Low Complexity Regularization of Inverse Problems

Cours #2
Recovery Guarantees

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Overview of the Course

- Course #1: Inverse Problems

- **Course #2: Recovery Guarantees**

- Course #3: Proximal Splitting Methods

Overview

- Low-complexity Regularization with Gauges

- Performance Guarantees

- Grid-free Regularization

Inverse Problem Regularization

$Observations:\ y = \Phi x_0 + w \in \mathbb{R}^P.$

$Estimator:\ x(y)$ depends only on → observations y ; → parameter λ

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Data fidelity — Regularity

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Noise level $\|w\|$	⟷	Regularity of x_0 $J(x_0)$

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No noise: $\lambda \to 0^+$, minimize $x^\star \in \underset{x \in \mathbb{R}^Q, \mathcal{K}x = y}{\text{argmin}} J(x)$

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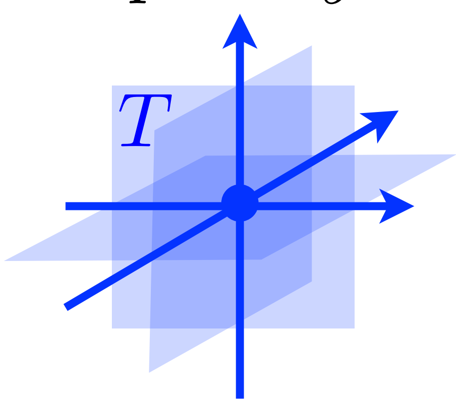
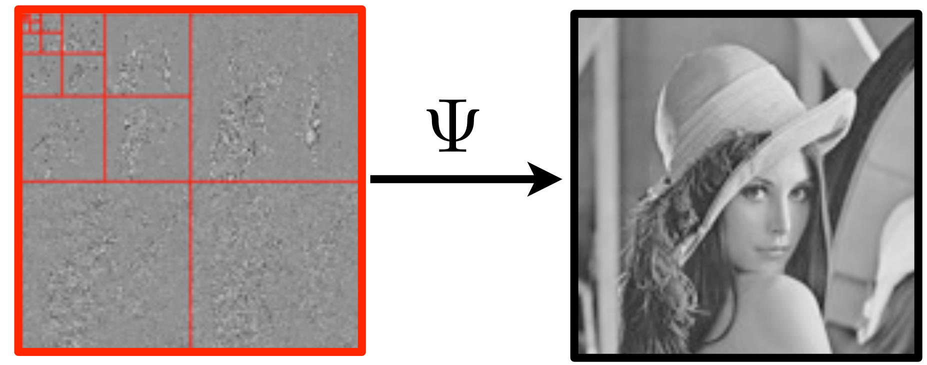
This course:
- **Performance analysis.**
- **Fast computational scheme.**

Union of Linear Models for Data Processing

Union of models: $T \in \mathcal{T}$ linear spaces.

Synthesis sparsity:





Coefficients x Image Ψx

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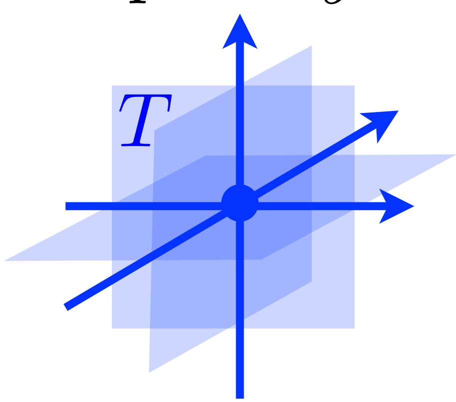

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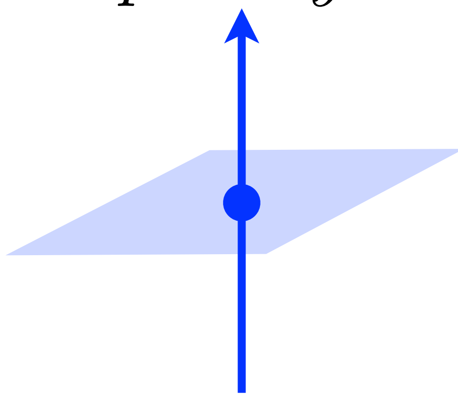

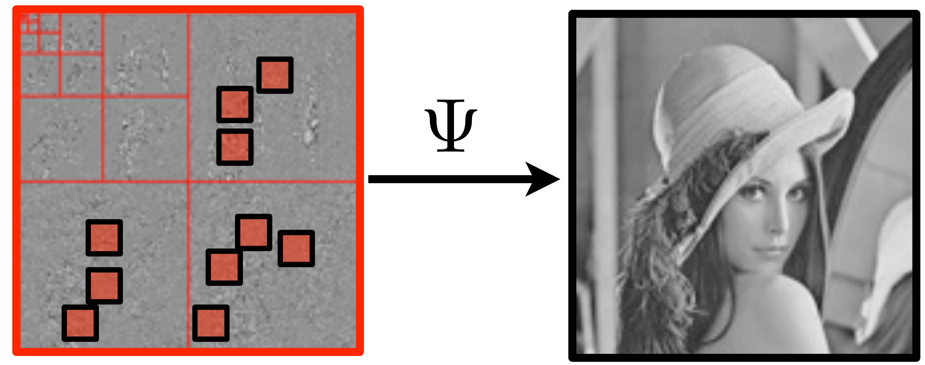

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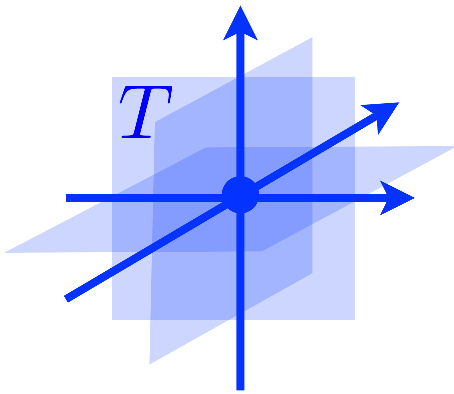

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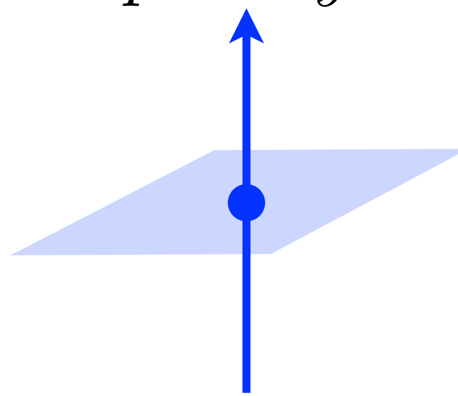

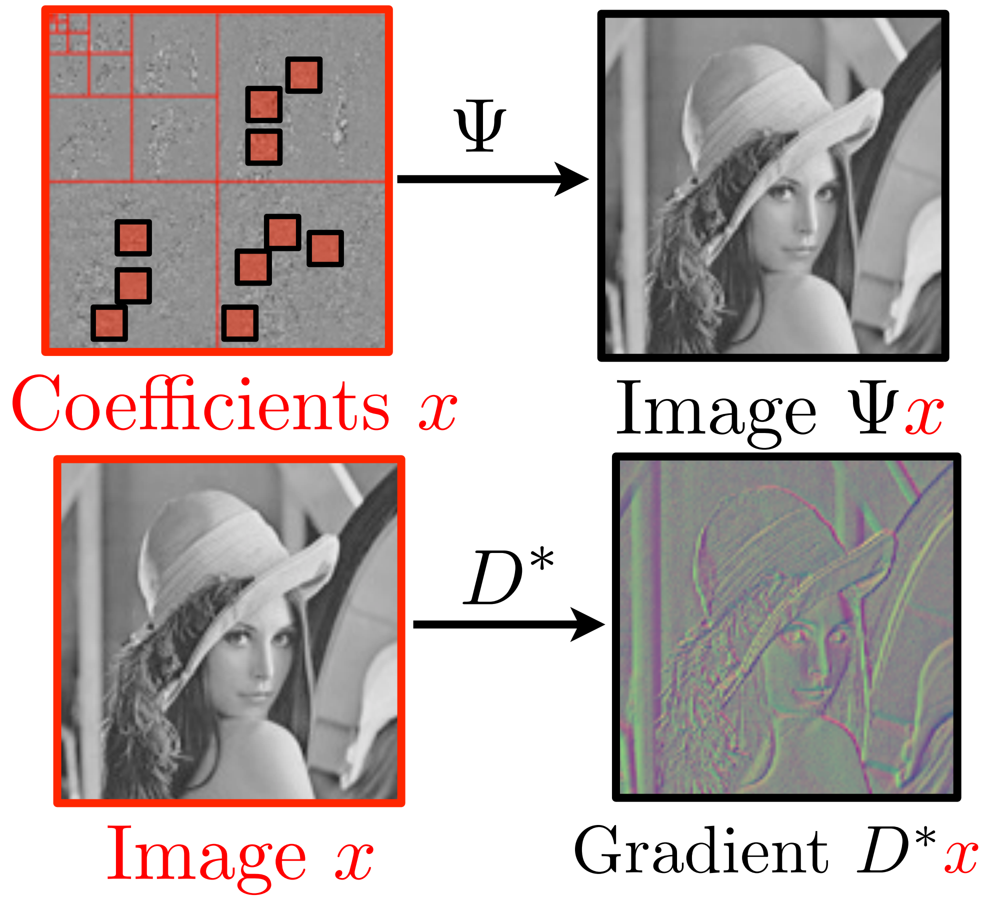

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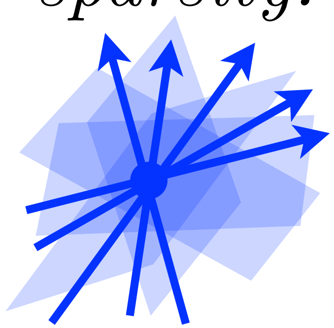

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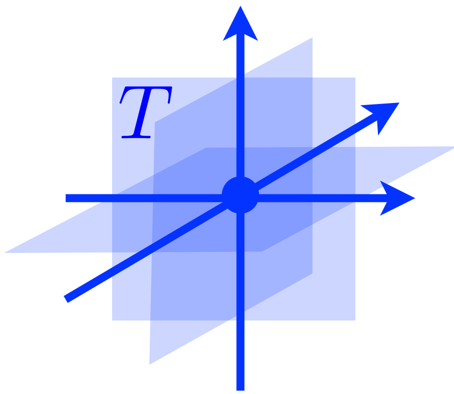

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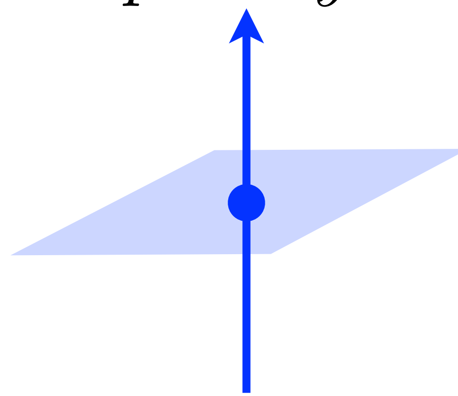

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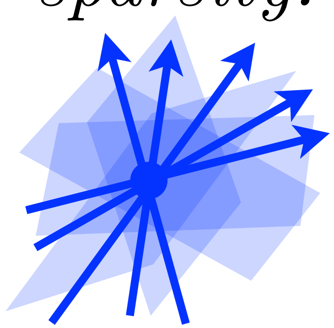
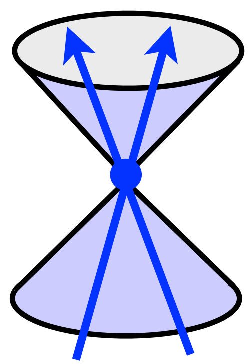
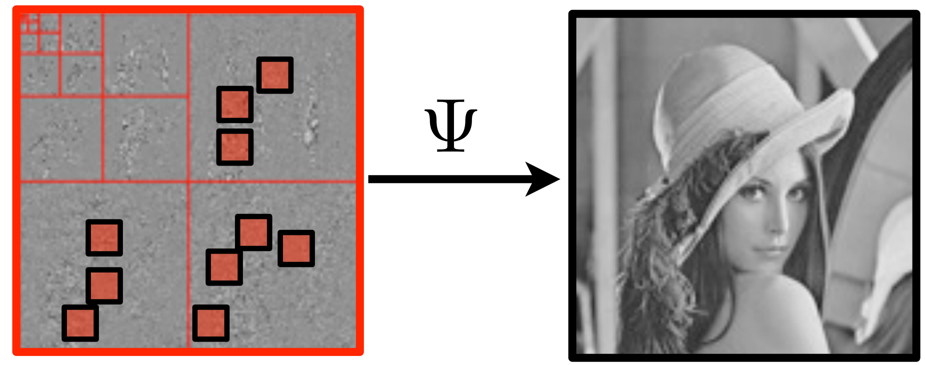


