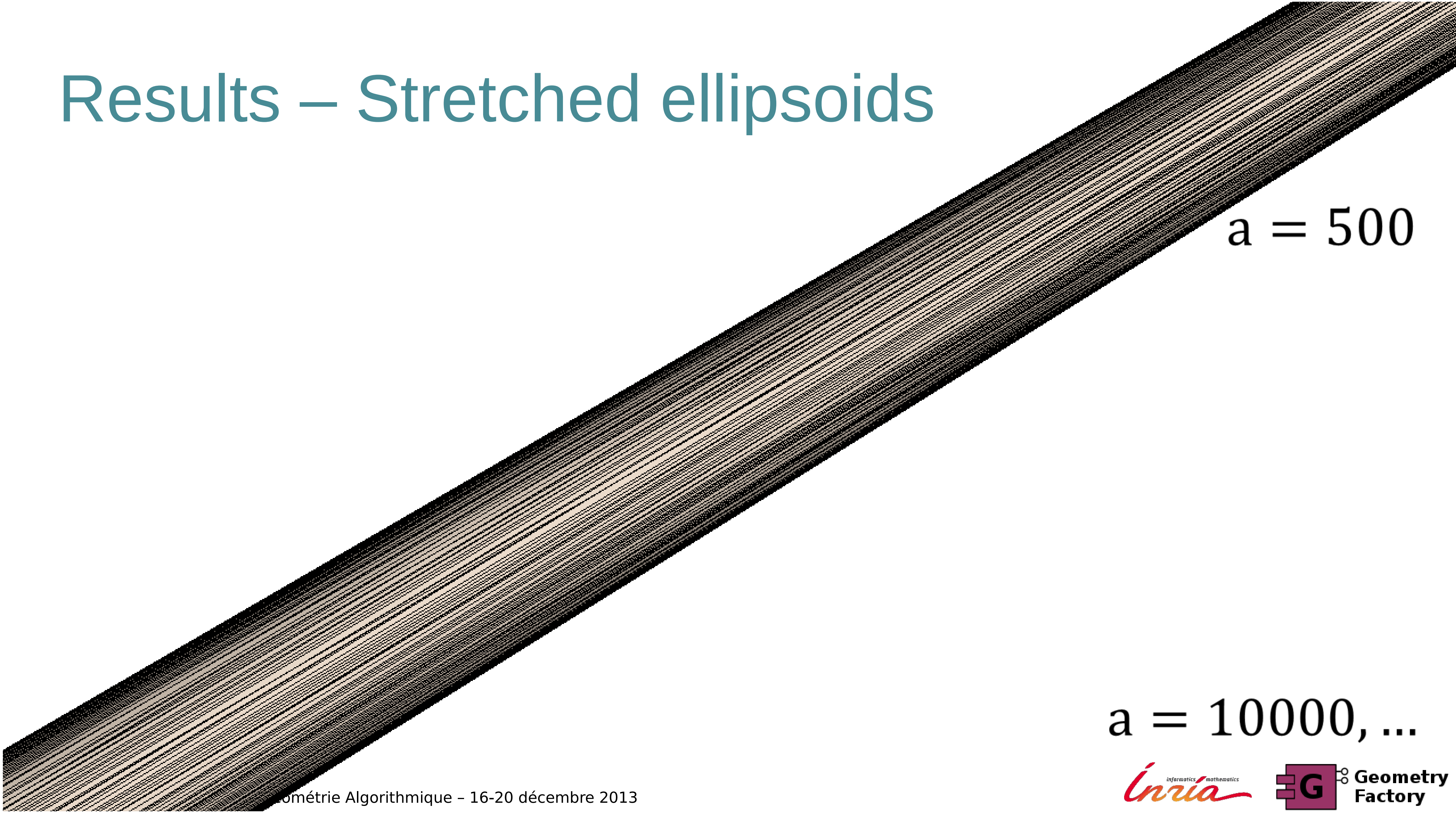
| a | 10 | 50 | 100 | 200 | 500 | 1000 | 10000 |
|---|---|---|---|---|---|---|---|
| #iso | 24872 | 126298 | 250941 | 502272 | 1257633 | 2512782 | ~20m |
| /surf.unit | 250.85 | 255.88 | 254.24 | 254.45 | 254.85 | 254.60 | ~250 |
| #aniso | 6502 | 6734 | 6866 | 6900 | 7119 | 7300 | 7835 |
| /surf.unit | 65.58 | 13.64 | 6.96 | 3.50 | 1.44 | 0.74 | 0,079 |

[Heckbert,Garland'99] nbv asymptotically optimal when anisotropy ratio is $\frac{\sqrt{\kappa_1}}{\sqrt{\kappa_2}}$

