

# On the reconstruction of convex bodies from random normal measurements

Quentin Mérigot

LJK / CNRS / Université de Grenoble

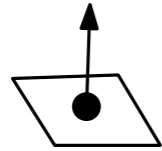
Joint work with Hiba Abdallah, Université de Grenoble

Journées de géométrie algorithmique 2013

# Motivation

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- ▶ CEA-Leti has designed small sensors that can measure their orientation

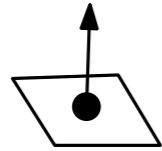


accelerometer + magnetometer  $\implies$  oriented normal

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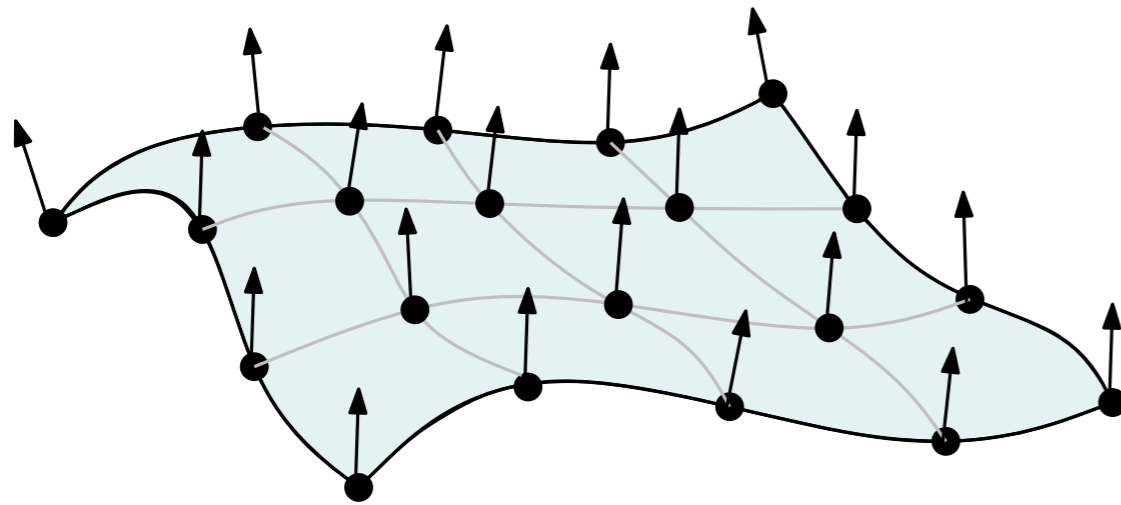
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already a non-convex inverse problem

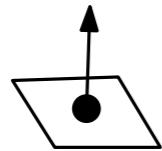
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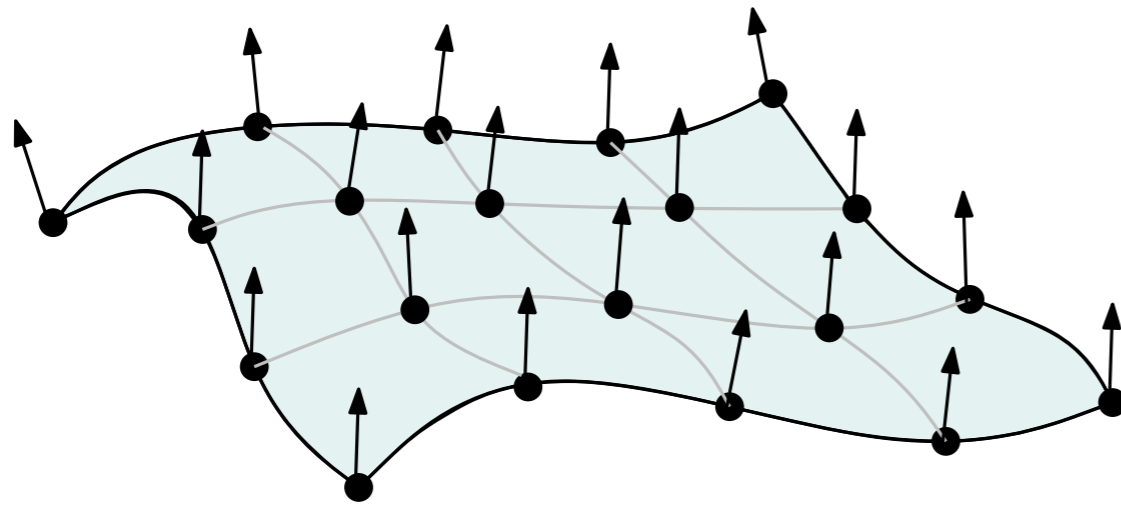
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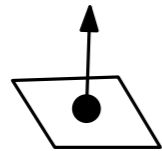
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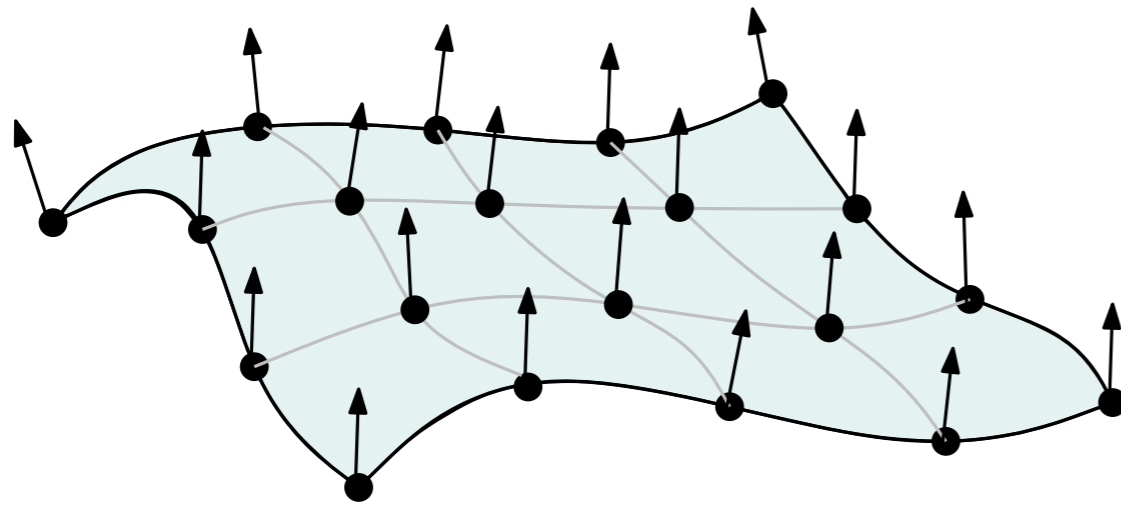
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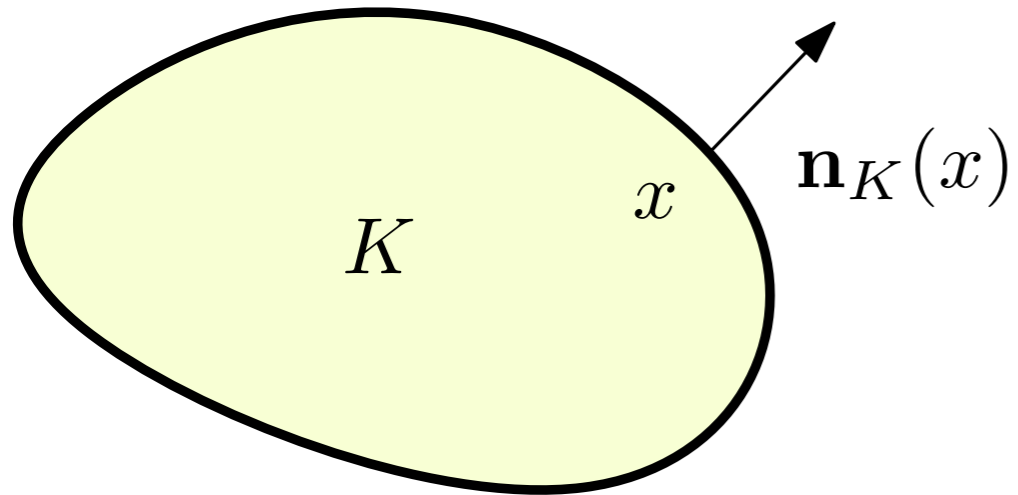
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- ▶ Our goal:  $S =$  boundary of a convex body, random sampling.