Low-rank:





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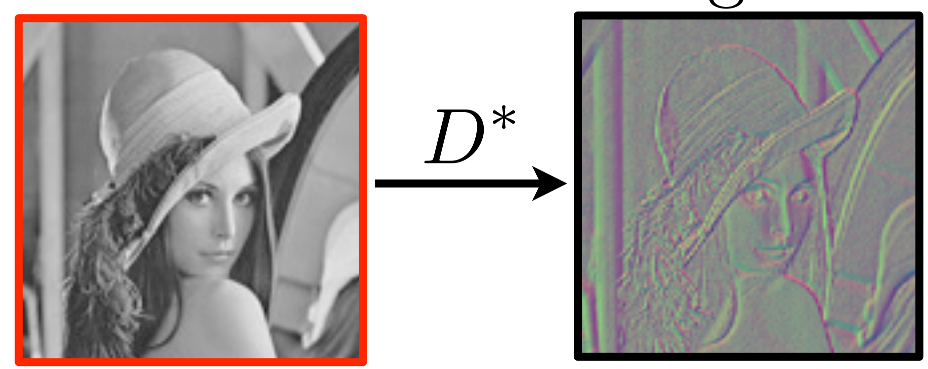
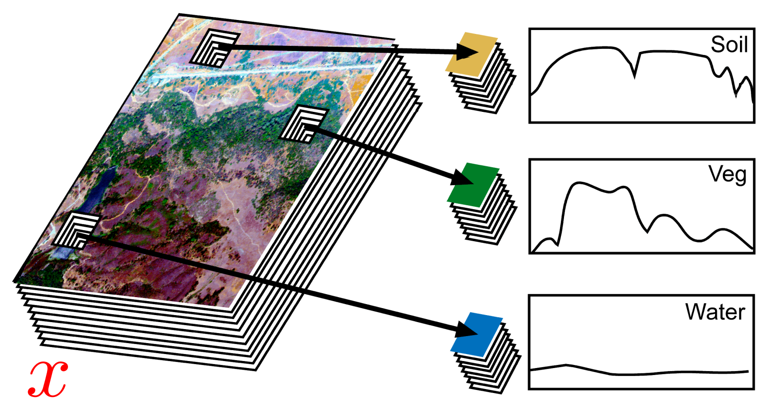


Image x Gradient $D^* x$

Multi-spectral imaging:

$$x_{i,\cdot} = \sum_{j=1}^{r} A_{i,j} S_{j,\cdot}$$



Gauges for Union of Linear Models

$$\textit{Gauge:} \quad J : \mathbb{R}^N \to \mathbb{R}^+ \quad \left| \begin{array}{l} \text{Convex} \\ \forall\, \alpha \in \mathbb{R}^+,\ J(\alpha x) = \alpha J(x) \end{array} \right.$$

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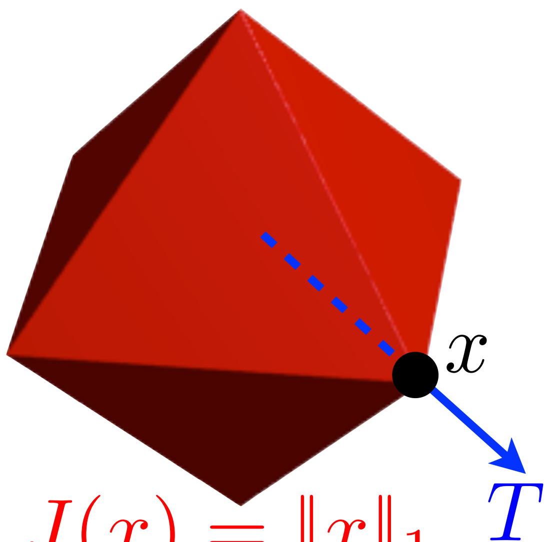
Piecewise regular ball $\Leftrightarrow$ Union of linear models $(T)_{T \in \mathcal{T}}$



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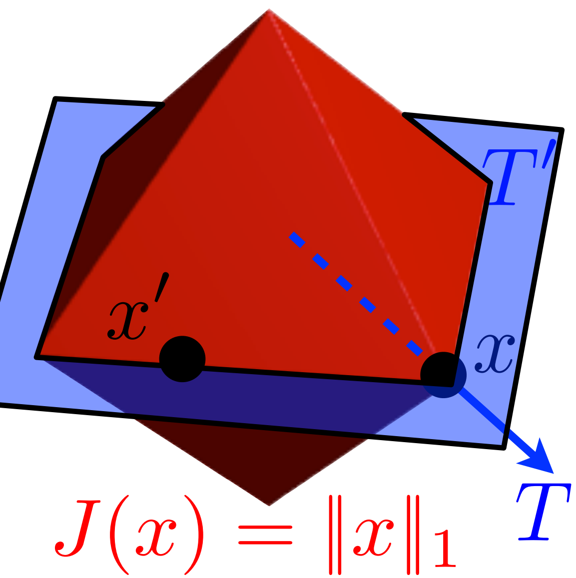

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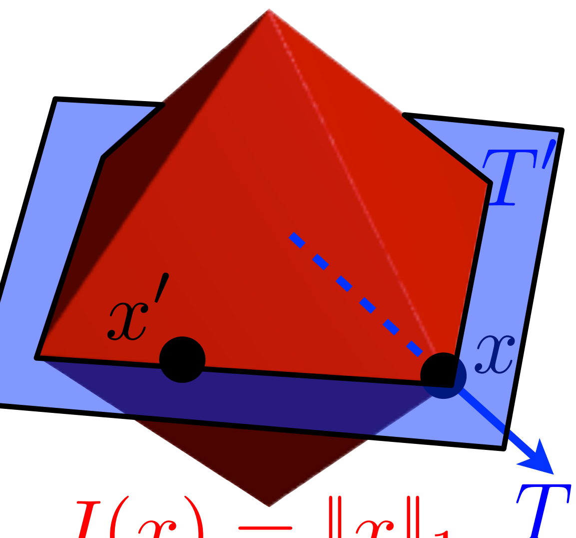
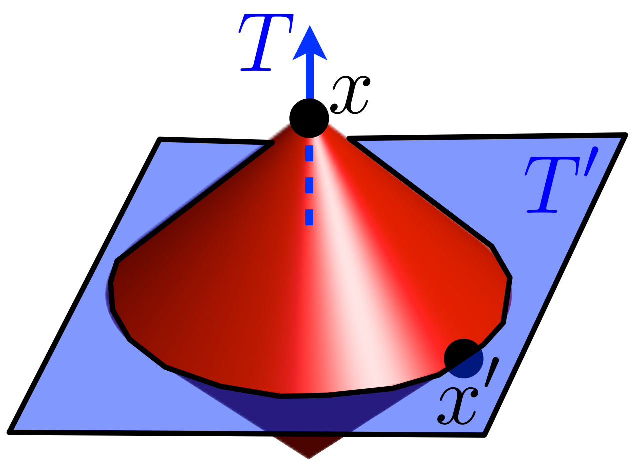


$J(x) = \|x\|_1$

$\mathcal{T}$ = sparse vectors



$|x_1| + \|x_{2,3}\|$

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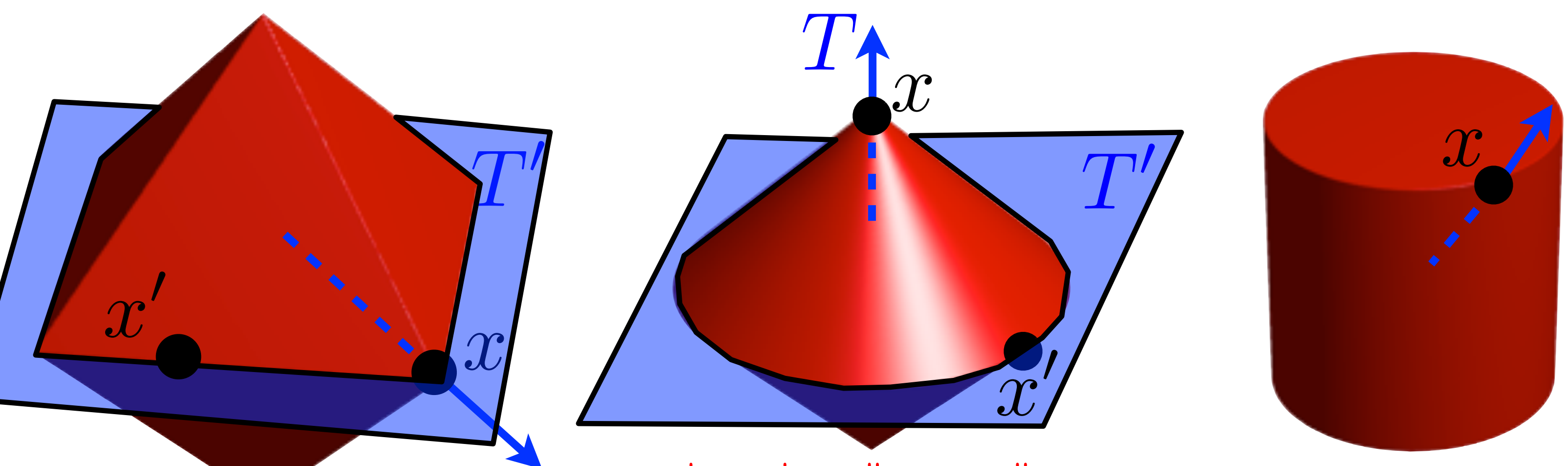

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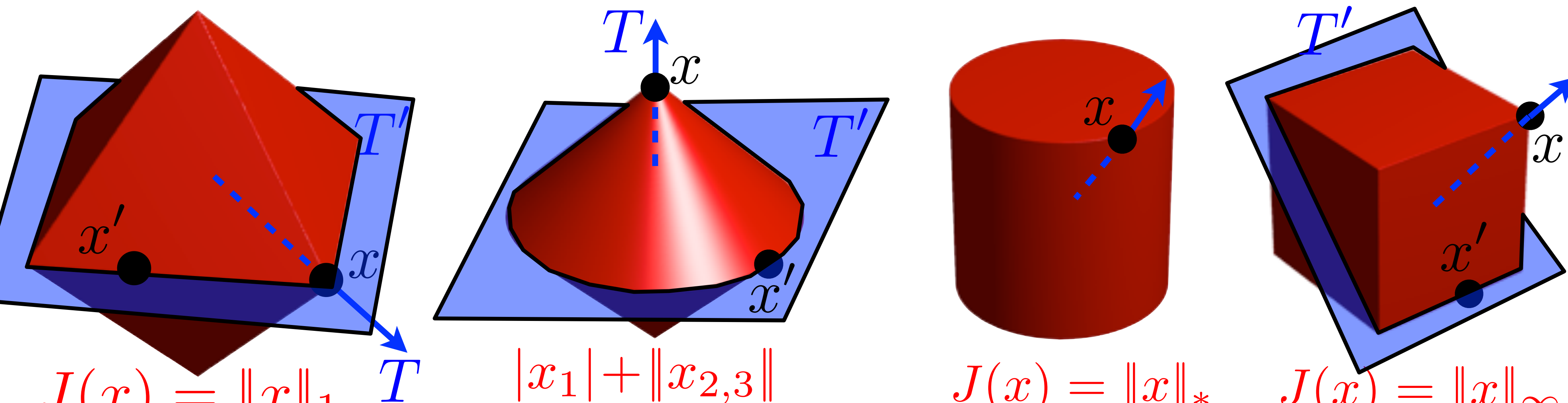

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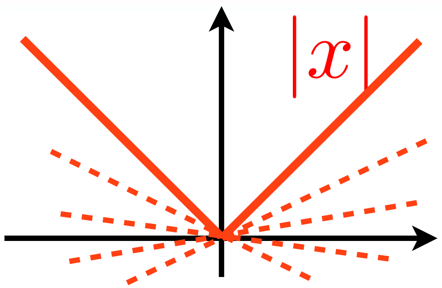
$J(x) = \|x\|_\infty$ — $\mathcal{T}$ = anti-sparse vectors