# Results – Stretched ellipsoids



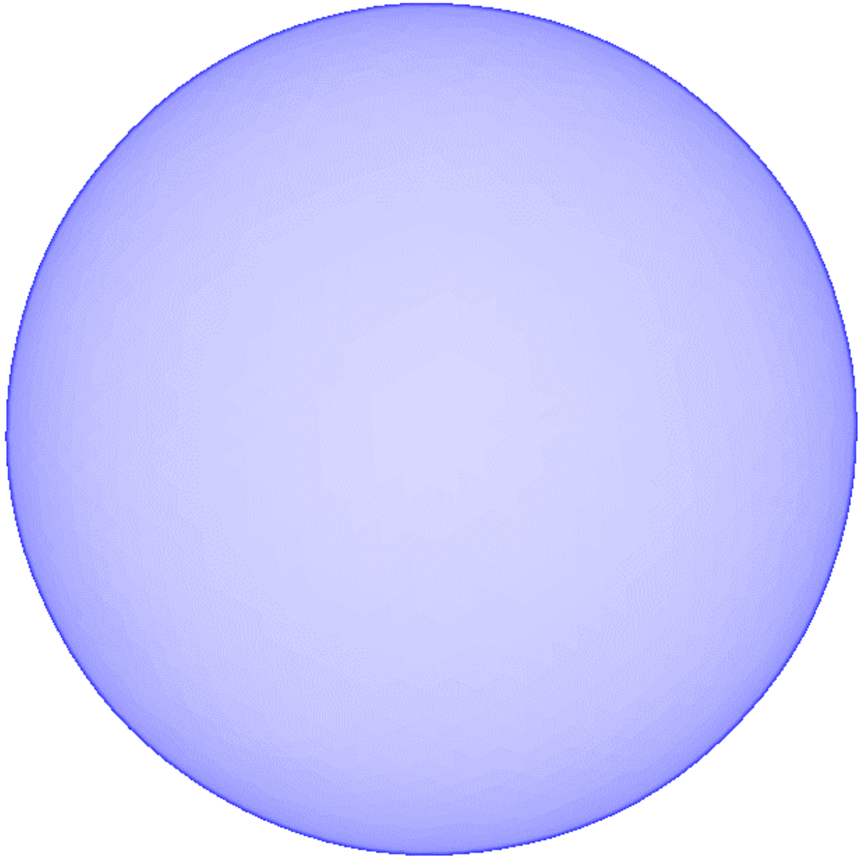
# Results – Stretched ellipsoids

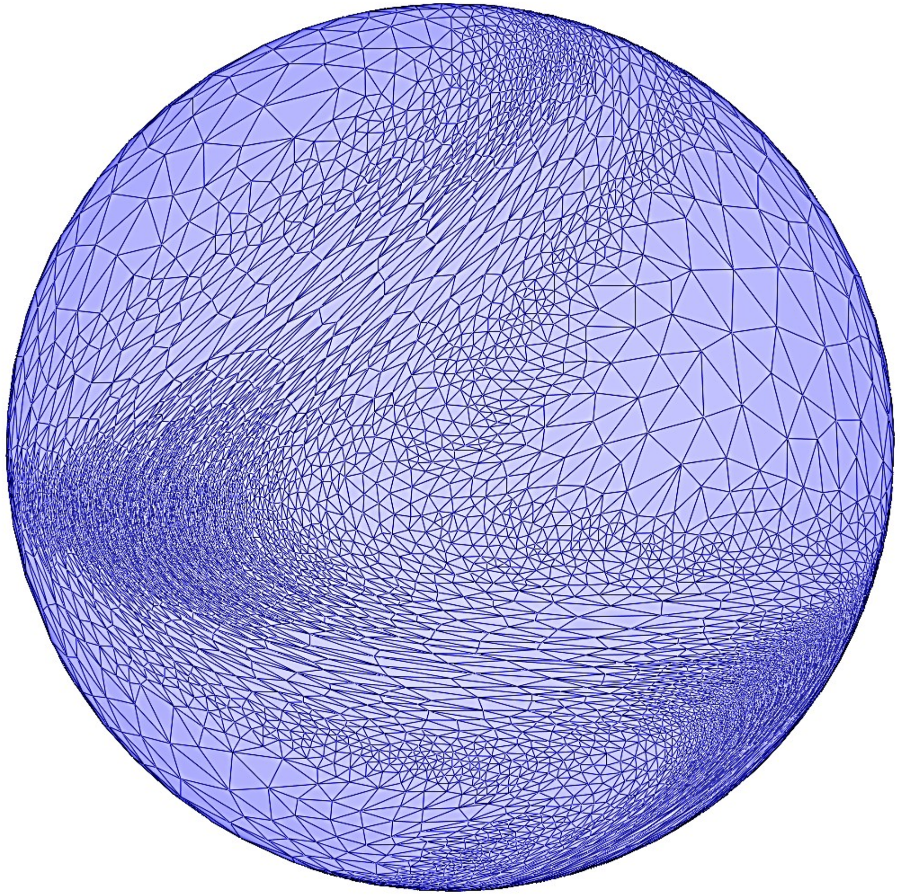


$a = 500$

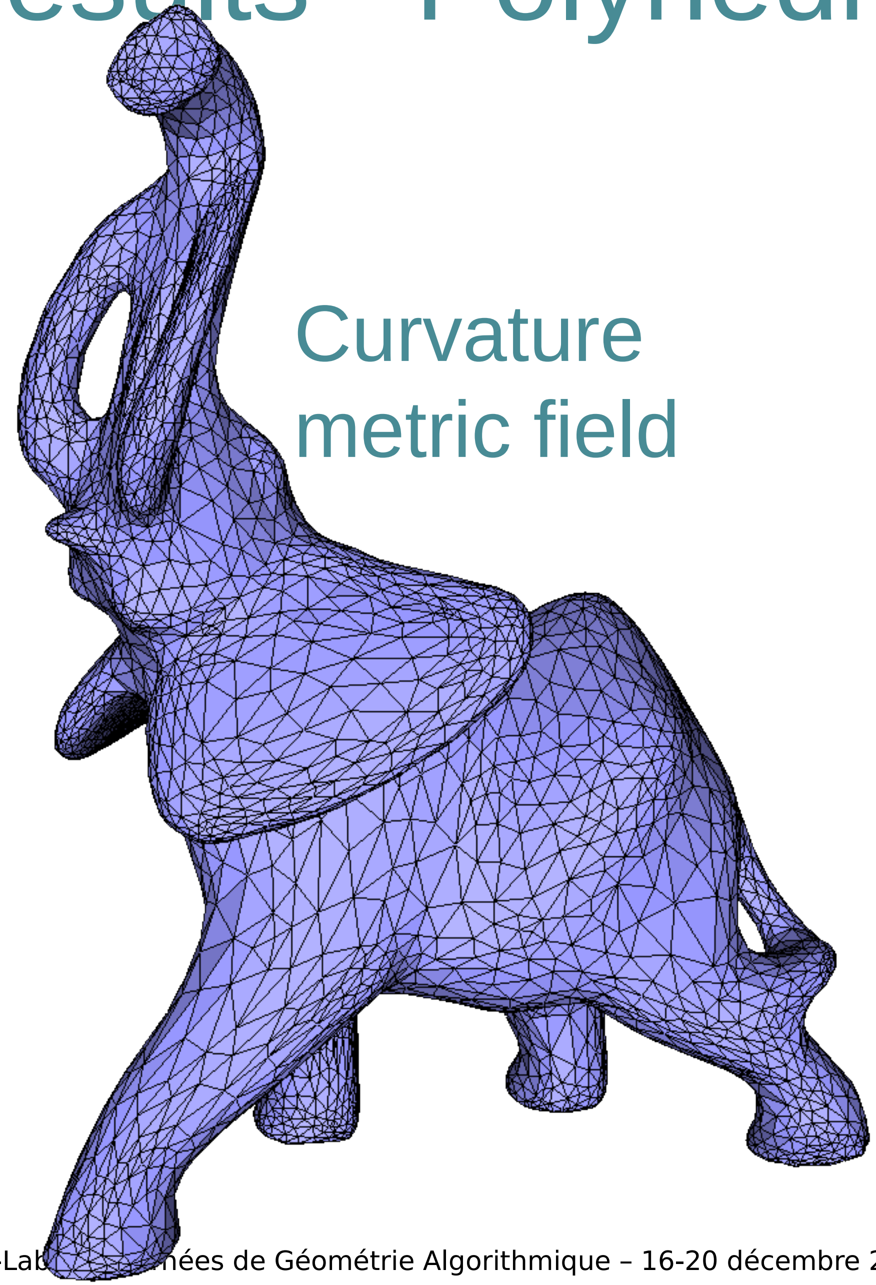
$a = 10000, \ldots$

# Results - Implicit surface & Scalar field

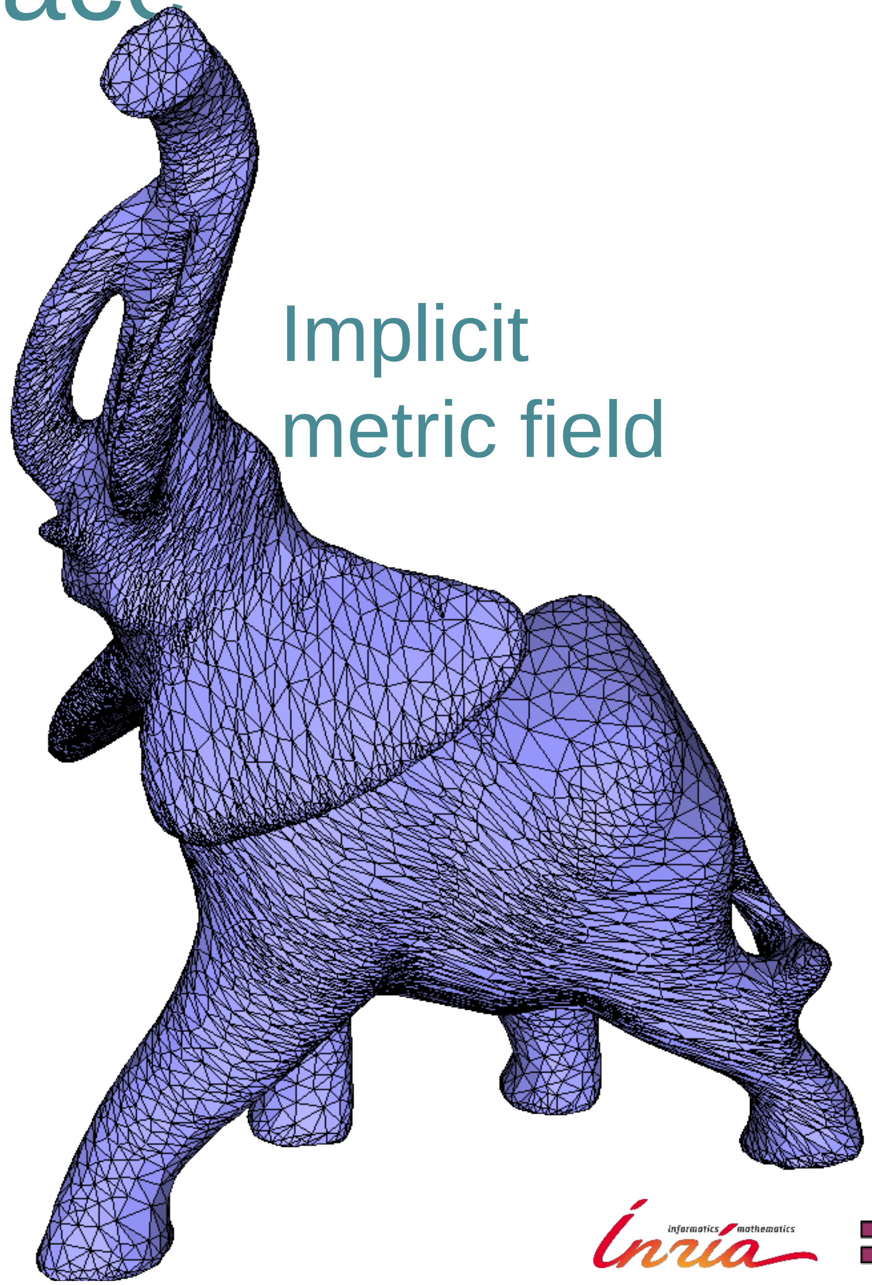




# Results - Polyhedral surface



Curvature
metric field



Implicit
metric field

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- **Conclusions & Perspectives**

# Conclusions

## Anisotropic surface mesh generator

- provably correct
- conceptually simple
  (Delaunay triangulation & restricted Delaunay)
- works in any dimension

## Advantages

- much fewer points than isotropic mesh for same approx.
- control shape, approximation, and size

# Wait a minute...!

## Conveniently overlooked:

- negative curvatures values...?

# Wait a minute...!

## Conveniently overlooked:

- negative curvatures values... Convexify the metric field

# Wait a minute...!

## Conveniently overlooked:

- negative curvatures values... Convexify the metric field
- parabolic points...?