# 1. Minkowski problem

# Surface area measure

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$K = \text{convex body in } \mathbb{R}^d$

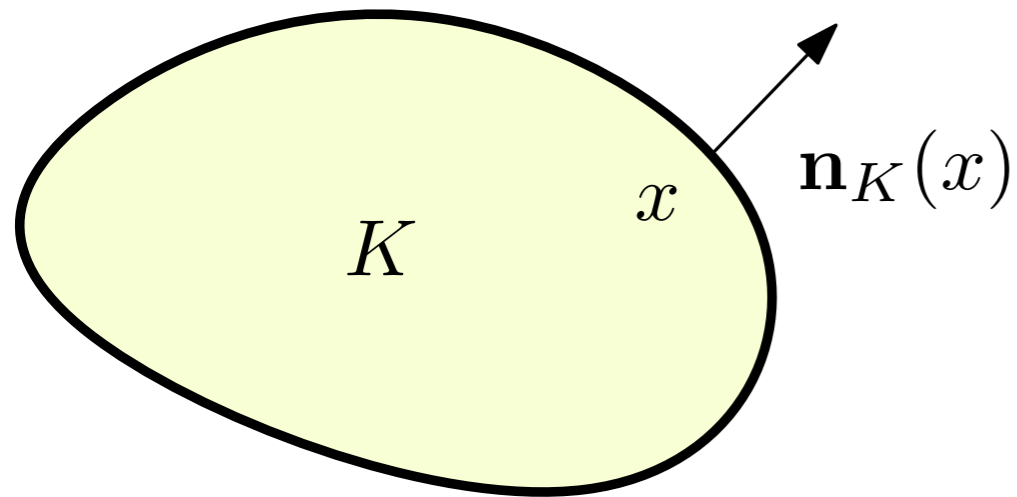


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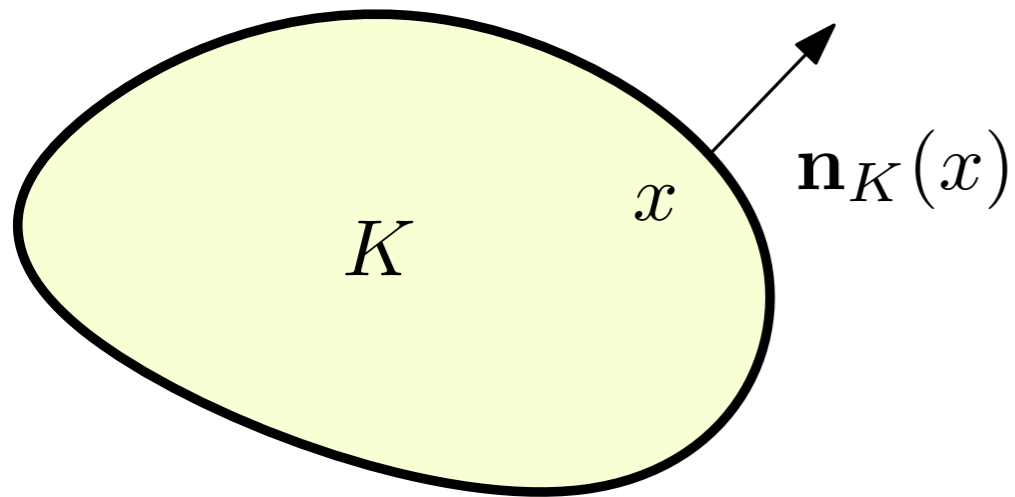
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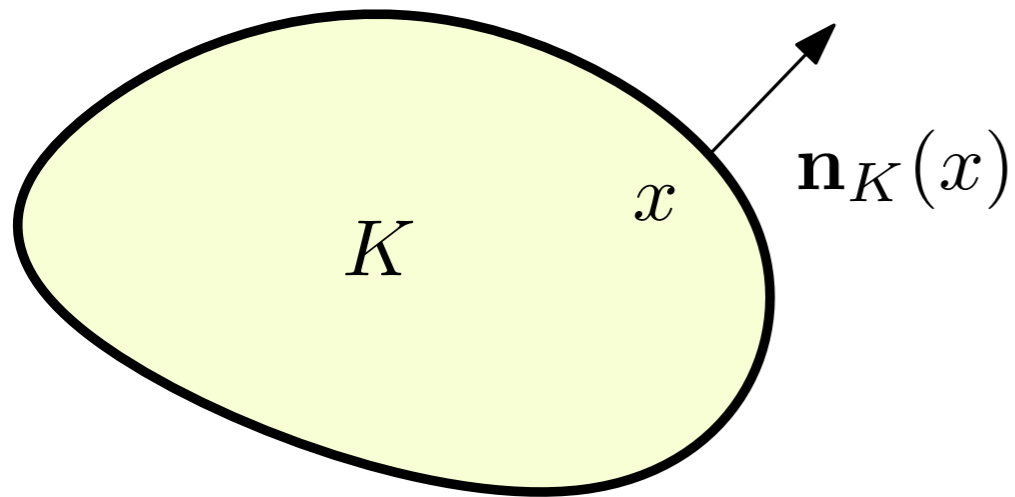
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In particular,  $\mu_K(\mathcal{S}^{d-1}) = \text{area}(\partial K)$ .