Subdifferentials and Models

$$\partial J(x) = \{\eta \setminus \forall\, y, J(y) \geqslant J(x) + \langle \eta,\, y - x\rangle\}$$



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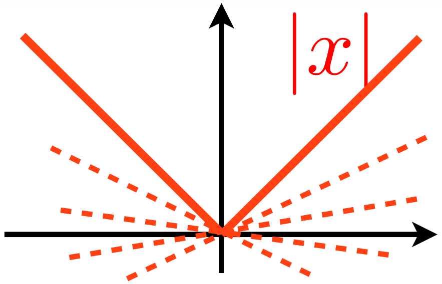

Example: $J(x) = \|x\|_1$

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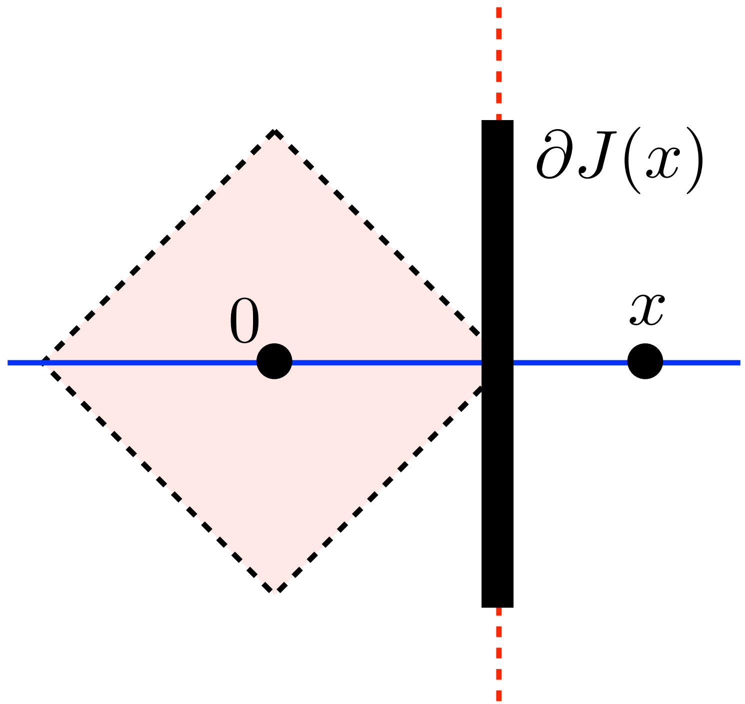

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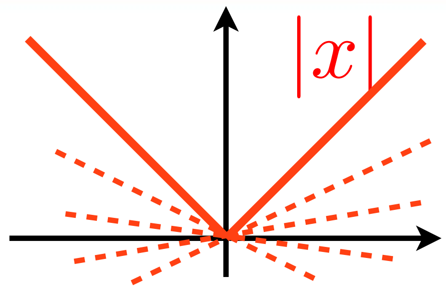

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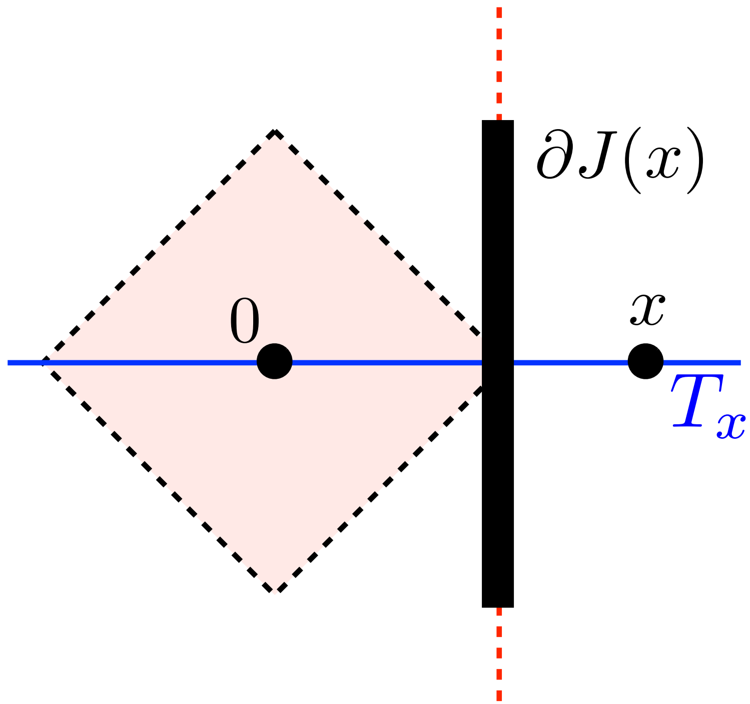

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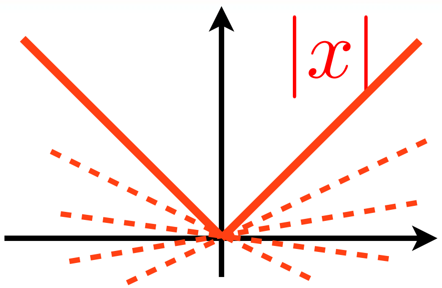

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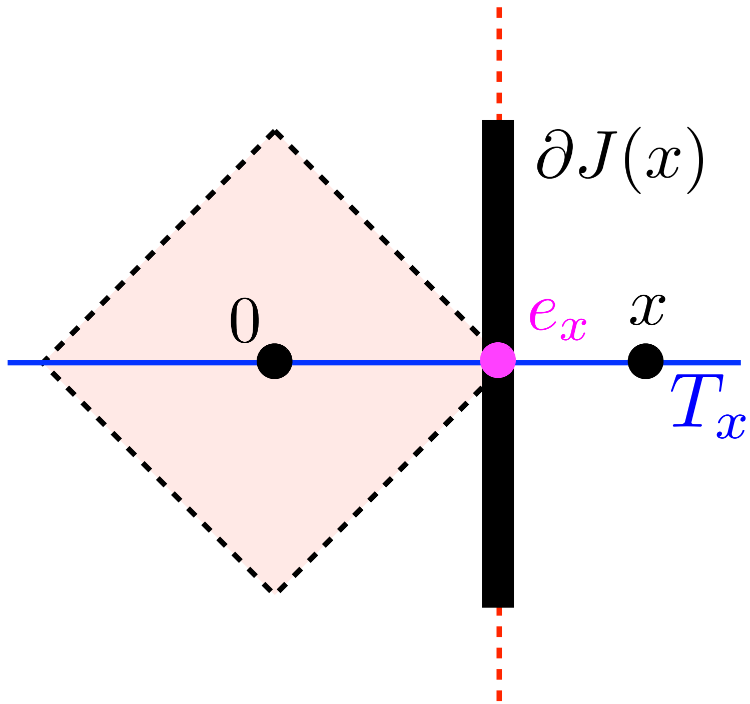

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$$\eta \in \partial J(x) \implies \operatorname{Proj}_{T_x}(\eta) = e_x$$

Examples

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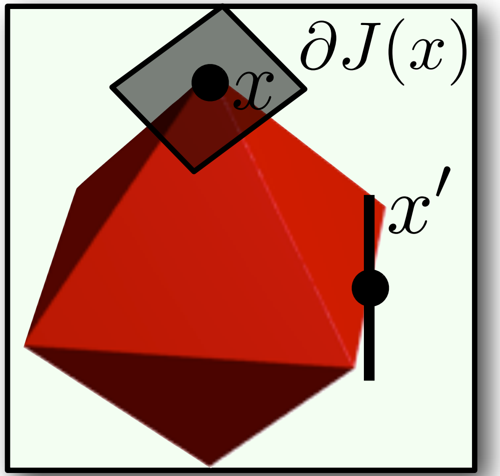

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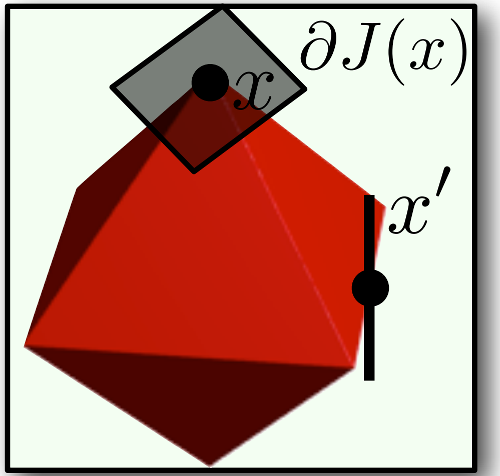

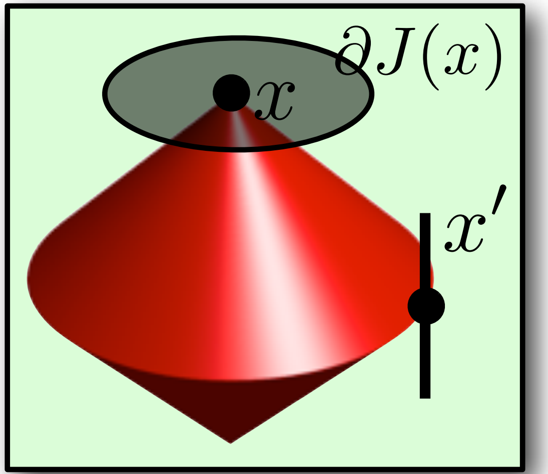

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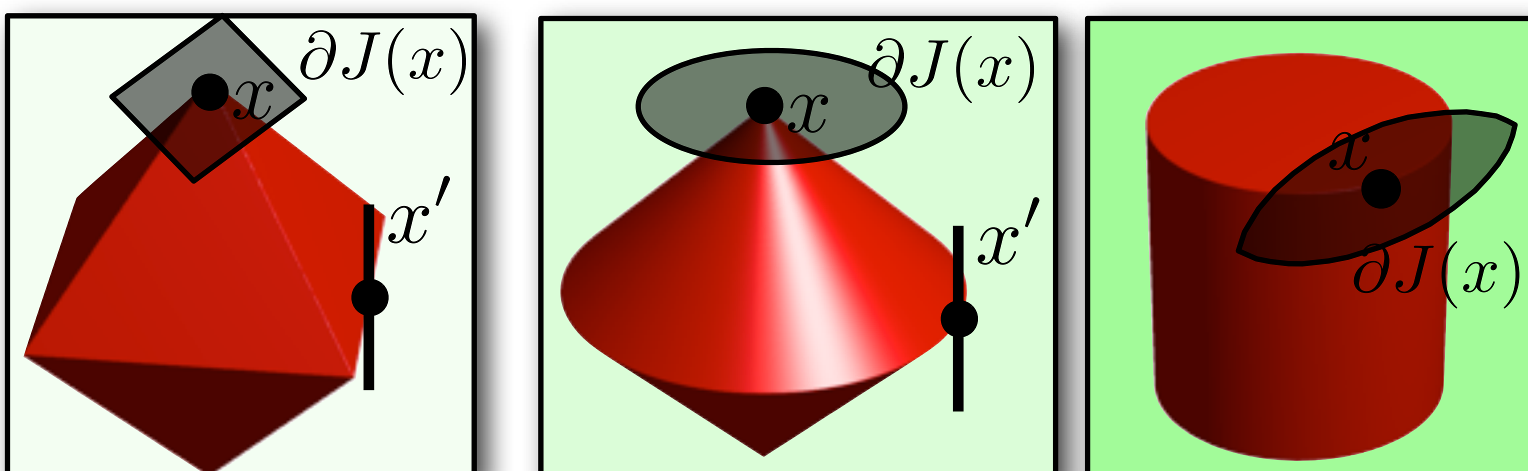
Nuclear norm: $J(x) = \|x\|_*$ $\qquad$ *SVD:* $x = U \Lambda V^*$

$e_x = UV^*$ $\qquad$ $T_x = \left\{UA + BV^* \setminus (A, B) \in (\mathbb{R}^{n \times n})^2\right\}$



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Anti-sparsity: $J(x) = \|x\|_\infty$ $\quad I = \{i \setminus |x_i| = \|x\|_\infty\}$