# Wait a minute...!

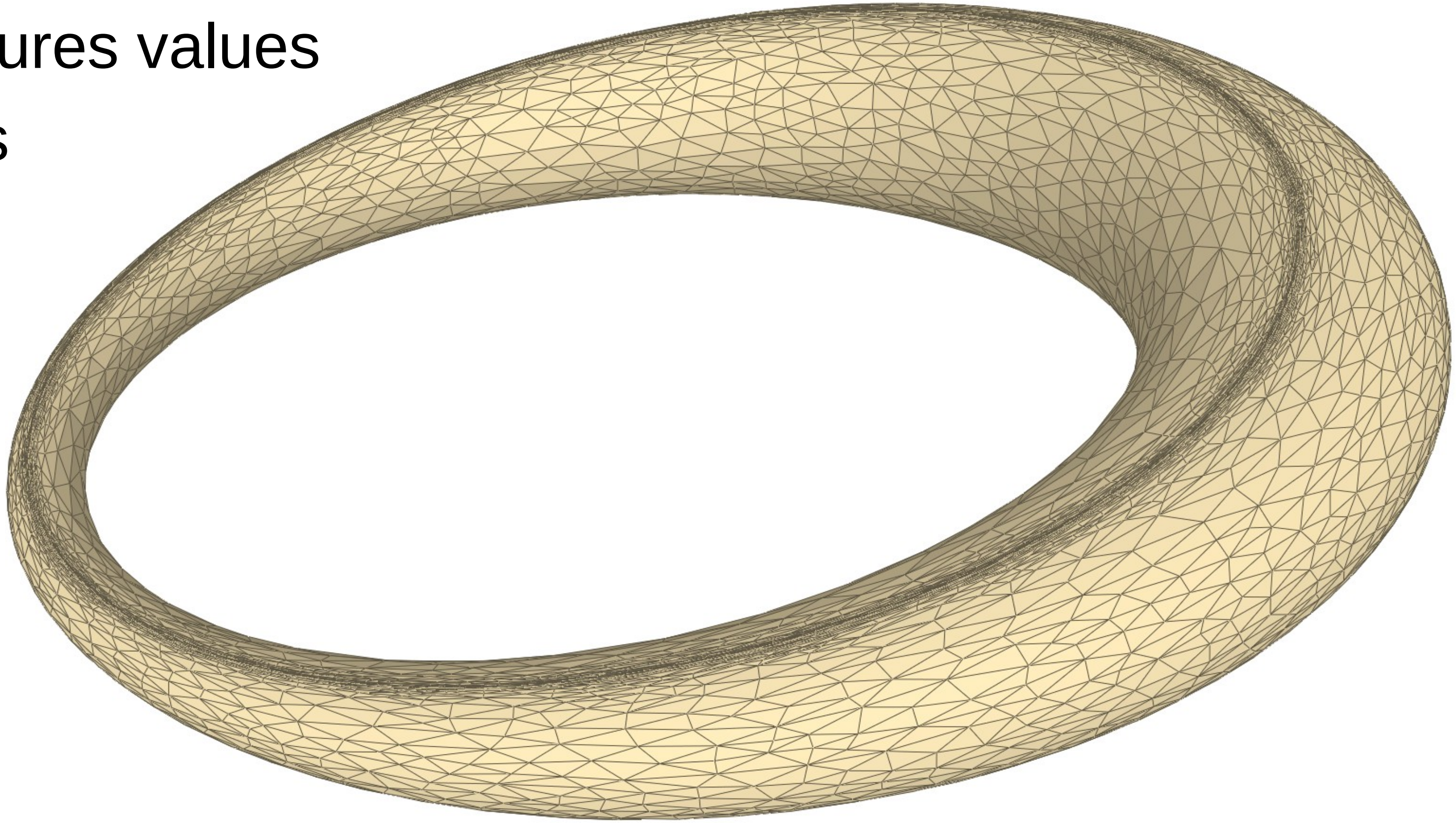
## Conveniently overlooked:

- negative curvatures values... Convexify the metric field
- parabolic points... Prevent null curvatures with an ε

# Wait a minute...!

## Conveniently overlooked:

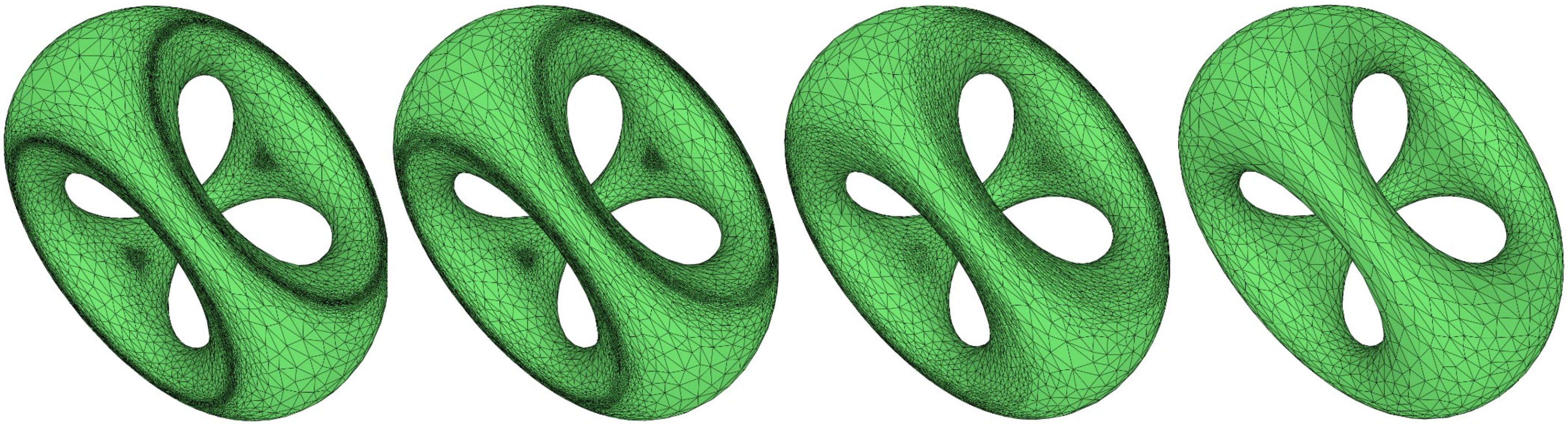
- negative curvatures values
- parabolic points
- optimal ratio...?