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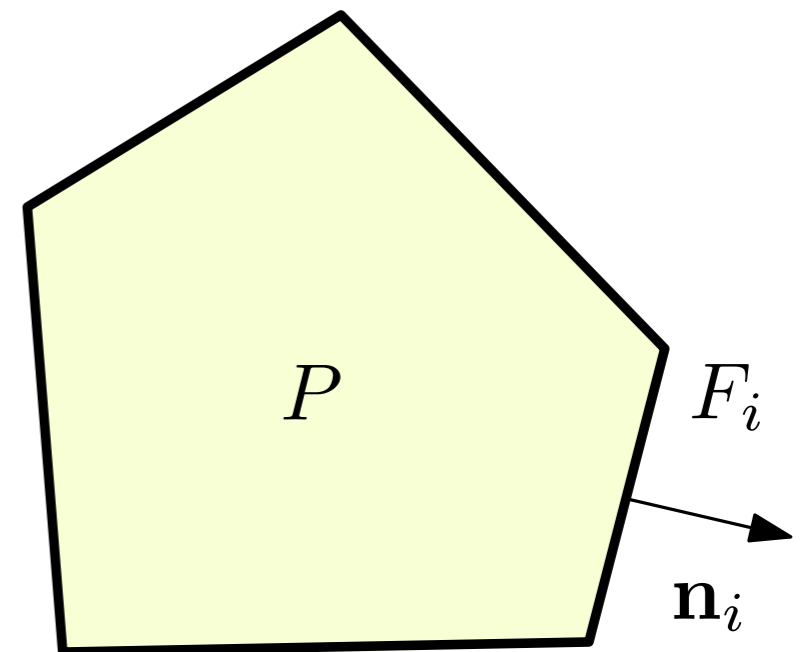
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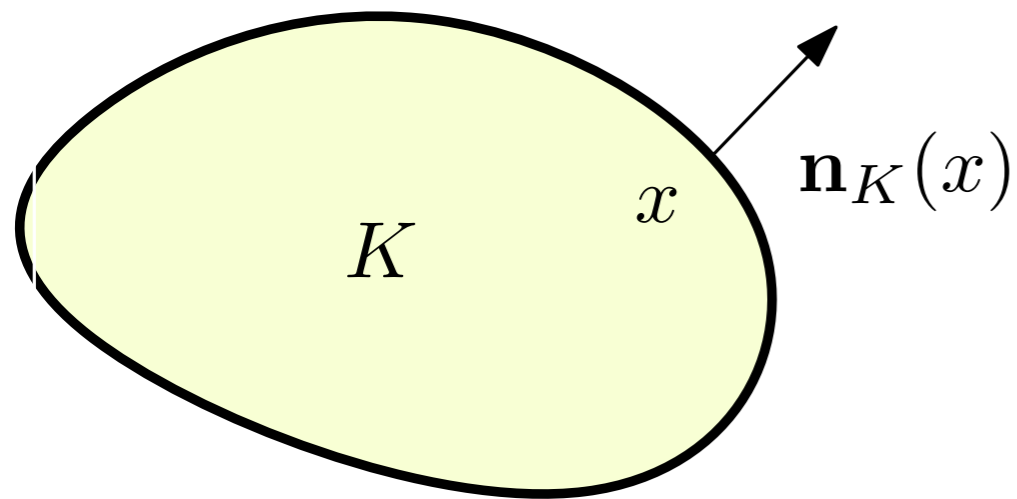
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$\mathbf{n}_i = \text{unit normal}$



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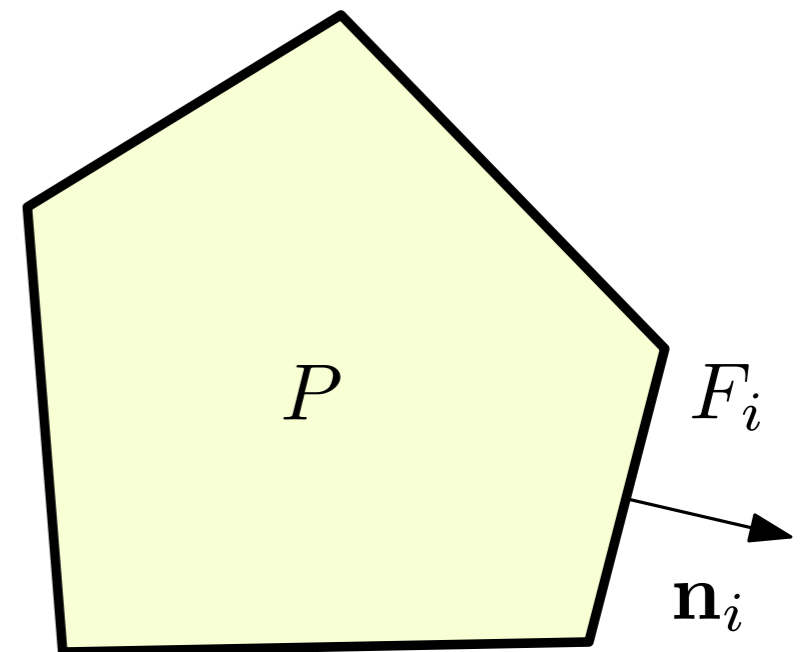
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**Example:**  $P \subseteq \mathbb{R}^d$  is a polyhedron

$$\mu_P = \sum_{i=1}^N \text{area}(F_i) \delta_{\mathbf{n}_i}$$

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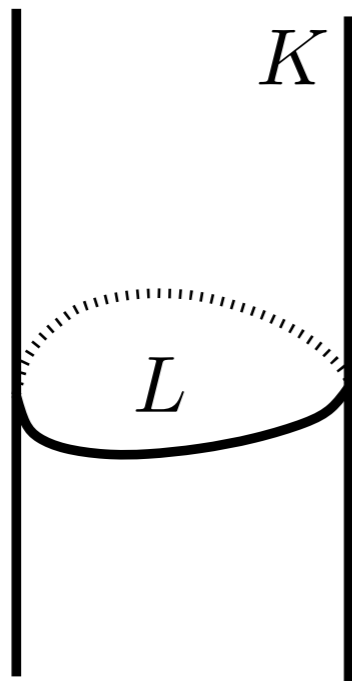
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Avoids this situation:

If  $L \subseteq \mathbb{R}^{d-1}$ , and  $K = L \times \mathbb{R}$ , then  $\mathbf{n}_K(\partial K) \subseteq \mathcal{S}^{d-2} \times \{0\}$