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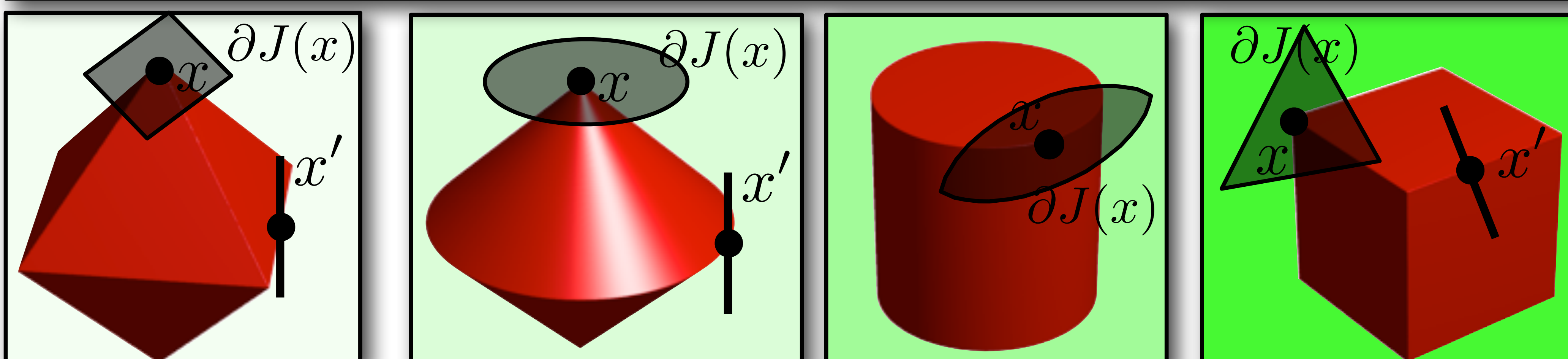

Overview

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- Performance Guarantees

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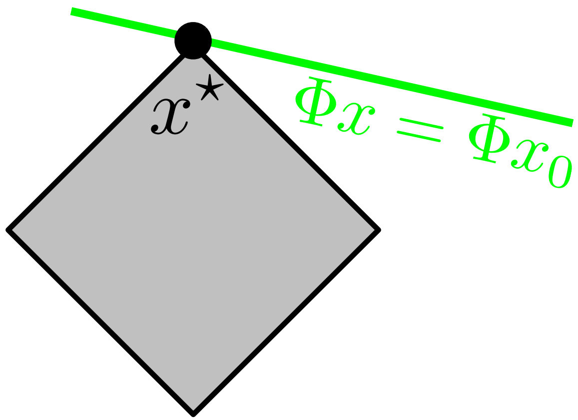
Dual Certificates

Noiseless recovery: $\min_{\Phi x = \Phi x_0} J(x) \qquad (\mathcal{P}_0)$



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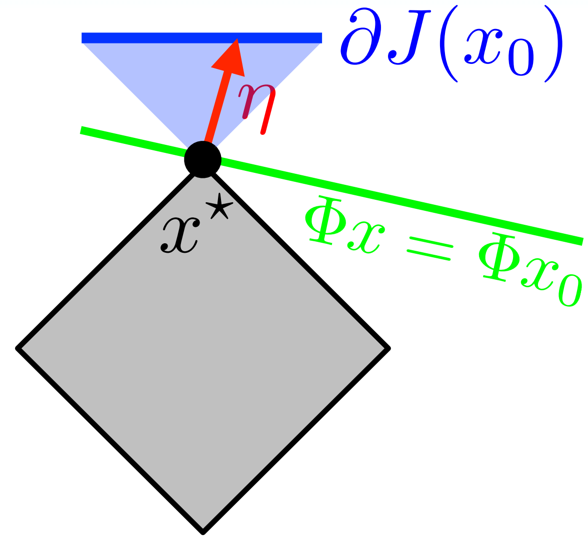

Proposition:
x_0 solution of $(\mathcal{P}_0) \iff \exists\, \eta \in \mathcal{D}(x_0)$

Dual certificates: $\mathcal{D}(x_0) = \mathrm{Im}(\Phi^*) \cap \partial J(x_0)$

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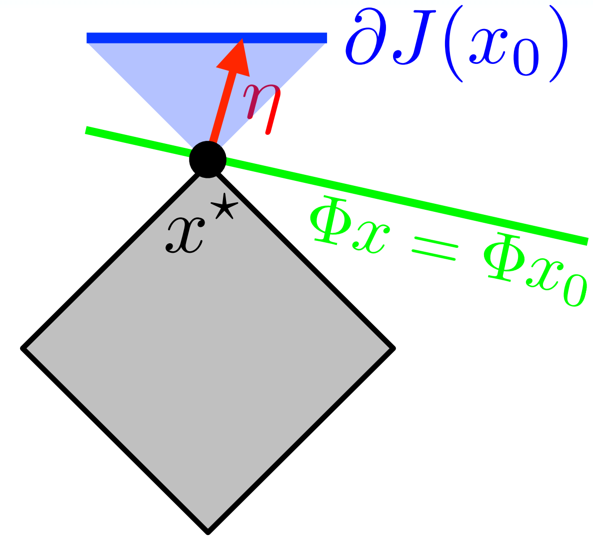

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Proof: $(\mathcal{P}_0) \iff \min_{\delta \in \ker(\Phi)} J(x_0 + \delta)$

$\forall\, (\eta, \delta) \in \partial J(x_0) \times \ker(\Phi), \quad J(x_0 + \delta) \geqslant J(x_0) + \langle \delta,\, \eta \rangle$

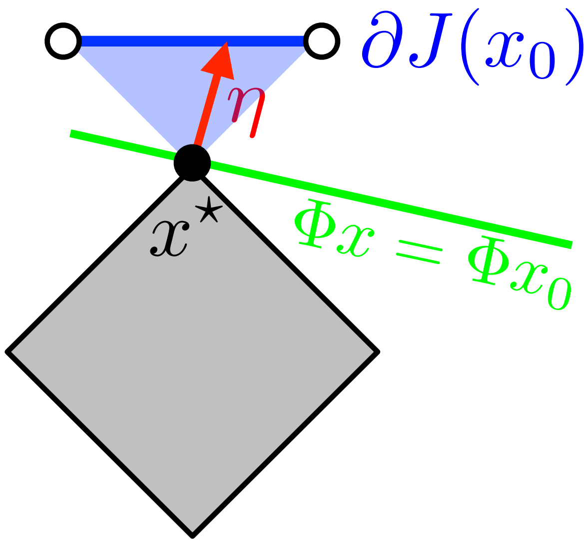
$\eta \in \mathrm{Im}(\Phi^*) \implies \langle \delta,\, \eta \rangle = 0 \implies x_0$ solution.

x_0 solution $\implies \forall\, \delta,\ \langle \delta,\, \eta \rangle \leqslant 0 \implies \eta \in \ker(\Phi)^\perp$.

Dual Certificates and L2 Stability

Tight dual certificates:

$$\bar{\mathcal{D}}(x_0) = \mathrm{Im}(\Phi^*) \cap \mathrm{ri}(\partial J(x_0))$$



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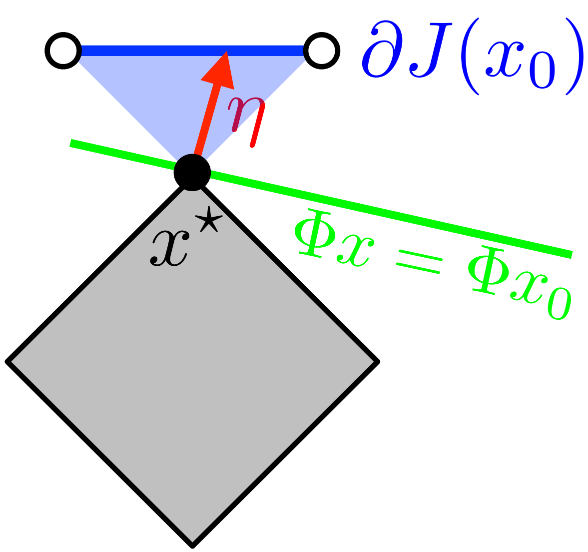

Theorem: [Fadili et al. 2013]

If $\exists\, \eta \in \bar{\mathcal{D}}(x_0)$, for $\lambda \sim \|w\|$ one has $\|x^\star - x_0\| = O(\|w\|)$

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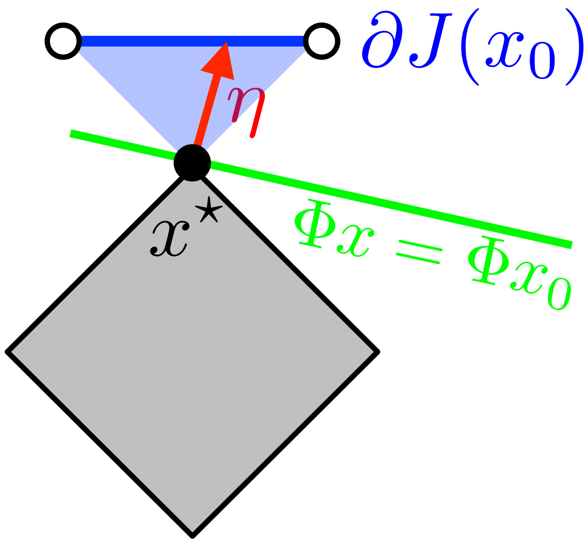


Theorem: [Fadili et al. 2013]

If $\exists\, \eta \in \bar{\mathcal{D}}(x_0)$, for $\lambda \sim \|w\|$ one has $\|x^\star - x_0\| = O(\|w\|)$

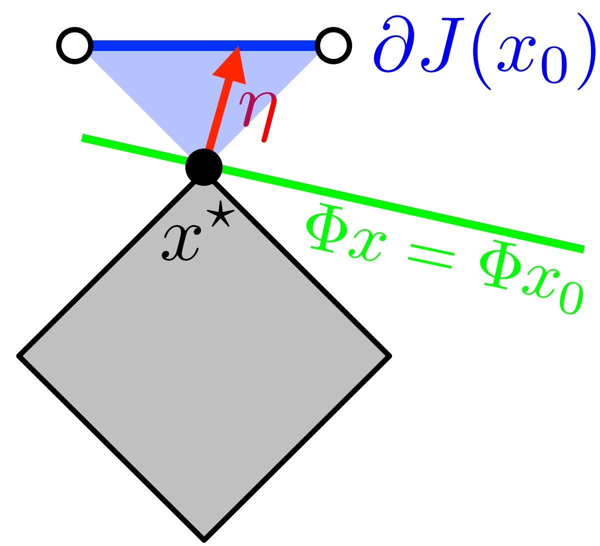
[Grassmair, Haltmeier, Scherzer 2010]: $J = \|\cdot\|_1$.

[Grassmair 2012]: $J(x^\star - x_0) = O(\|w\|)$.

Dual Certificates and L2 Stability

Tight dual certificates:

$$\bar{\mathcal{D}}(x_0) = \mathrm{Im}(\Phi^*) \cap \mathrm{ri}(\partial J(x_0))$$



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$\longrightarrow$ The constants depend on N ...

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Random matrix: $\Phi \in \mathbb{R}^{P \times N}, \quad \Phi_{i,j} \sim \mathcal{N}(0,1)$, i.i.d.

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[Rudelson, Vershynin 2006]
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Theorem: Let $s = \|x_0\|_0$. If

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Then $\exists \eta \in \bar{\mathcal{D}}(x_0)$ with high probability on Φ.

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Low-rank matrices: $J = \|\cdot\|_*$.

