Discrete Systolic Inequalities and Decompositions of Triangulated Surfaces

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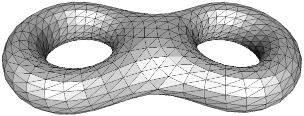
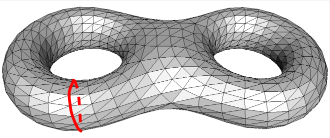


A primer on surfaces

We deal with ***connected***, ***compact*** and ***orientable*** surfaces of ***genus*** g without boundary.

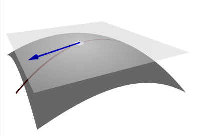
Discrete metric

Triangulation G.
Length of a curve $|\gamma|_G$:
Number of edges.



Riemannian metric

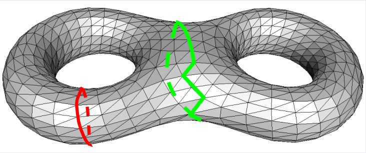
Scalar product m on the tangent space.
Riemannian length $|\gamma|_m$.



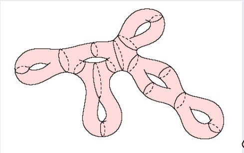
Systoles and surface decompositions

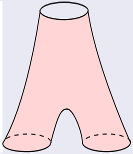
We study the length of topologically interesting curves and graphs, for discrete and continuous metrics.

Non-contractible curves

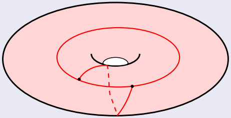


Pants decompositions





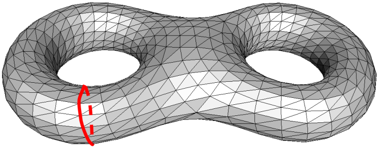
Cut-graphs



Fundamental objects in algorithm design for surface embedded graphs, texture mapping, and many other applications.

Discrete Setting: Topological graph theory

The ***edgewidth*** of a triangulated surface is the length of the shortest ***noncontractible*** cycle.

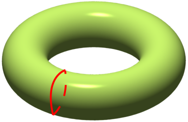


Theorem (Hutchinson '88)

The edgewidth of a triangulated surface with n triangles of genus g is $O(\sqrt{n/g}\log g)$.

- Hutchinson conjectured that the right bound is $\Theta(\sqrt{n/g})$.
- Disproved by Przytycka and Przytycki '90-97 who achieved $\Omega(\sqrt{n/g}\sqrt{\log g})$, and conjectured $\Theta(\sqrt{n/g}\log g)$.
- How about non-separating cycles ?

The ***systole*** of a Riemannian surface is the length of the shortest ***noncontractible*** cycle.



Theorem (Gromov '83, Katz and Sabourau '04)

The systole of a Riemannian surface of genus g and area A is $O(\sqrt{A/g}\log g)$.

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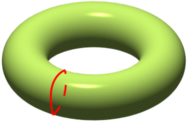

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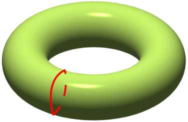

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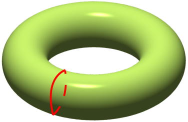

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- Buser and Sarnak '94 introduced ***arithmetic surfaces*** achieving the lower bound $\Omega(\sqrt{A/g}\log g)$.
- Larry Guth: "Arithmetic hyperbolic surfaces are remarkably hard to picture."

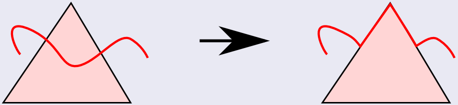
A two way street: From discrete to continuous.

Theorem (Colin de Verdière, Hubard, de Mesmay '13)

Let (S, G) be a triangulated surface of genus g, with n triangles. There exists a Riemannian metric m on S with area n such that for every closed curve γ in (S, m) there exists a homotopic closed curve γ' on (S, G) with

$$|\gamma'|_G \leq (1+\delta)\sqrt[4]{3}\ |\gamma|_m \quad \text{for some arbitrarily small } \delta.$$

Proof.

- Glue Euclidean triangles of area 1 (and thus side length $2/\sqrt[4]{3}$) on the triangles.
- Smooth the metric.
- 

□

Corollaries

Corollary

Let (S, G) be a triangulated surface with genus g and n triangles.

1. *Some non contractible cycle has length $O(\sqrt{n/g}\log g)$.*
2. *Some non separating cycle has length $O(\sqrt{n/g}\log g)$.*

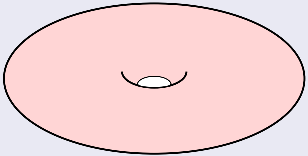
- (1) shows that Gromov $\Rightarrow$ Hutchinson and improves the best known constant.
- (2) is new.

Theorem (Colin de Verdière, Hubard, de Mesmay '13)

Let (S, m) be a Riemannian surface of genus g and area A. There exists a triangulated surface (S, G) embedded on S with n triangles, such that every closed curve γ in (S, G) satisfies

$|\gamma|_m \leq (1+\delta)\sqrt{\frac{32}{\pi}}\sqrt{A/n}\ |\gamma|_G$ *for some arbitrarily small δ.*

Proof.