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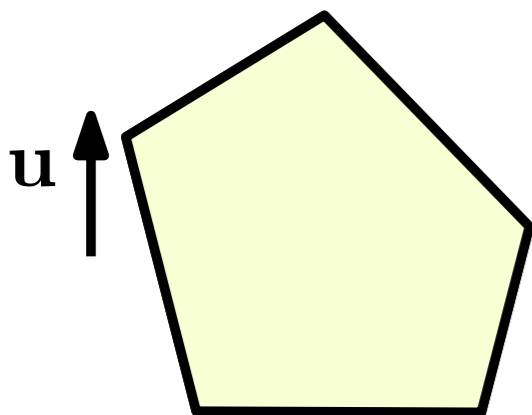
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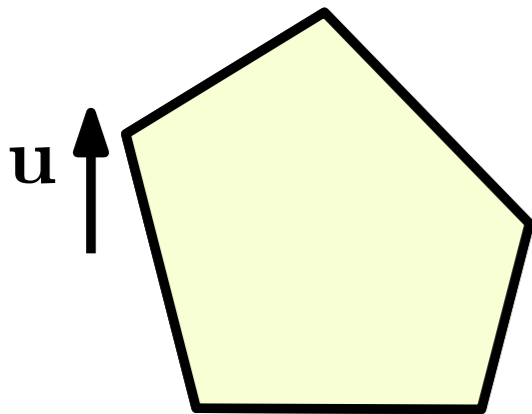
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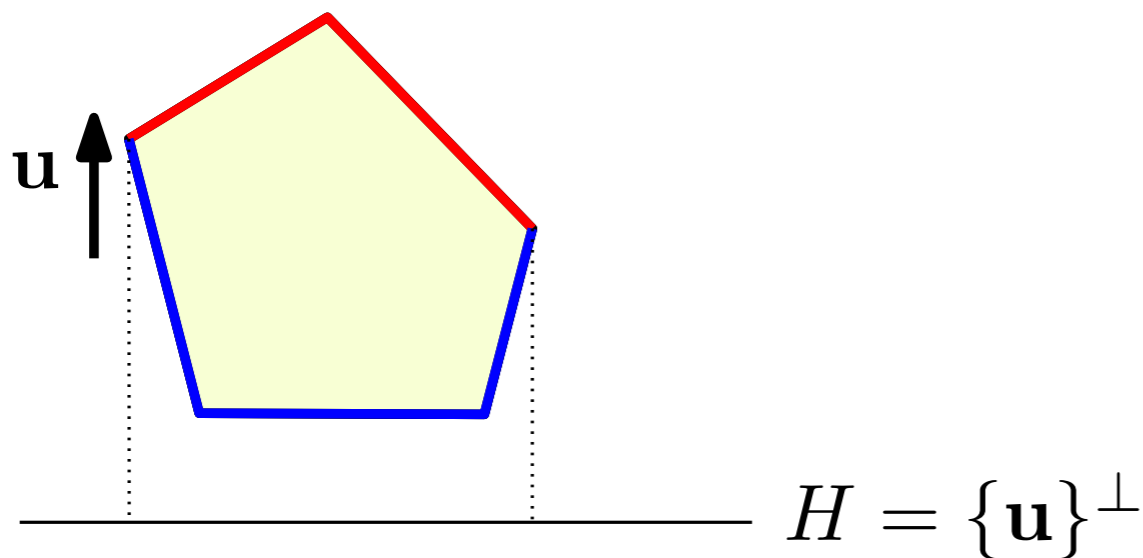
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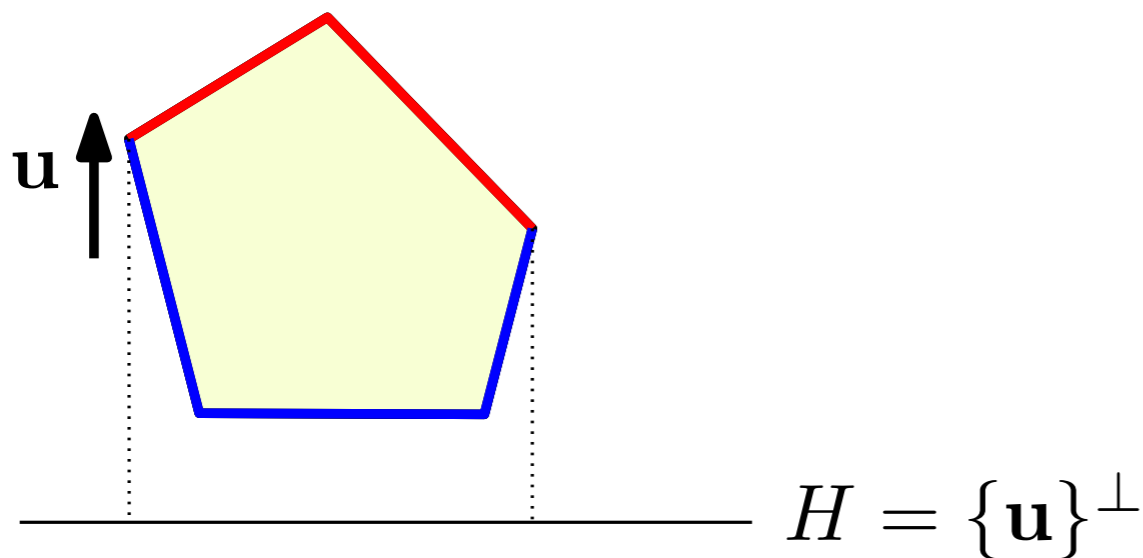
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**Theorem (Minkowski-Alexandrov):** Given any measure  $\mu$  on  $\mathcal{S}^{d-1}$ , which satisfies **(non-degeneracy)** and **(zero-mean)**, there exists a convex body  $K$  with  $\mu = \mu_K$ . This body is unique up to translation.

# Stability in Minkowski theorem

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**Definition:**  $d_{\text{bL}}(\mu, \nu) = \max_{f \in \text{BL}_1} \left| \int_{\mathcal{S}^{d-1}} f \, d\mu - \int_{\mathcal{S}^{d-1}} f \, d\nu \right|$ , where  
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$d_{\text{bL}}(\mu, \nu) = \text{Wass}_1(\mu, \nu)$  when  $\mu(\mathcal{S}^{d-1}) = \nu(\mathcal{S}^{d-1})$ .

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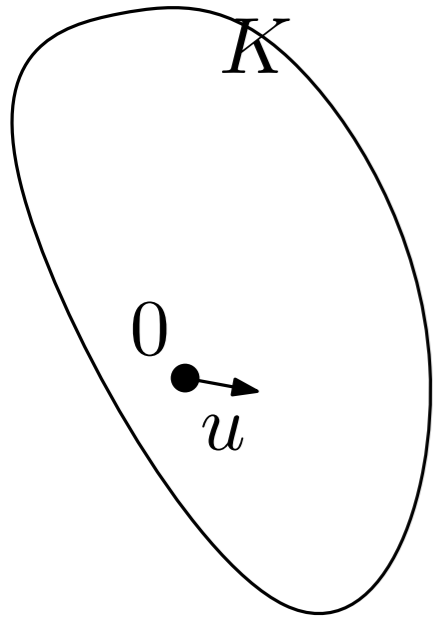
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  - ▶ Reconstruction result under random sampling

## 2. Improved stability theorem

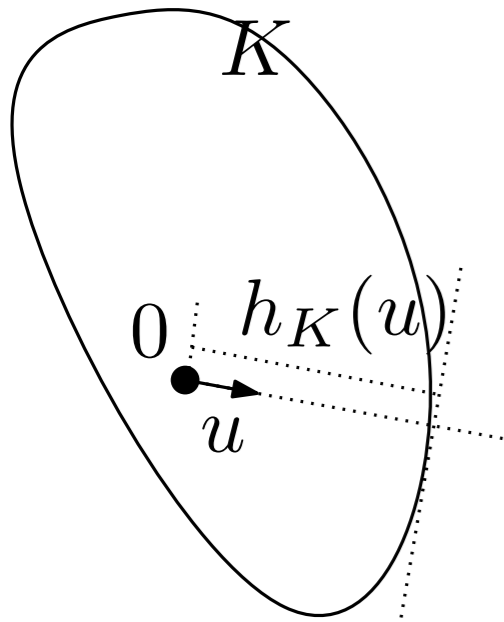
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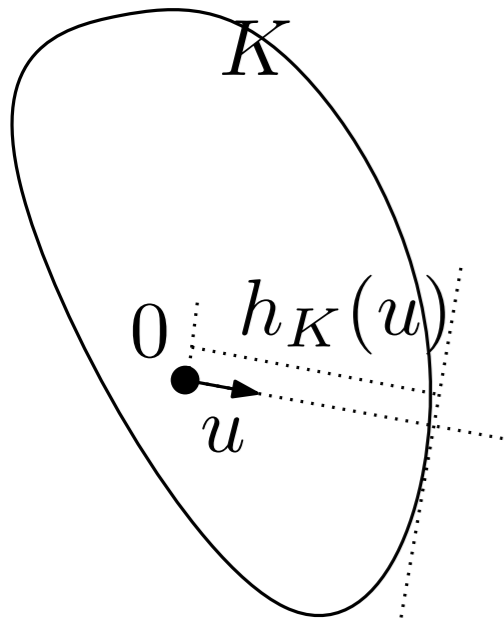