Theorem: Let $r = \text{rank}(x_0)$. If

[Chandrasekaran et al. 2011]

$$P \geqslant 3r(N_1 + N_2 - r)$$

$x_0 \in \mathbb{R}^{N_1 \times N_2}$

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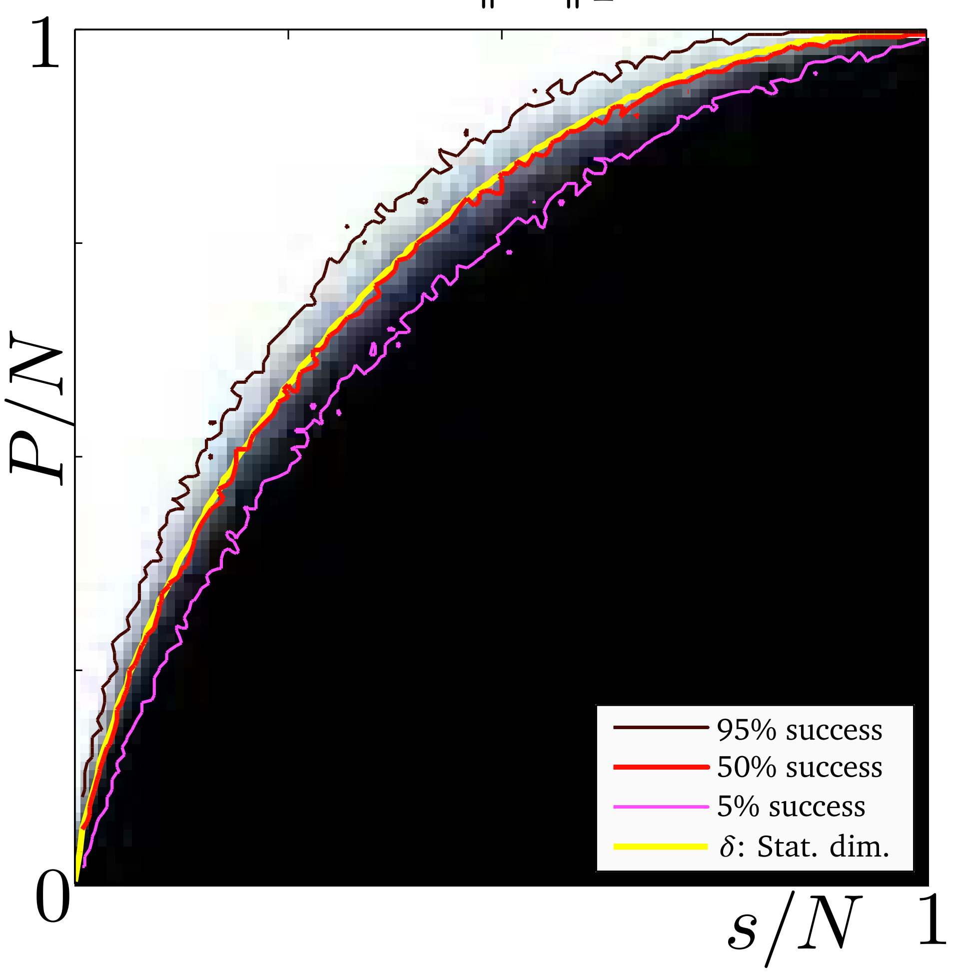
Phase Transitions

$J = \|\cdot\|_1$



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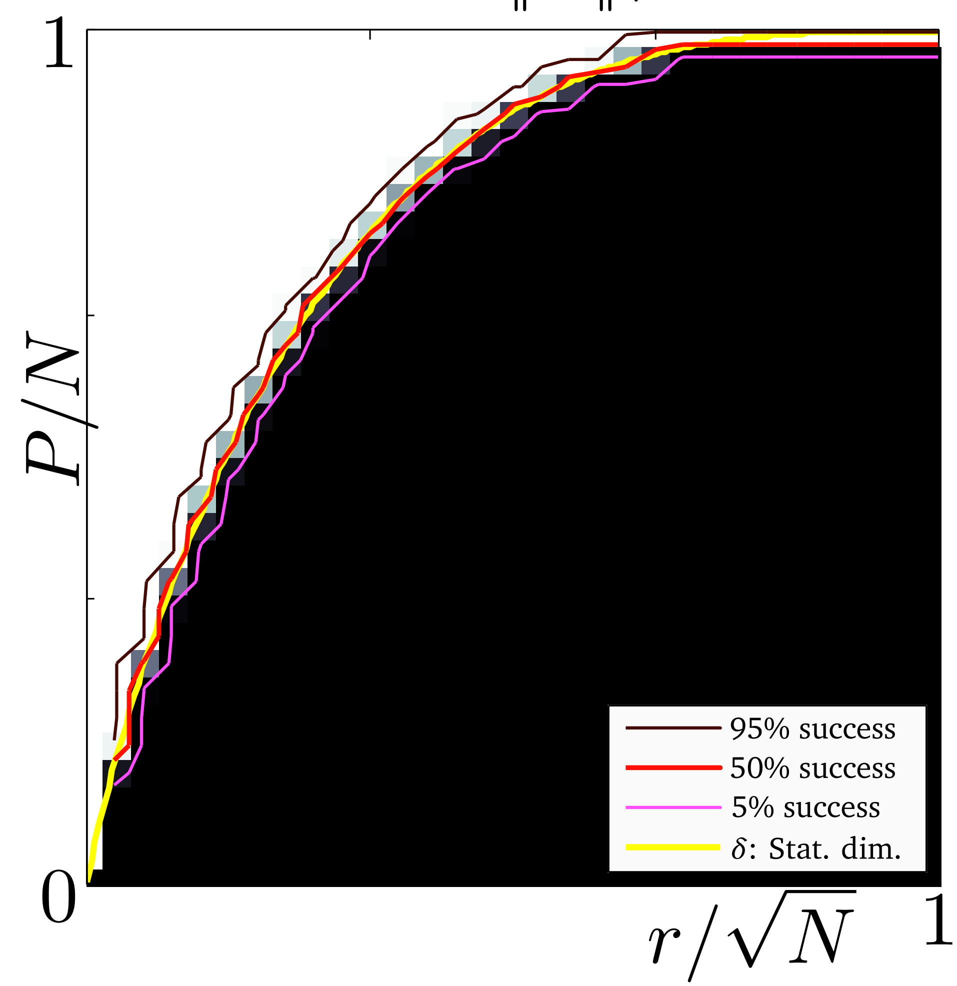

From [Amelunxen et al. 20013]

Minimal-norm Certificate

$$\eta \in \mathcal{D}(x_0) \quad \Longrightarrow \quad \begin{cases} \eta = \Phi^* q \\ \text{Proj}_T(\eta) = e \end{cases}$$

$T = T_{x_0}$
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$T_{x^\star} = T_{x_0}$ and $\|x^\star - x_0\| = O(\|w\|)$ [Vaiter et al. 2013]

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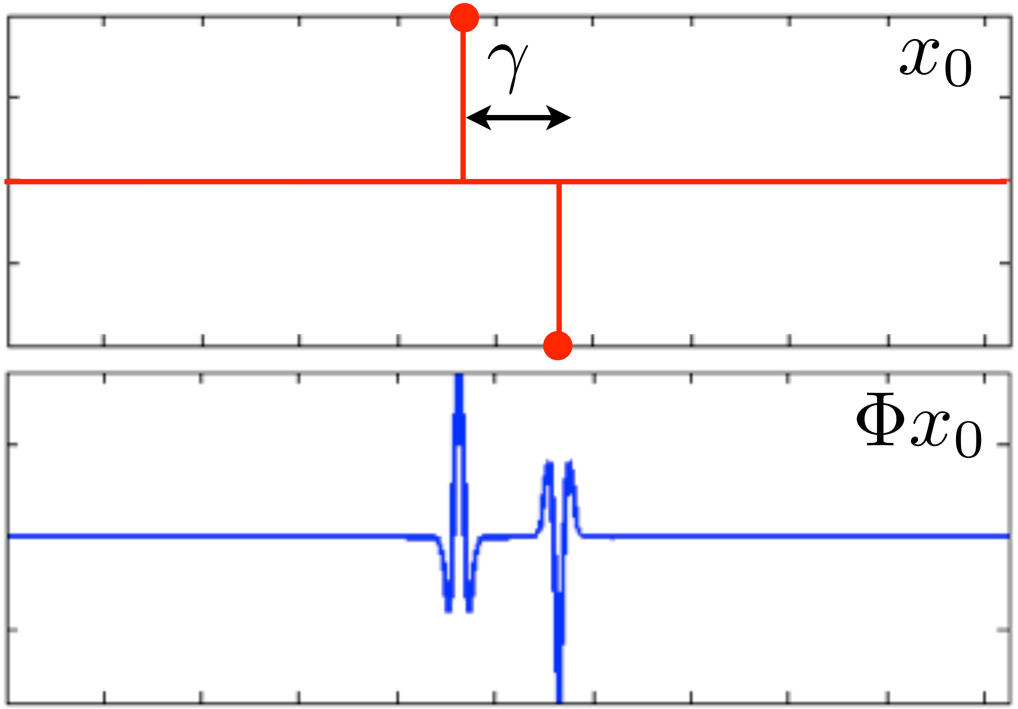
$\longrightarrow$ Similar results for $\|\cdot\|_{1,2}, \|\cdot\|_*, \|\cdot\|_\infty$.

$\longrightarrow$ Not using RIP technics (non-uniform result on x_0).

1-D Sparse Spikes Deconvolution

$$\Phi x = \sum_i x_i \varphi(\cdot - \Delta i)$$

$$J(x) = \|x\|_1$$



Increasing Δ:

$\rightarrow$ reduces correlation.

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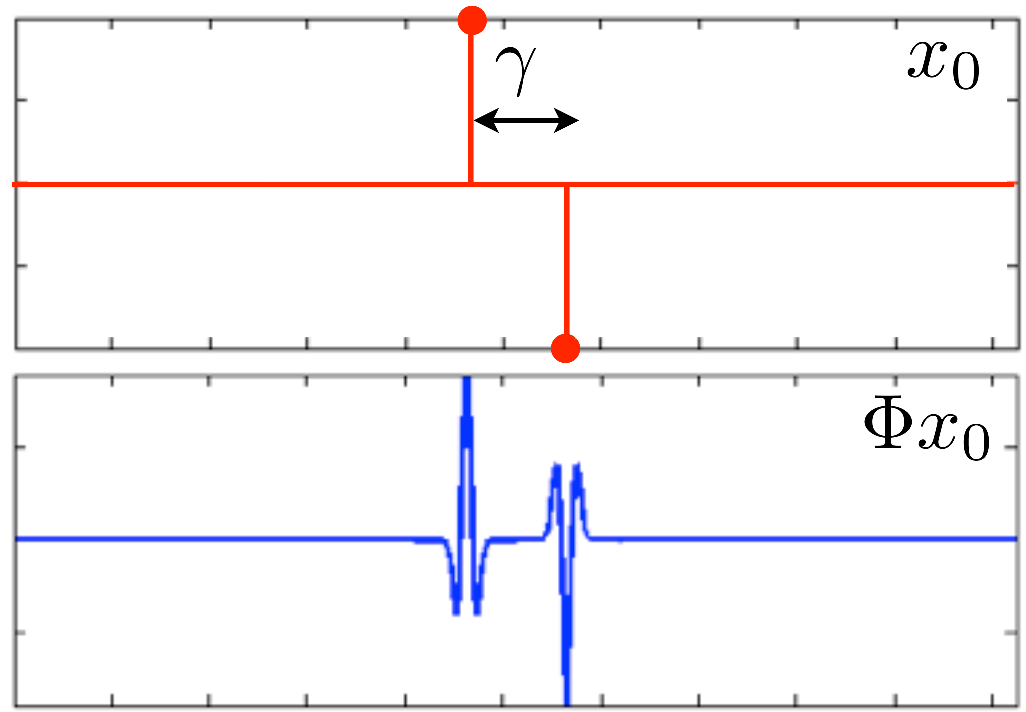
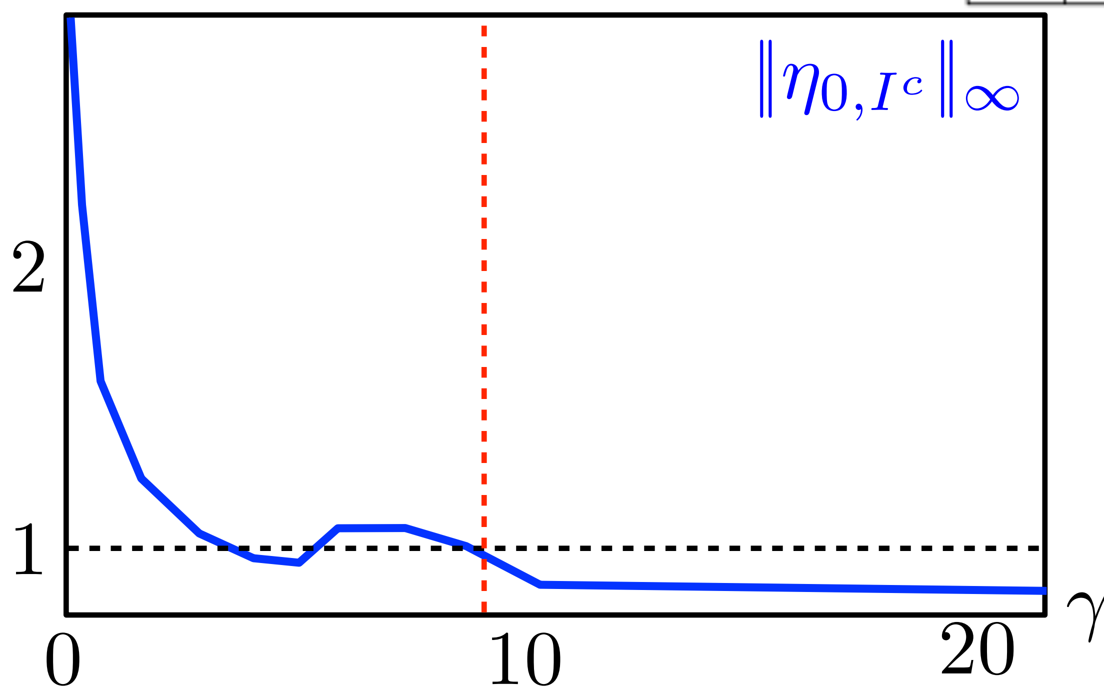




$$I = \{ j \setminus x_0(j) \neq 0 \}$$

$$\|\eta_{0,I^c}\|_\infty < 1$$

$$\Longleftrightarrow$$

$$\eta_0 \in \bar{\mathcal{D}}(x_0)$$

$$\Longleftrightarrow$$

support recovery.