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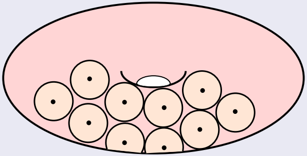

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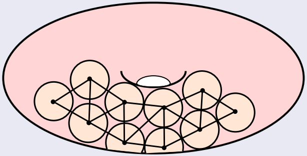

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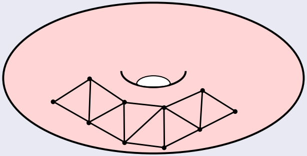

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Corollaries

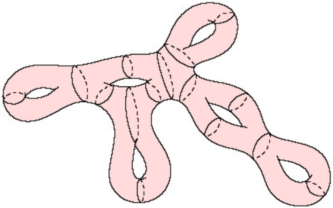
- This shows that Hutchinson $\Rightarrow$ Gromov.
- Proof of the conjecture of Przytycka and Przytycki:

Corollary

There exist arbitrarily large g and n such that the following holds: There exists a triangulated combinatorial surface of genus g, with n triangles, of edgewidth at least $\frac{1-\delta}{6}\sqrt{n/g}\log g$ *for arbitrarily small δ.*

Pants decompositions

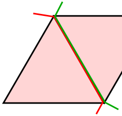
- A ***pants decomposition*** of a triangulated or Riemannian surface S is a family of cycles Γ such that cutting S along Γ gives pairs of pants, e.g., spheres with three holes.



- A pants decomposition has $3g - 3$ curves.
- Complexity of computing a shortest pants decomposition on a triangulated surface: in NP, not known to be NP-hard.

Pathologies

- Several curves may run along the same edge:



  $\rightarrow$ Colin de Verdière and Lazarus '07 proved an $O(gn)$ bound on the length of the shortest pants decomposition.
- We have an $O(gn)$ algorithm to compute pants decomposition of length $O(g^{3/2}\sqrt{n})$, taking inspiration from Riemann(ian) surfaces. (skipped)

Random surfaces: Sample uniformly at random among the triangulated surfaces with n triangles.

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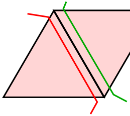

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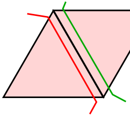

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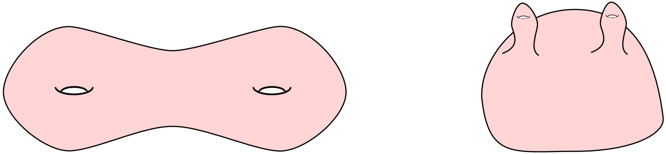
Random surfaces: Sample uniformly at random among the triangulated surfaces with n triangles.
These run-alongs happen a lot for random triangulated surfaces:

Theorem (Guth, Parlier and Young '11)

If (S, G) is a random triangulated surface with n triangles, and thus $O(n)$ edges, the length of the shortest pants decomposition of (S, G) is $\Omega(n^{7/6-\delta})$ w.h.p. for arbitrarily small δ

Cut-graphs with fixed combinatorial map

- What is the length of the shortest cut-graph with a fixed shape (combinatorial map) ?
- Useful to compute a homeomorphism between two surfaces.



- Example: Canonical systems of loops (Lazarus et al '01) have $\Theta(gn)$ length.

- Can one find a better map ?

Theorem (Colin de Verdière, Hubard, de Mesmay '13)

If (S, G) is a random triangulated surface with n triangles and genus g, for any combinatorial map M, the length of the shortest cut-graph with combinatorial map M is $\Omega(n^{7/6-\delta})$ w.h.p. *for arbitrarily small δ.*

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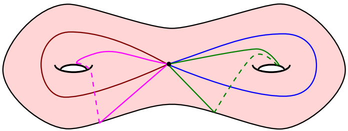

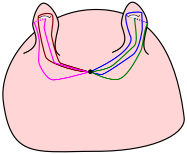

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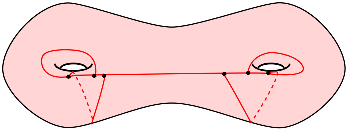

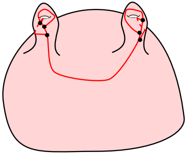

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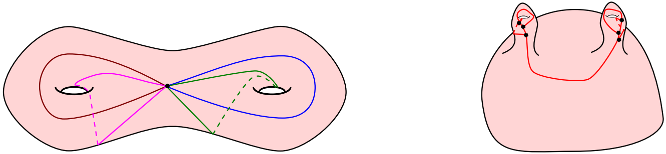

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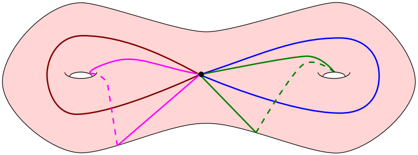

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Restated in a dual setting: What is the minimal number of crossings between two cellularly embedded graphs G_1 and G_2 with specified combinatorial maps ?

Corollary

For a fixed G_1, for most choices of trivalent G_2 with n vertices, there are $\Omega(n^{7/6-\delta})$ crossings in any embedding of G_1 and G_2 for arbitrarily small δ.

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Thank you ! Questions ?