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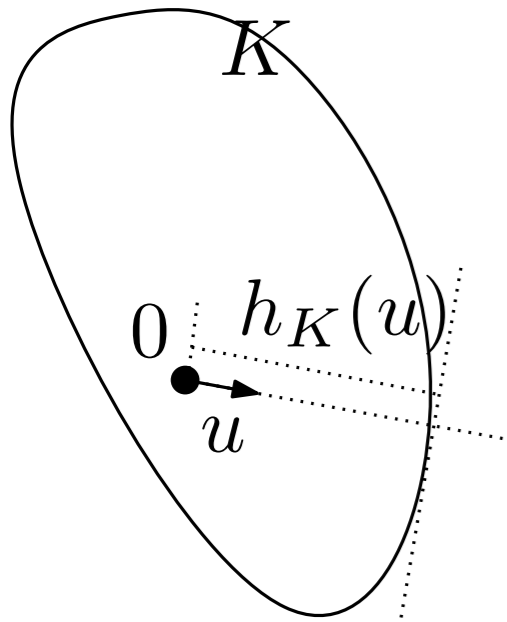
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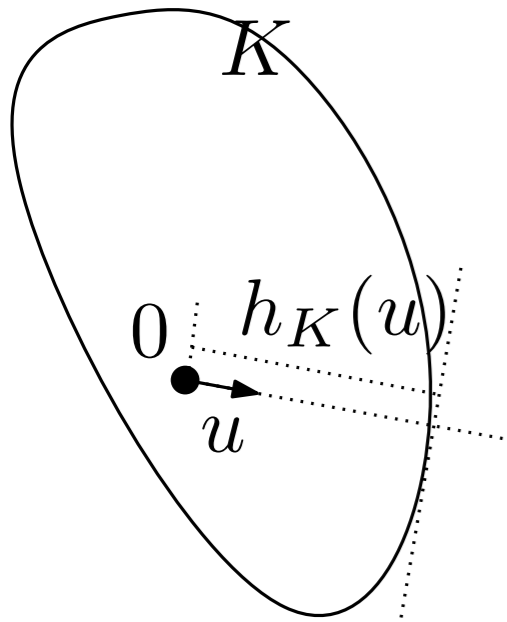
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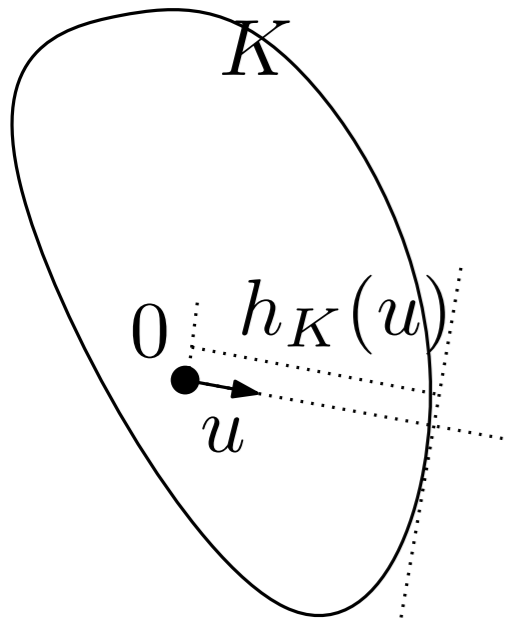
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**Definition: convex-dual distance** between measure  $\mu$  and  $\nu$  on  $\mathcal{S}^{d-1}$ :

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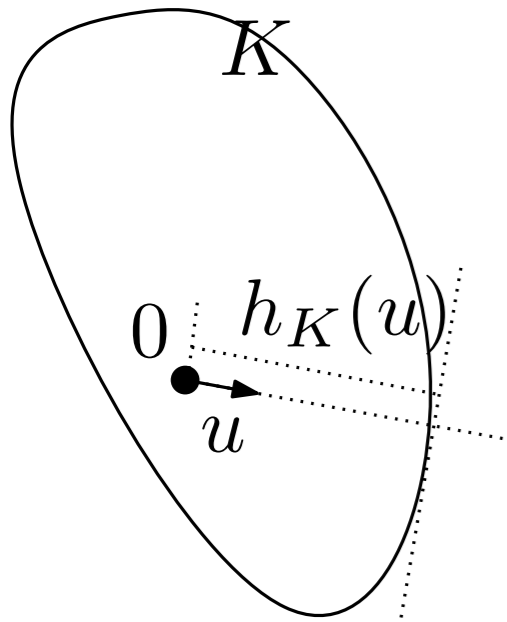
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- ▶ Since  $C_1 \subseteq BL_1$ ,  $d_C \leq d_{bL}$ ; a stability result for  $d_C$  is stronger.

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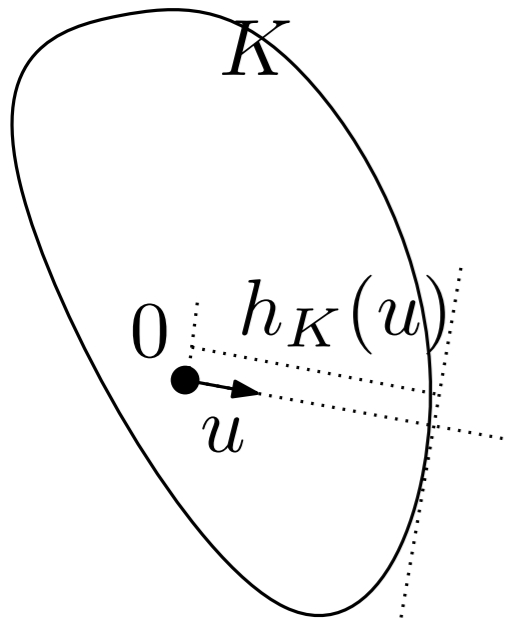
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# A new distance between surface area measures



**Definition:** Support function of  $K$ ,  $h_K : \mathcal{S}^{d-1} \rightarrow \mathbb{R}$

$$h_K(u) := \max_{p \in K} \langle u | p \rangle$$

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- ▶  $d_C$  satisfies the triangle inequality, but  $d_C(\mu, \nu) = 0 \not\Rightarrow \mu = \nu$ .
- ▶ However, for surface area measures,  $d_C(\mu_K, \mu_L) = 0 \Rightarrow \mu_K = \mu_L$ .