Overview

- Low-complexity Regularization with Gauges

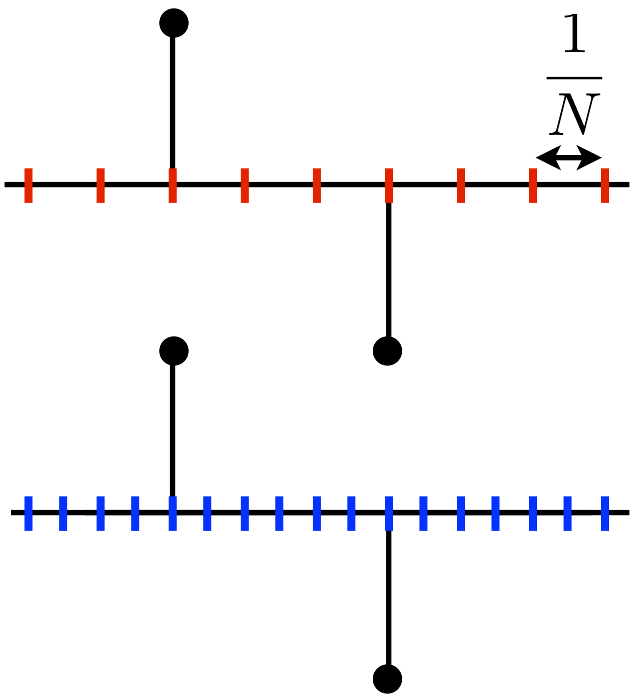
- Performance Guarantees

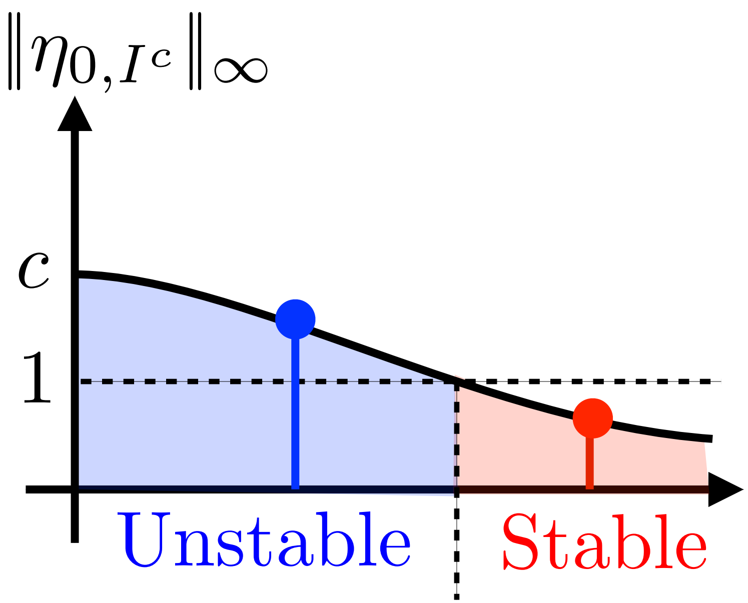
- **Grid-free Regularization**

Support Instability and Measures

When $N \to +\infty$, support is not stable:

$$\|\eta_{0,I^c}\|_\infty \underset{N \to +\infty}{\longrightarrow} c > 1.$$





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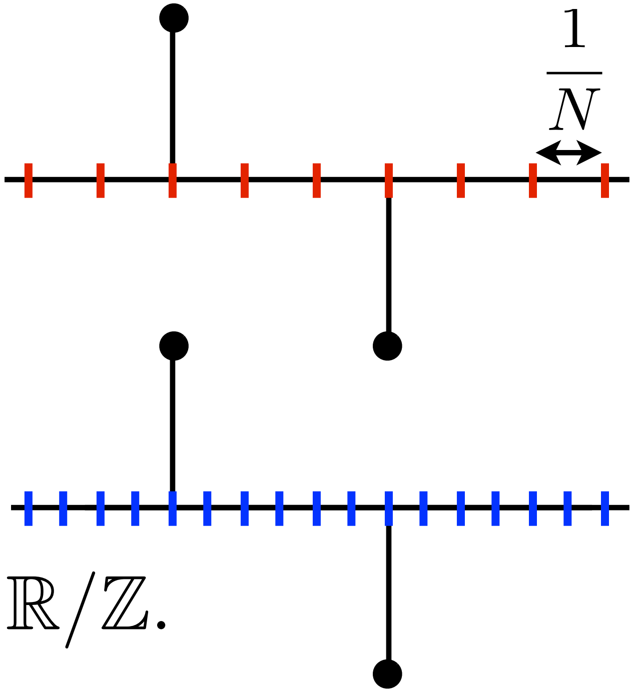
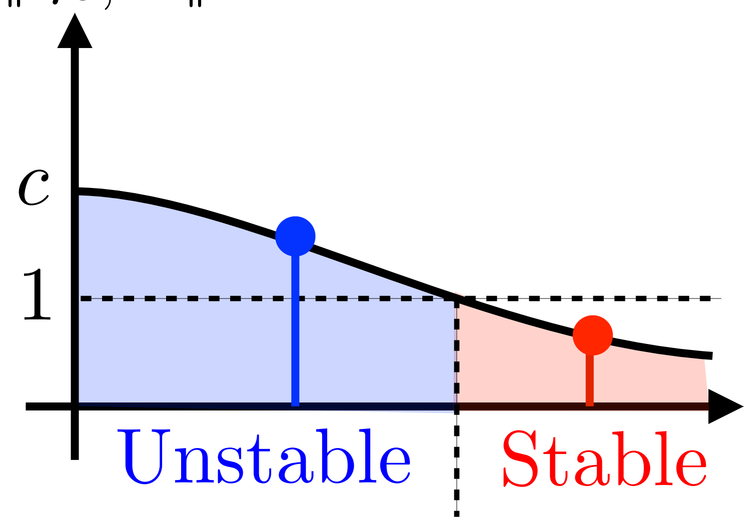


Intuition: spikes wants to move laterally.

$\longrightarrow$ Use Radon measures $m \in \mathcal{M}(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.



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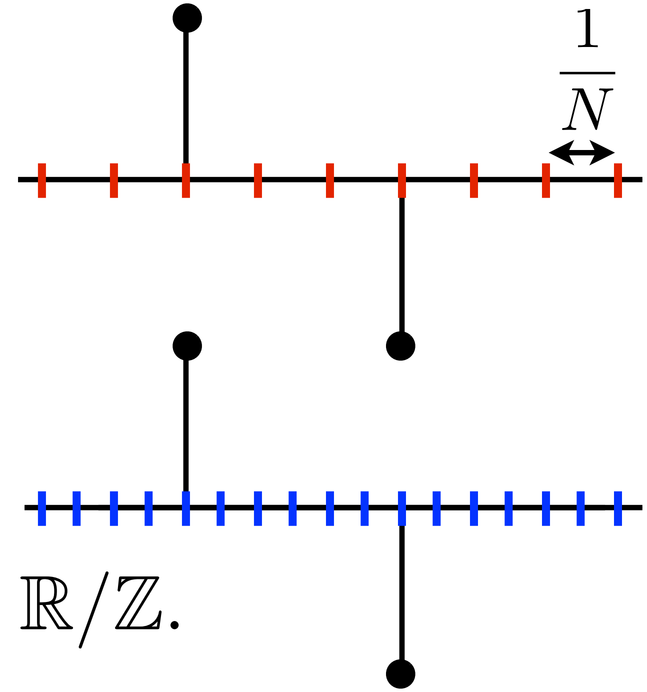

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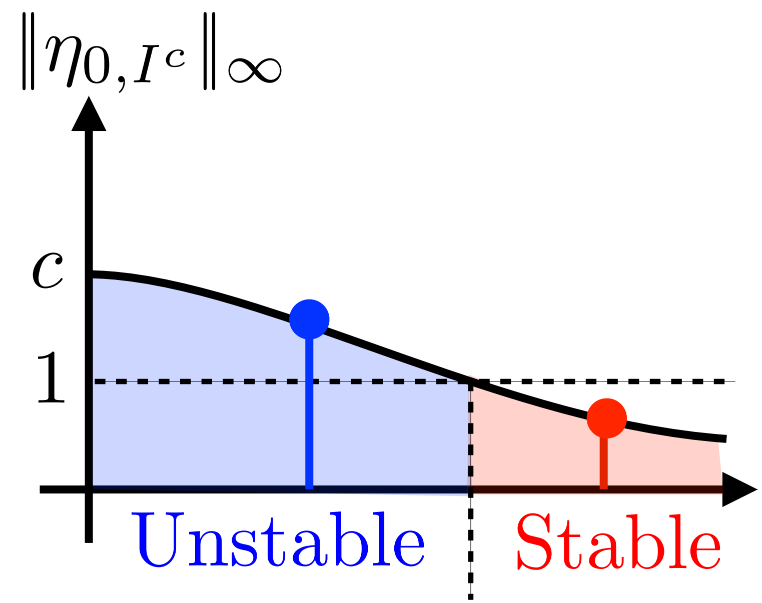
Extension of ℓ^1: total variation

$$\|m\|_{\text{TV}} = \sup_{\|g\|_\infty \leqslant 1} \int_{\mathbb{T}} g(x)\,\mathrm{d}m(x)$$



Discrete measure: $m_{x,a} = \sum_i a_i \delta_{x_i}$.

One has $\|m_{x,a}\|_{\text{TV}} = \|a\|_1$

Sparse Measure Regularization

Measurements: $y = \Phi(m_0) + w$ where $\begin{cases} m_0 \in \mathcal{M}(\mathbb{T}), \\ \Phi : \mathcal{M}(\mathbb{T}) \to \mathrm{L}^2(\mathbb{T}), \\ w \in \mathrm{L}^2(\mathbb{T}). \end{cases}$

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Acquisition operator:

$$\Phi(m)(x) = \int_{\mathbb{T}} \varphi(x, x') \mathrm{d}m(x') \quad \text{where} \quad \varphi \in C^2(\mathbb{T} \times \mathbb{T})$$

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$$\min_{m \in \mathcal{M}(\mathbb{T})} \frac{1}{2} \|\Phi(m) - y\|^2 + \lambda \|m\|_{\mathrm{TV}}$$

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$\longrightarrow$ Infinite dimensional convex program.

$\longrightarrow$ If $\dim(\mathrm{Im}(\Phi)) < +\infty$, dual is finite dimensional.

$\longrightarrow$ If Φ is a filtering, re-cast dual as SDP program.