# Ongoing work

## Curvature metric field variations

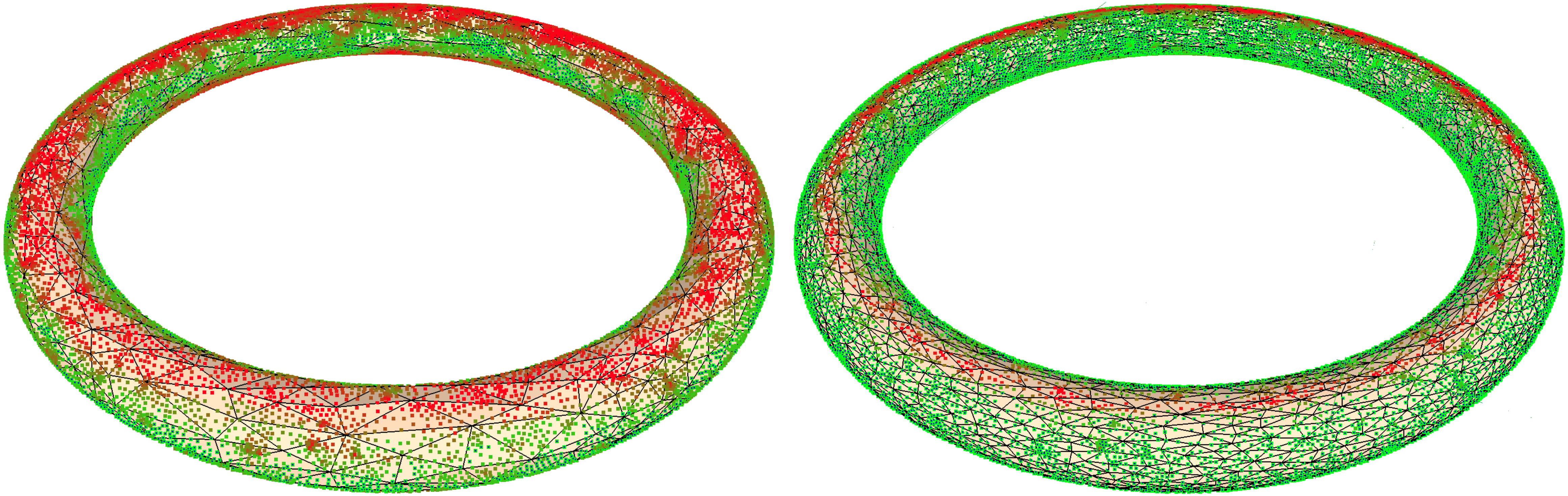
- bound on the eigenvalues: ε



# Ongoing work

## Curvature metric field variations

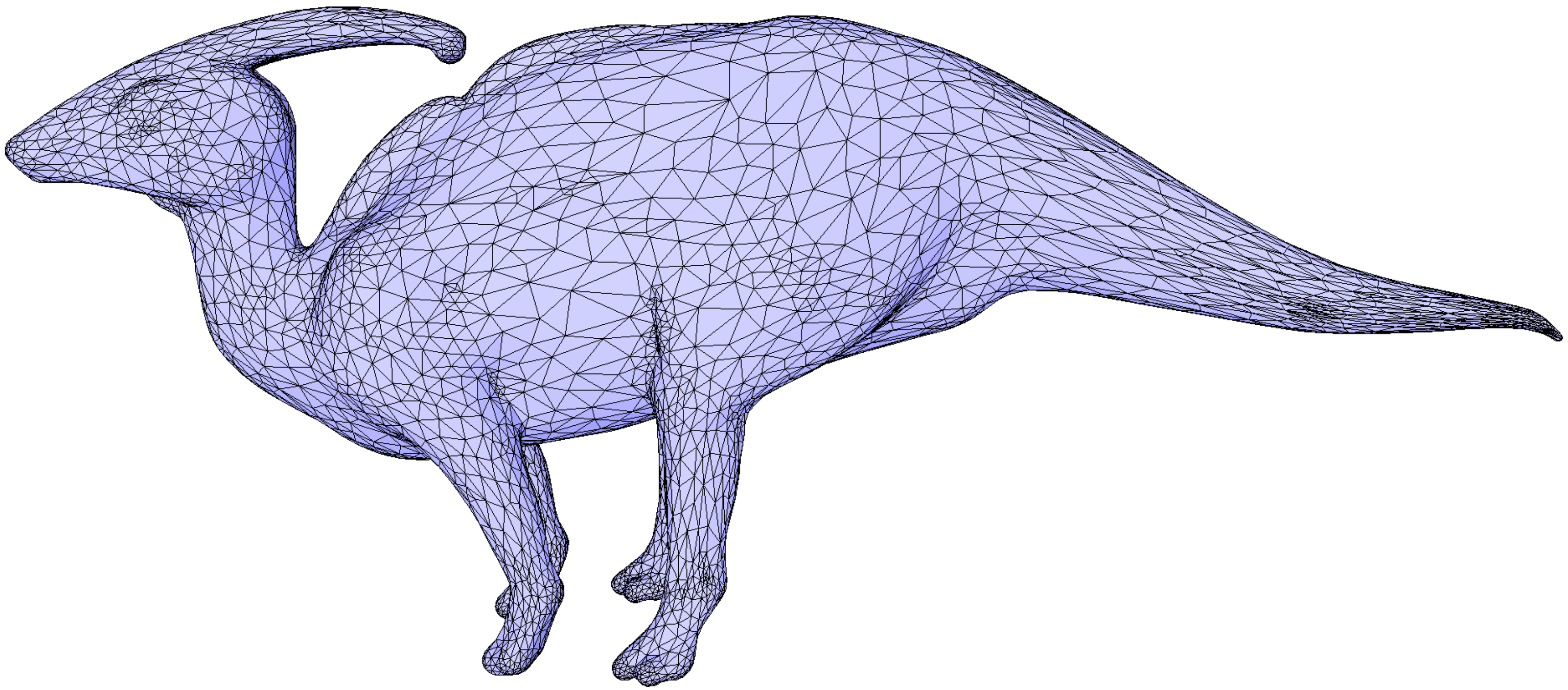
- heuristic approach: rebuild an adapted metric field