# Stability theorem for the convex-dual distance

**Theorem (Abdallah-M. '13):** Given a convex body  $K$  and  $\mu$  a measure on  $\mathcal{S}^{d-1}$  such that  $d_C(\mu_K, \mu) \leq \varepsilon_0(K)$ , there exists a convex body  $L$  such that  $\mu = \mu_L$  and moreover

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- ▶ exponent  $\frac{1}{d}$  is most likely non-optimal (best possible:  $\frac{1}{d-1}$ )
- ▶ to drop the requirement  $r \leq \text{inrad}(\mu_L)$ ,  $R \geq \text{circumrad}(\mu_L)$ , we introduce the **weak rotundity** and exploit a lemma of Cheng and Yau.



# Weak rotundity

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**Definition:** The weak rotundity of a measure  $\mu$  on  $\mathcal{S}^{d-1}$  is:

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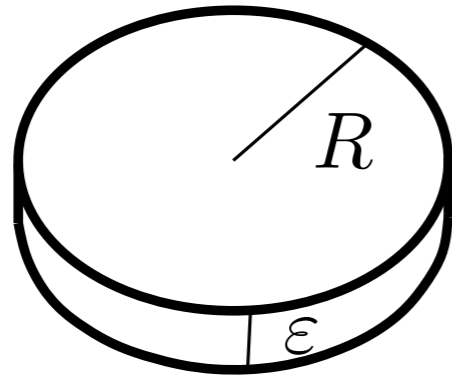
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$$\begin{aligned} \text{circumrad}(K) &\simeq R & \text{area}(\partial K) &\simeq R^{d-1} \\ \text{inrad}(K) &\simeq \varepsilon & \text{rotund}(\mu_K) &\simeq R^{d-1} \varepsilon \end{aligned}$$

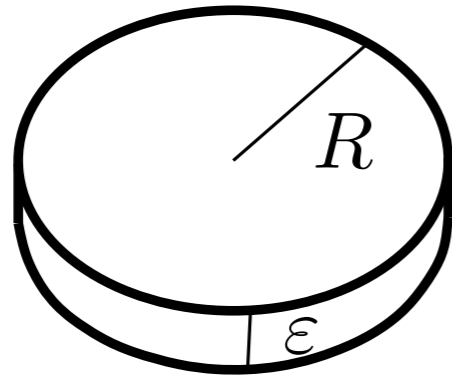
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► **Cheng-Yau Lemma:**

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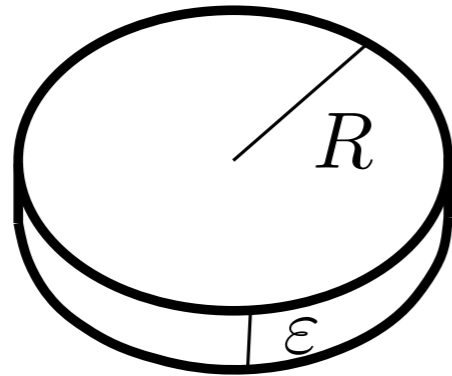
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► Is  $\text{rotund}(\mu)$  stable under perturbations of  $\mu$  ?