Fuchs vs. Vanishing Pre-Certificates

Measures:

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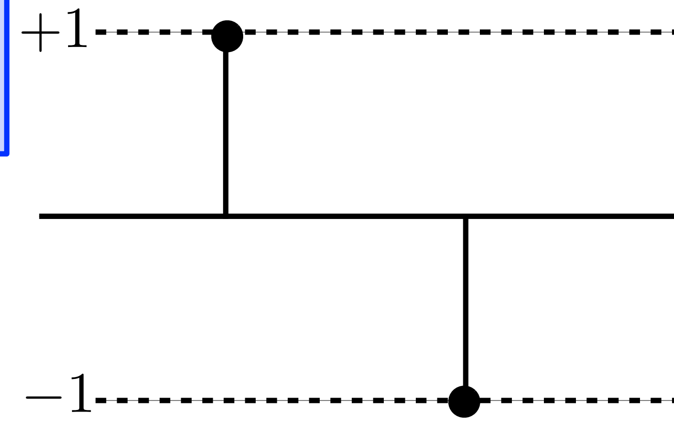

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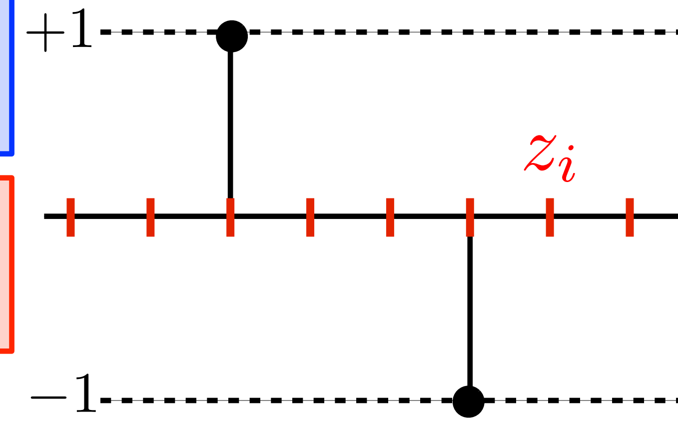

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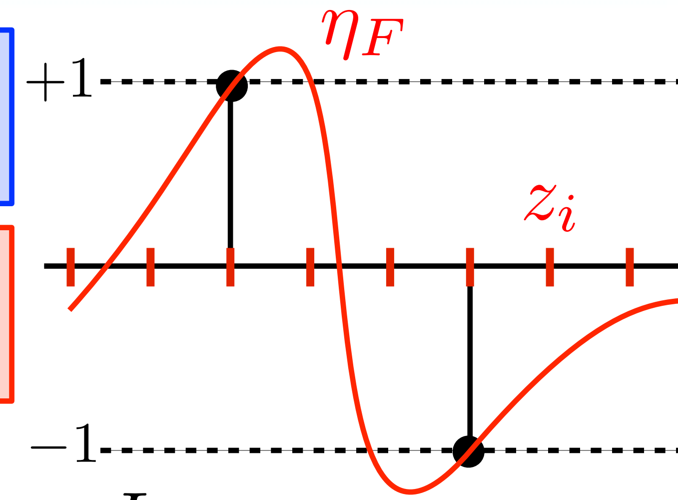

For $m_0 = m_{z,a_0}$, $\mathrm{supp}(m_0) = x_0$, $\mathrm{supp}(a_0) = I$:

$$\eta_F = \Phi^* \Phi_I^{*,+} \mathrm{sign}(a_{0,I})$$

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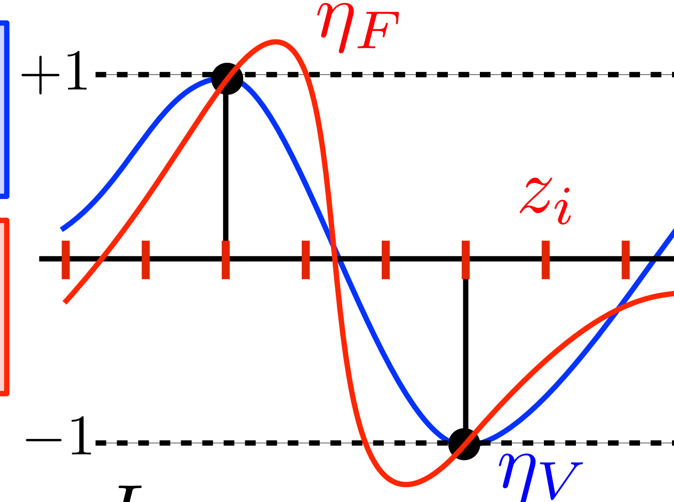

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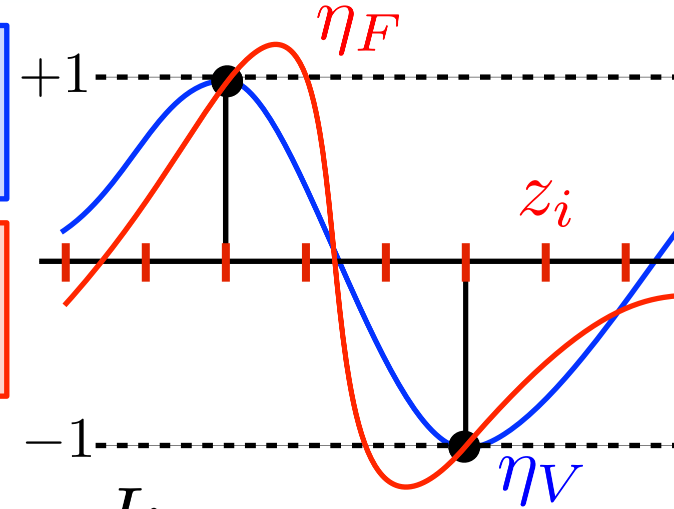

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Theorem: [Fuchs 2004]

If $\forall\, j \notin I, |\eta_F(x_j)| < 1$,

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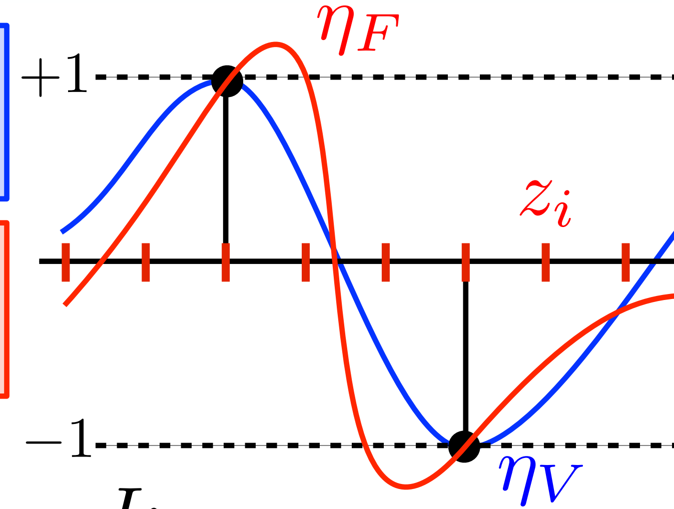

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Theorem: [Duval-Peyré 2013]
If $\forall\, t \notin x_0, |\eta_V(t)| < 1$,
then $m_\lambda = m_{x_\lambda, a_\lambda}$ with
$\|x_\lambda - x_0\|_\infty = O(\|w\|)$

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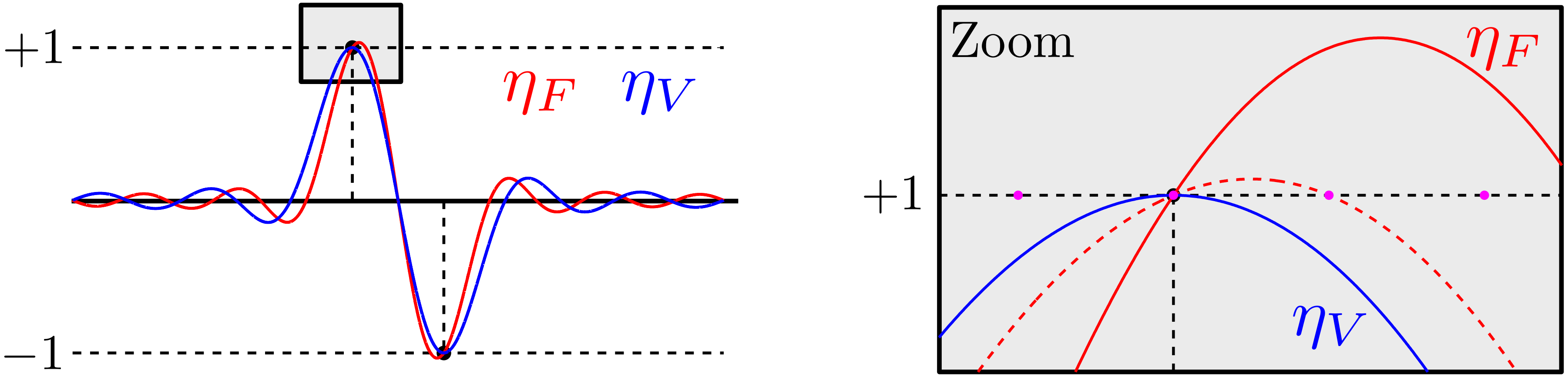
Numerical Illustration

Ideal low-pass filter: $\varphi(x, x') = \frac{\sin((2f_c+1)\pi(x-x'))}{\sin(\pi(x-x'))}$, $f_c = 6$.



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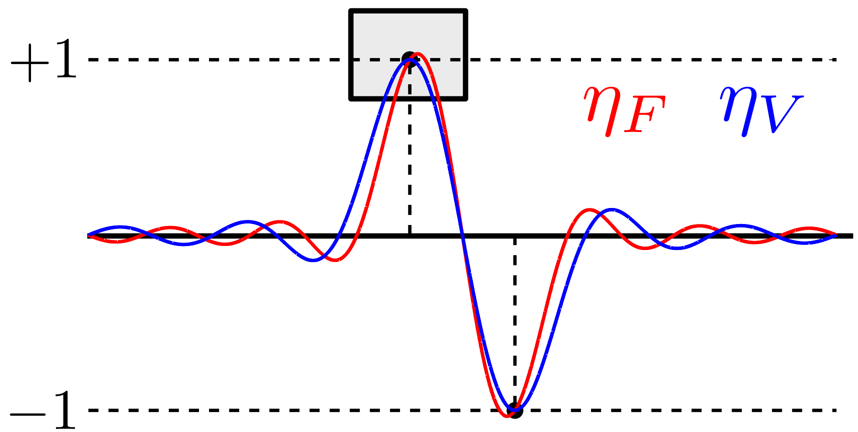