# Ongoing work

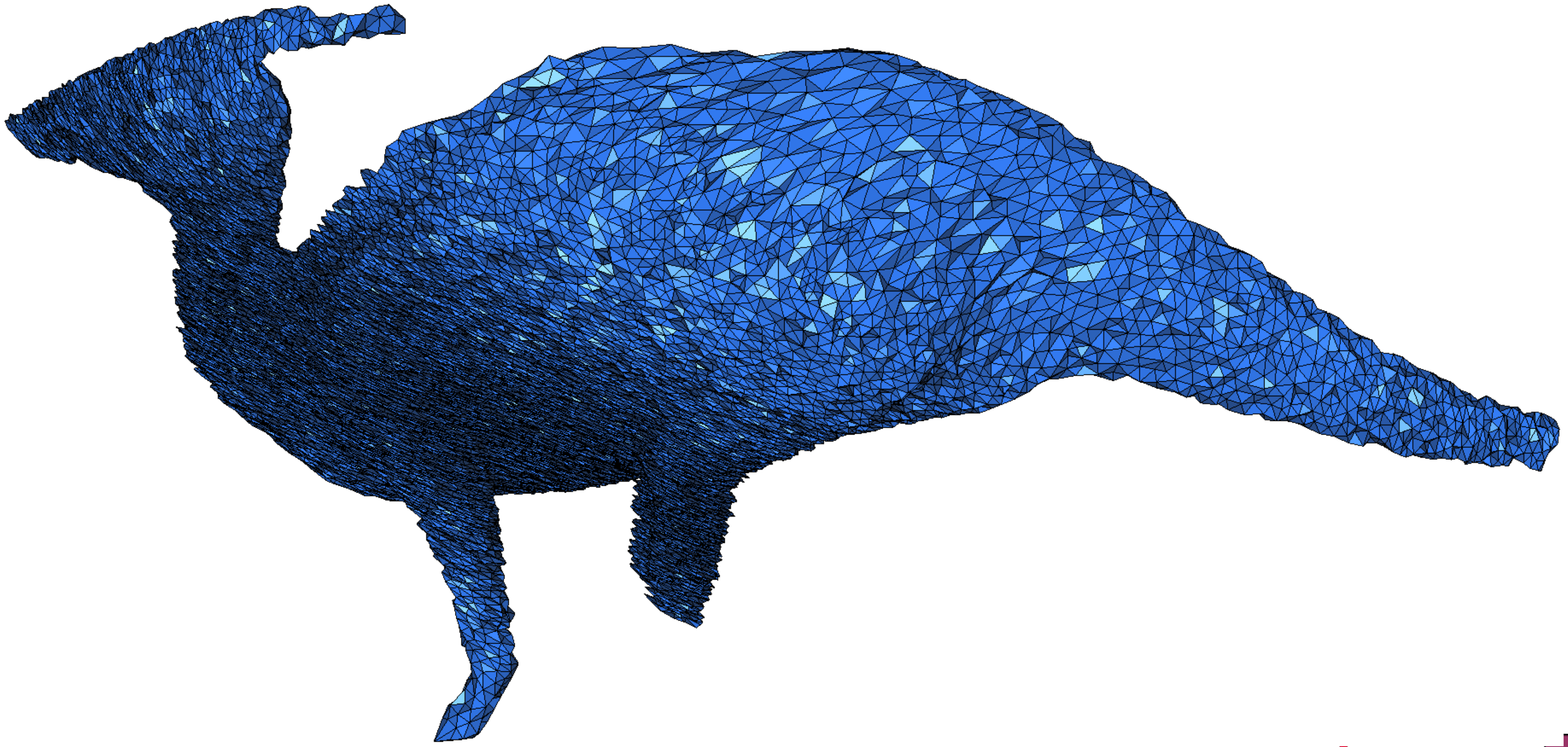
## Curvature metric field variations

- smoothing approach: locally smooth to lower the distortion to an “acceptable” value