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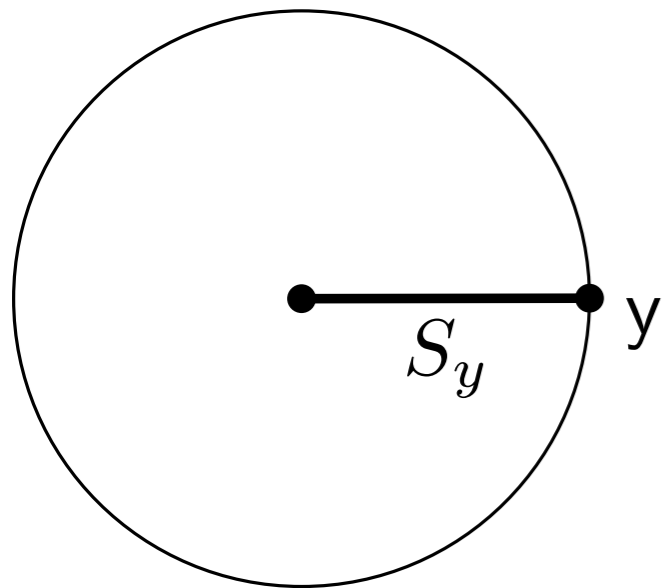
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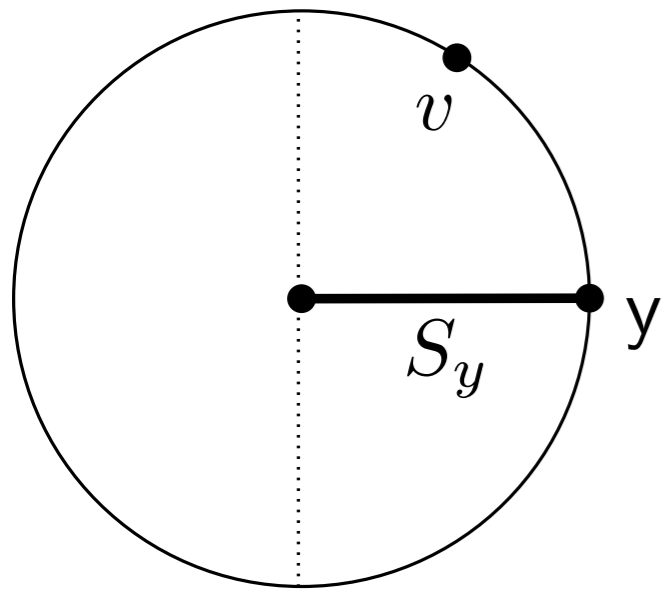
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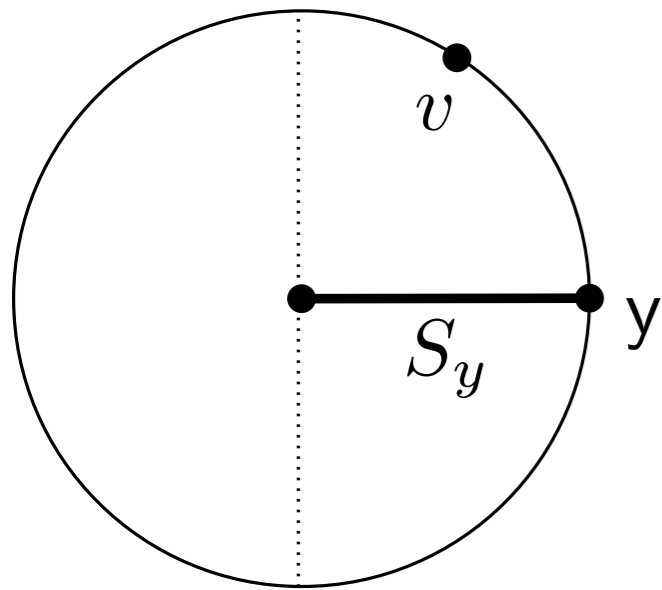
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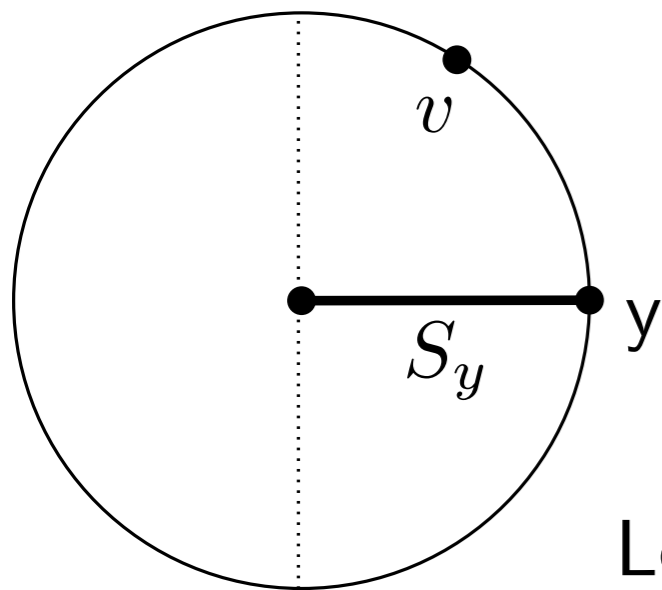
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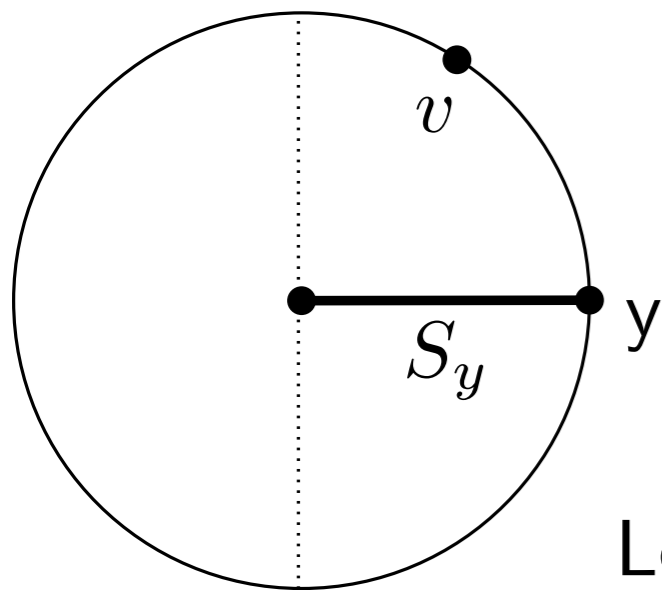
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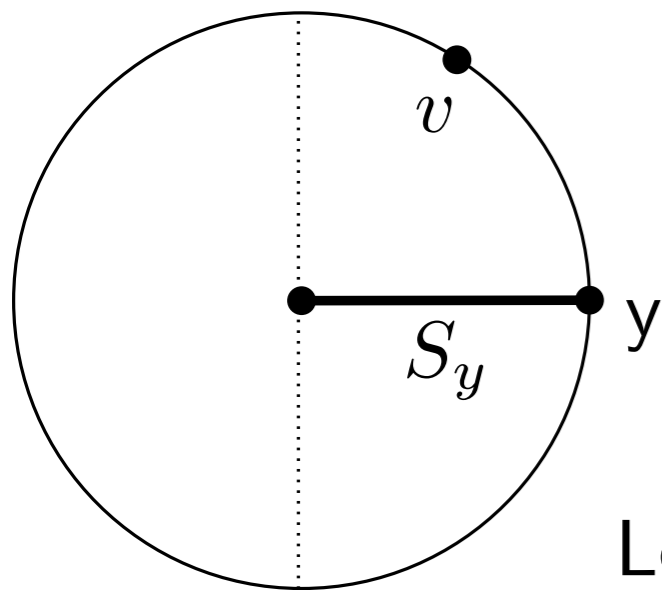
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One concludes using  $\text{rotund}(\mu) = \min_{y \in \mathcal{S}^{d-1}} f_\mu(y)$ .

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Therefore  $\mu$  satisfies **(non-degeneracy)** and **(zero-mean)**.

The result follows from Minkowski's theorem.

# 3. Reconstruction under random sampling



# Reconstruction theorem

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- ▶ Analysis using the bound involving  $d_{\text{bL}}$  would give  $\eta^{-6}$  and  $\eta^{-12}$ .

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**Bronshstein's theorem** imply  $\mathcal{N}(C_1, \varepsilon) \leq \exp(\text{const}(d)\varepsilon^{\frac{1-d}{2}})$ .

(NB:  $\mathcal{N}(BL_1, \varepsilon) = \Omega(\exp(\varepsilon^{1-d}))$ )



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**Proposition:**  $\mathbb{P}(d_C(\mu_{K,N}, \mu_K) \geq \varepsilon) \leq 1 - 2 \exp(\text{const}(d)\varepsilon^{\frac{1-d}{2}} - N\varepsilon^2)$

**Proof:** Using Chernoff's bound and the union bound, one has

$$\mathbb{P}(d_C(\mu_{K,N}, \mu_K) \geq \varepsilon) \leq 1 - 2\mathcal{N}(C_1, \varepsilon) \exp(-2N\varepsilon^2)$$

where  $\mathcal{N}(C_1, \varepsilon)$  = covering number of  $C_1$  for  $\|\cdot\|_\infty$ . Moreover,