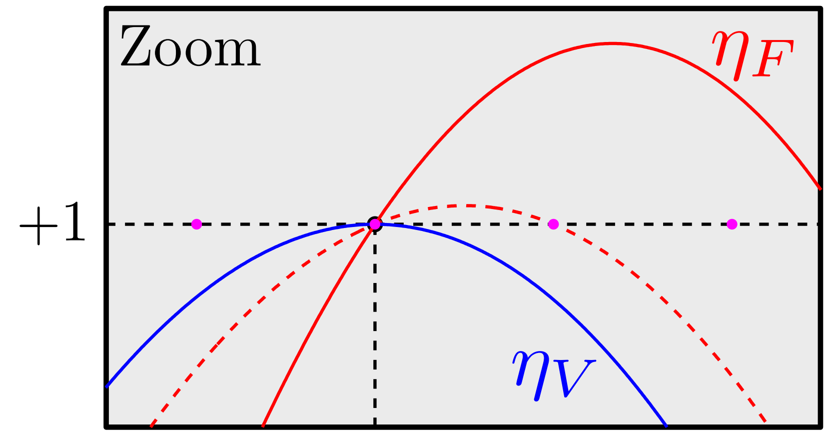

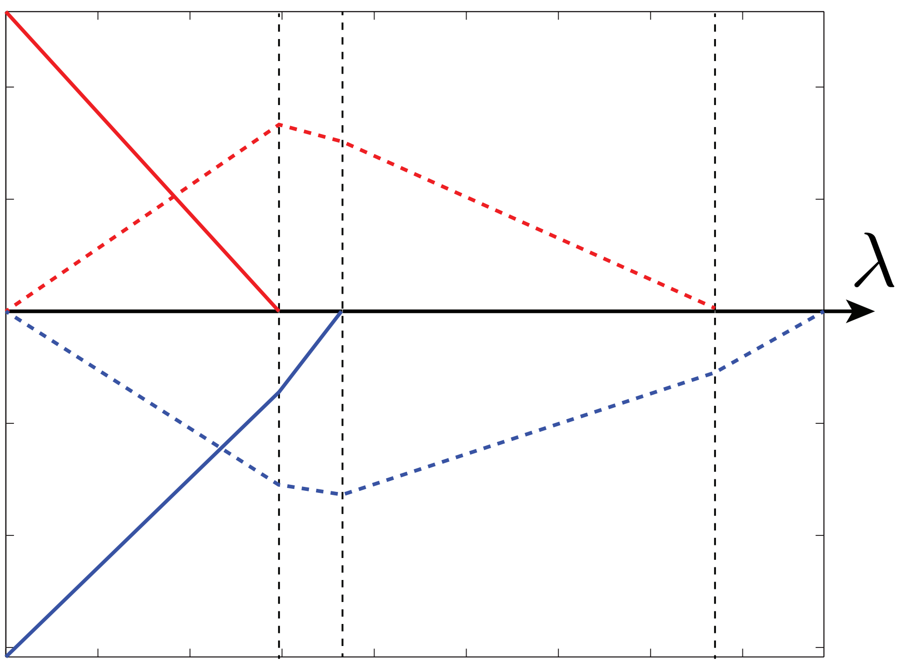

Solution path $\lambda \mapsto a_\lambda$

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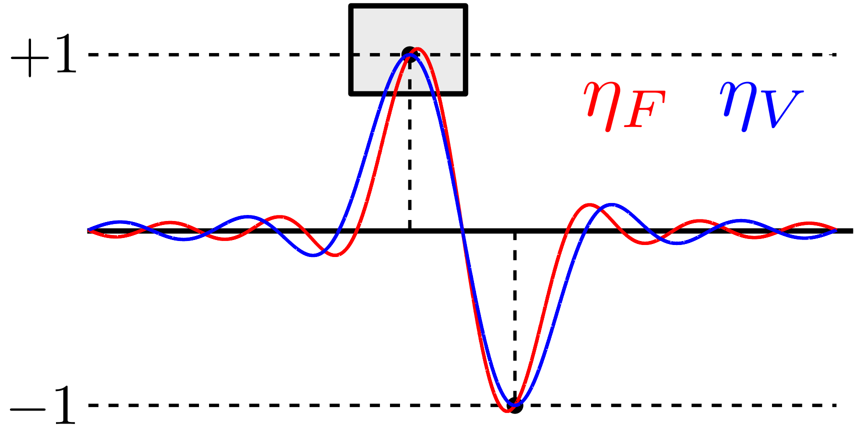
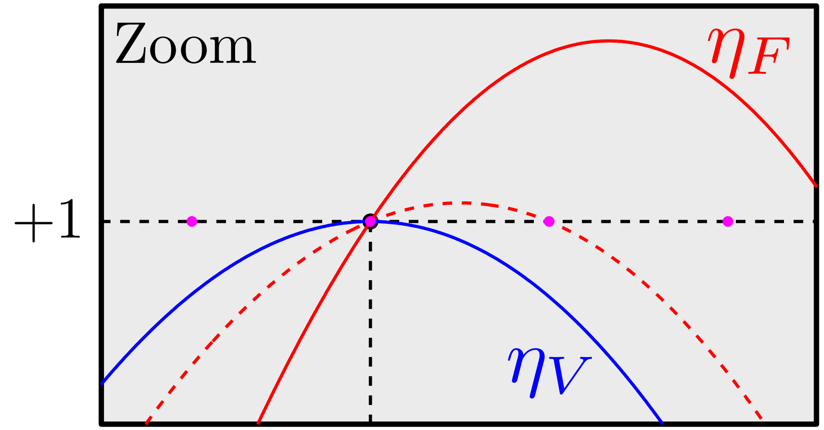
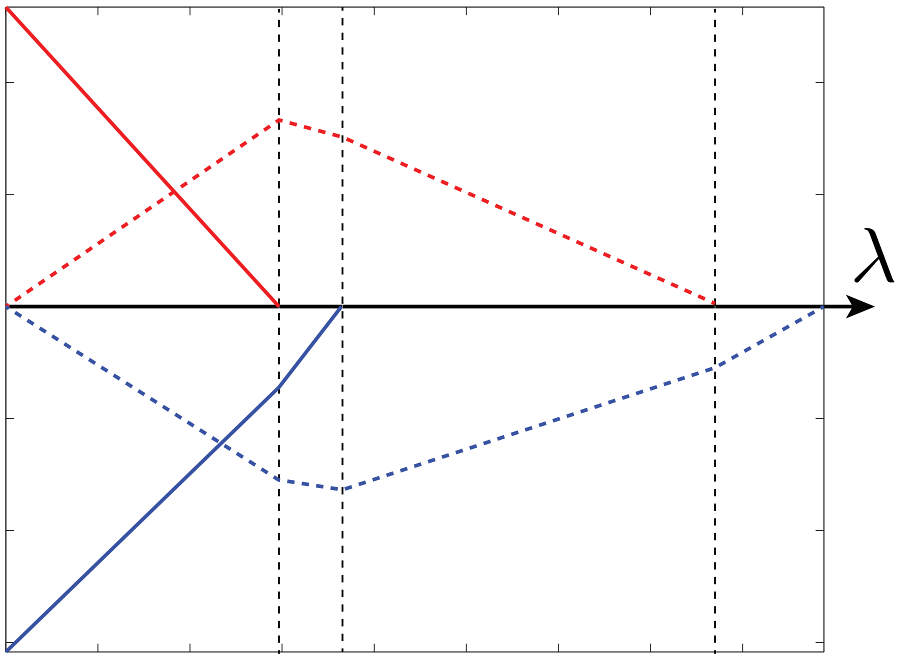
Ideal low-pass filter: $\varphi(x,x') = \frac{\sin((2f_c+1)\pi(x-x'))}{\sin(\pi(x-x'))}$, $f_c = 6$.







Solution path $\lambda \mapsto a_\lambda$

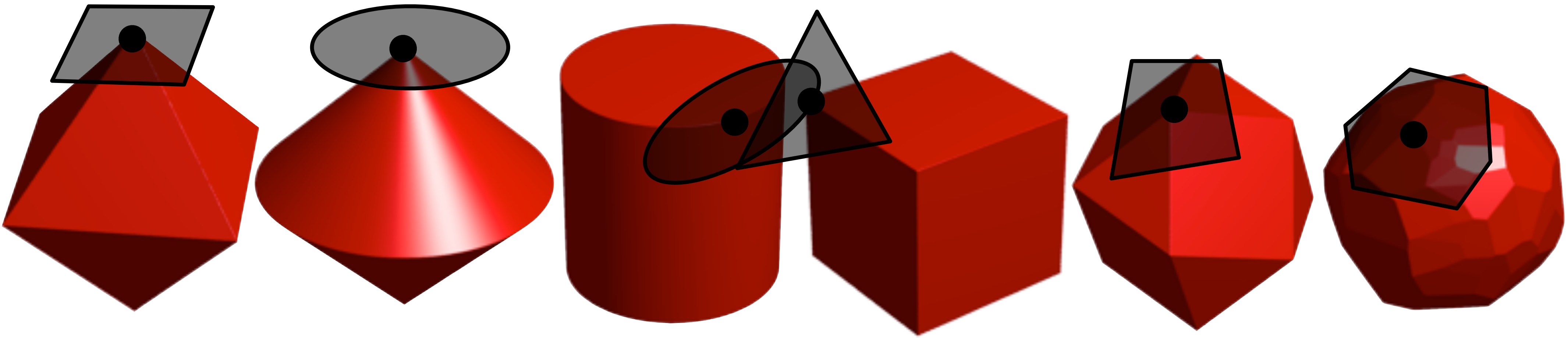
Discrete → continuous:

Theorem: [Duval-Peyré 2013]
If η_V is valid, then a_λ is supported on pairs of neighbors around $\mathrm{supp}(m_0)$.

(holds for $\lambda \sim \|w\|$ small enough.

Conclusion

Gauges: encode linear models as singular points.



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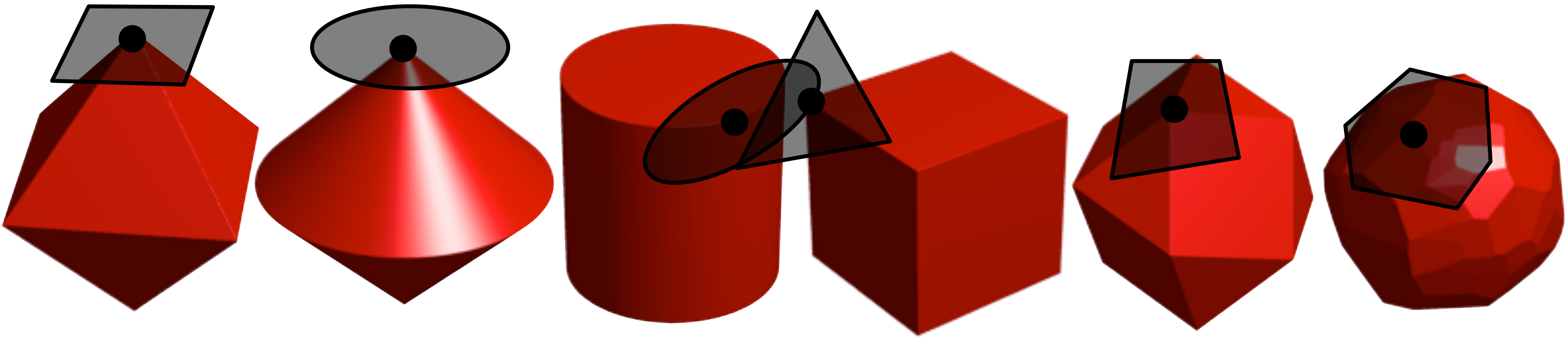
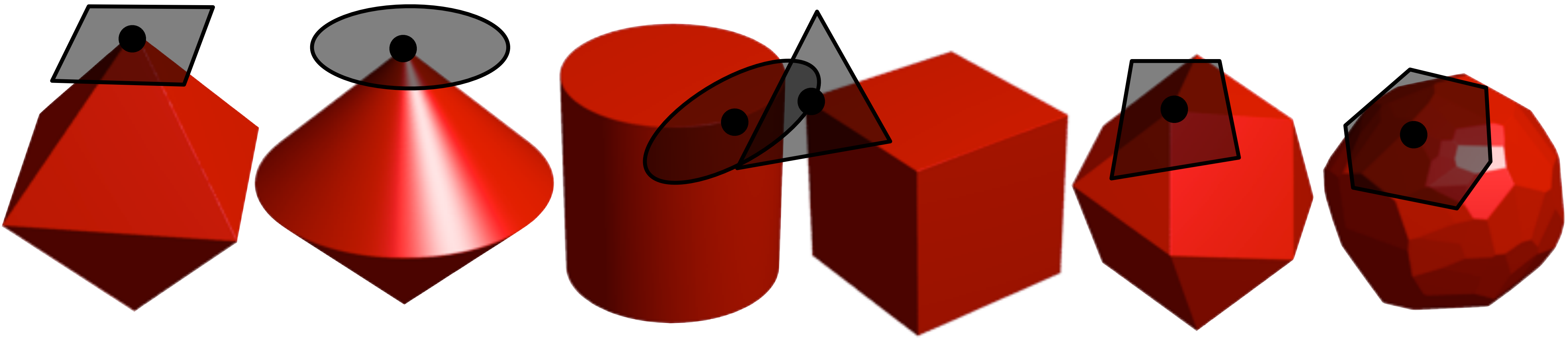


Performance measures → L^2 error / model → different CS guarantees

Conclusion

Gauges: encode linear models as singular points.



Performance measures → L^2 error / model → different CS guarantees

model → Specific certificate: $\eta_0, \eta_F, \eta_V, \ldots$

Conclusion

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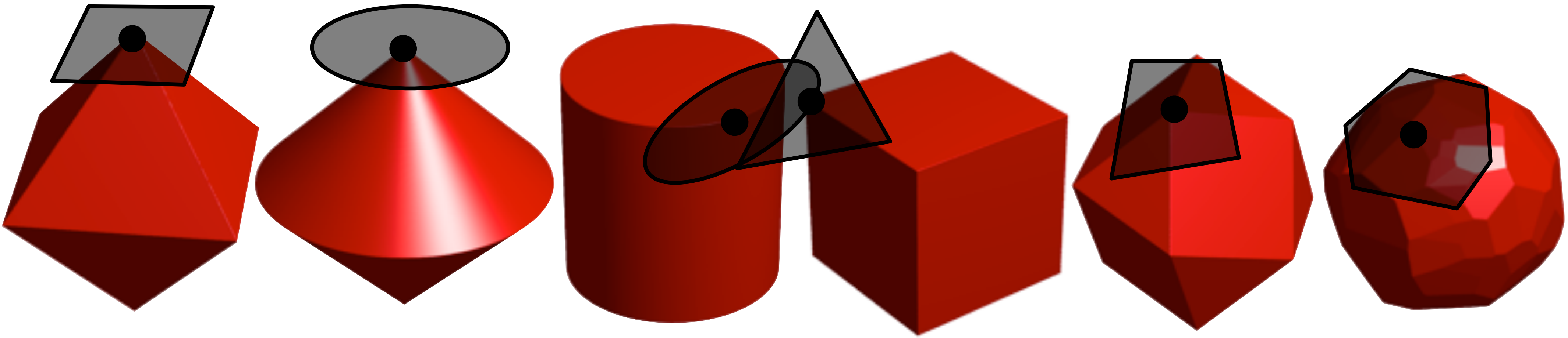

Performance measures → L^2 error / **model** → different CS guarantees

model → Specific certificate: $\eta_0, \eta_F, \eta_V, \ldots$

Open problems:

– CS performance with arbitrary gauges.

– Approximate model recovery $T_{x^\star} \approx T_{x_0}$.
(e.g. grid-free recovery)