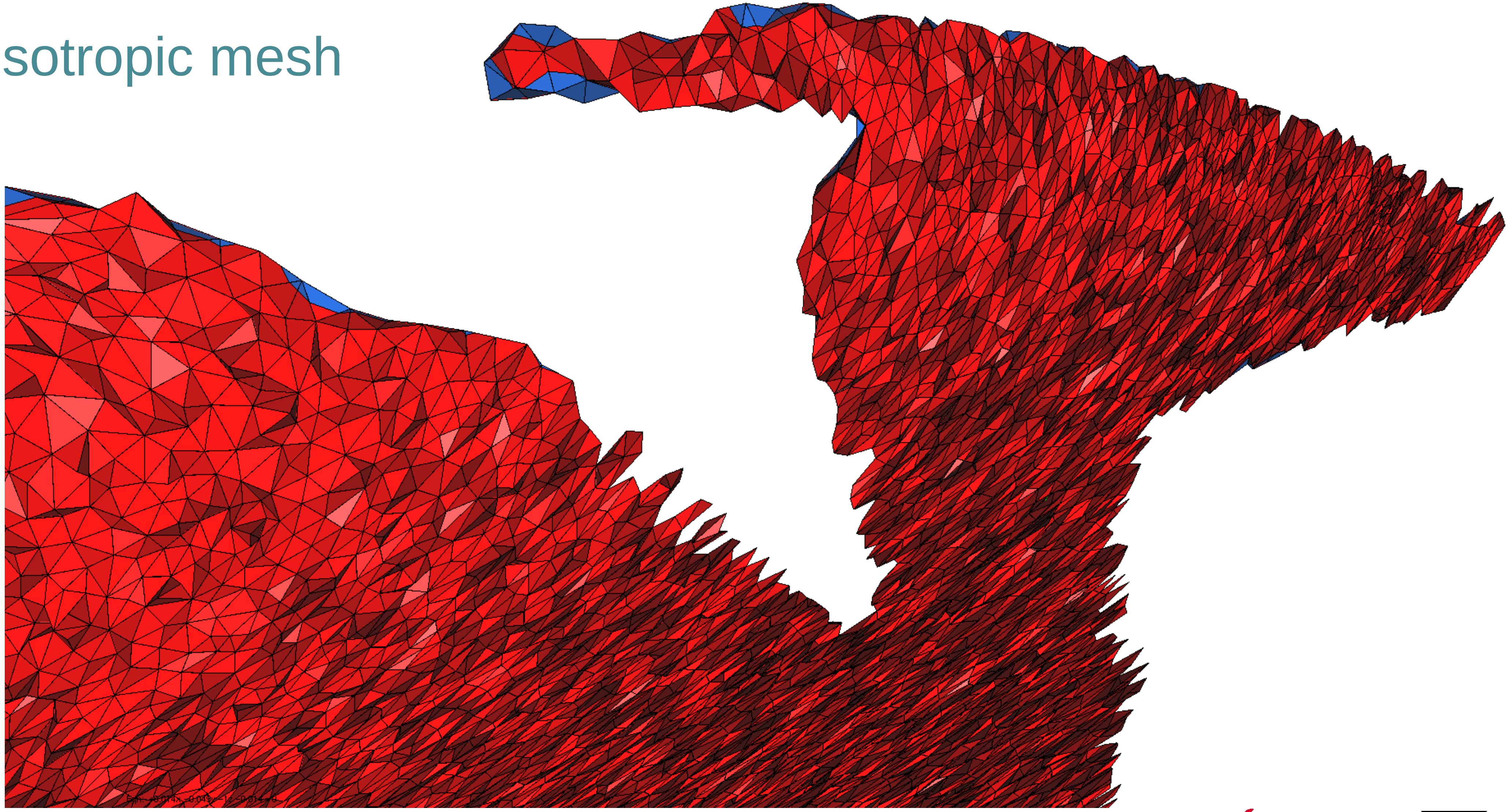
# Ongoing & Future work

## 3D anisotropic mesh



# Ongoing & Future work

## 3D anisotropic mesh



# Future work

- handle boundaries and sharp features
- handle discontinuous metric fields
- parallelize

# Thank you !