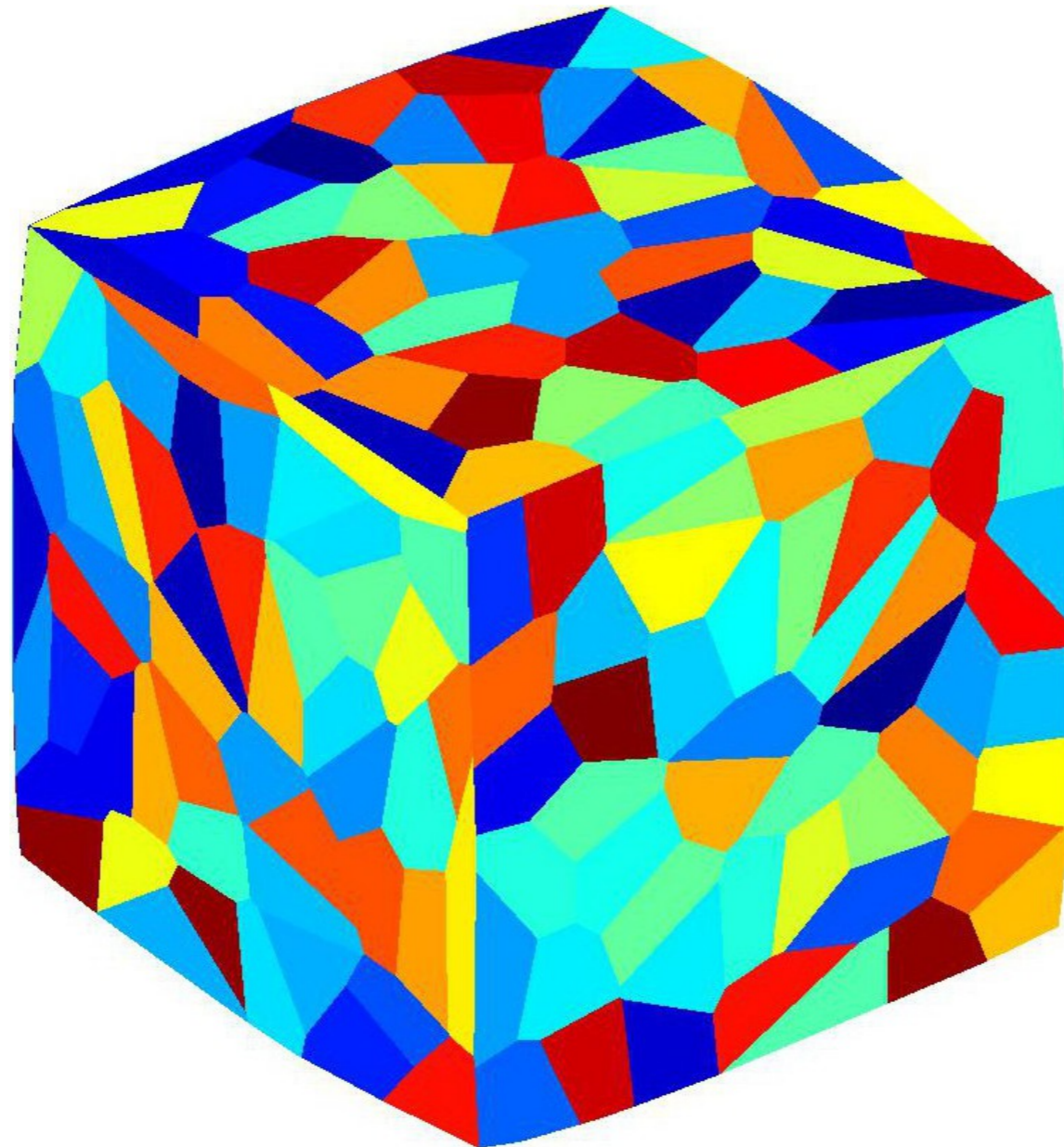
**Bronshstein's theorem** imply  $\mathcal{N}(C_1, \varepsilon) \leq \exp(\text{const}(d)\varepsilon^{\frac{1-d}{2}})$ .

$$\text{(NB: } \mathcal{N}(\text{BL}_1, \varepsilon) = \Omega(\exp(\varepsilon^{1-d}))\text{)}$$

- ▶ Conclusion uses the  $d_C$ -stability of the weak rotundity and mean, and the stability theorem using the convex-dual distance.

# Reconstruction theorem: an example

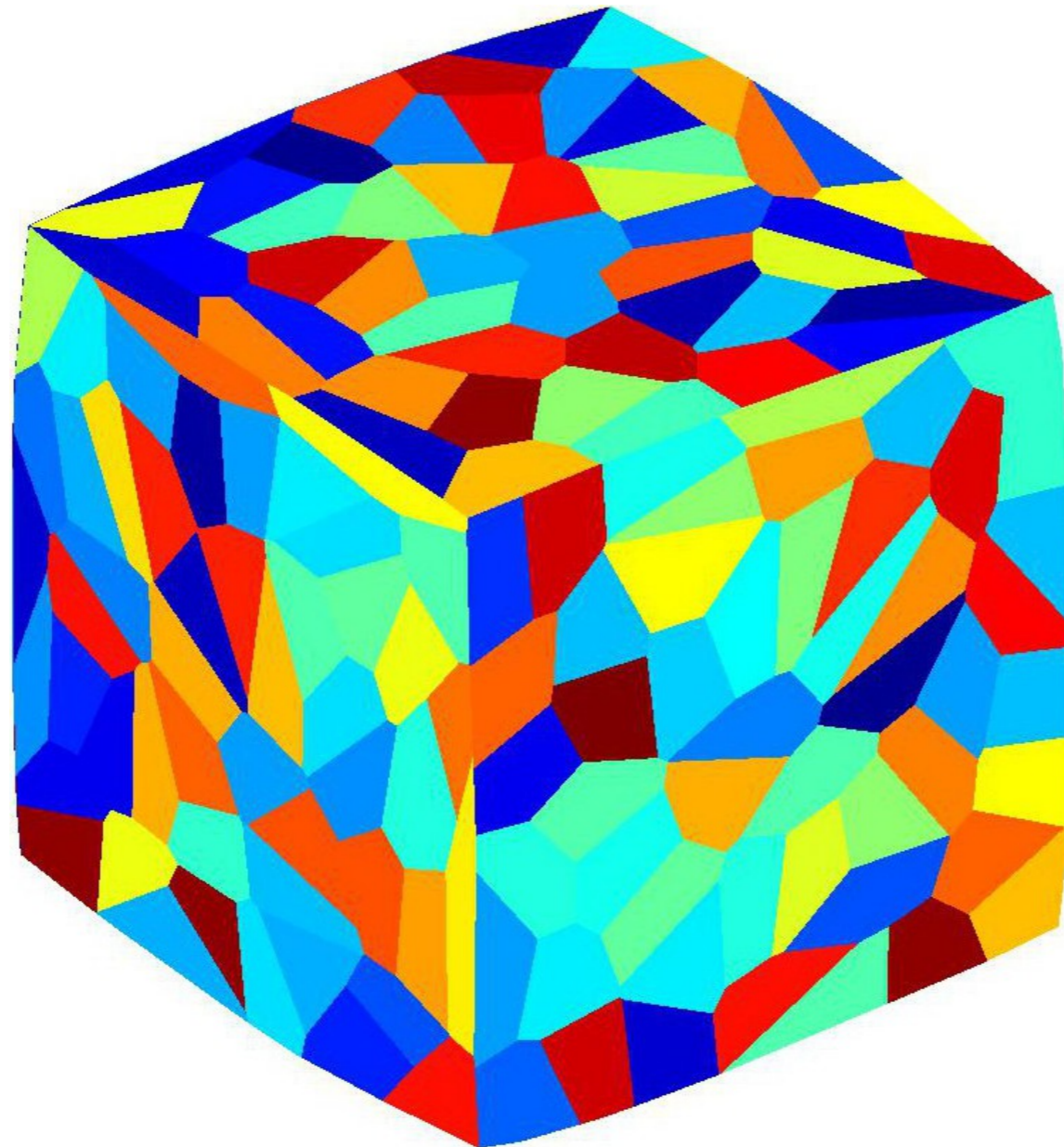
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- Reconstruction of a unit cube from  $N = 300$  random normal measurements, with a uniform noise of 0.05 on the normals.

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- ▶ Computation: variational characterization of solutions to Minkowski problem.

# Conclusion

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**Main result:** In order to reconstruct a convex body  $K$  with Hausdorff error  $\eta$  and with probability 99%, one needs  $N$  random normal measurements with

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$$F_{K,L} = \{h_K, h_L\} \cup \{h_{S_y}; y \in \mathcal{S}^{d-1}\} \cup \{h_{\{y\}}; y \in \mathcal{S}^{d-1}\}.$$

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**Thank you!**