On the reconstruction of convex bodies from random normal measurements

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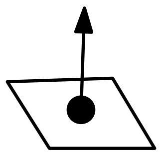
Joint work with Hiba Abdallah, Université de Grenoble

Journées de géométrie algorithmique 2013

Motivation

- CEA-Leti has designed small sensors that can measure their orientation



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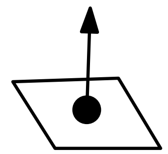

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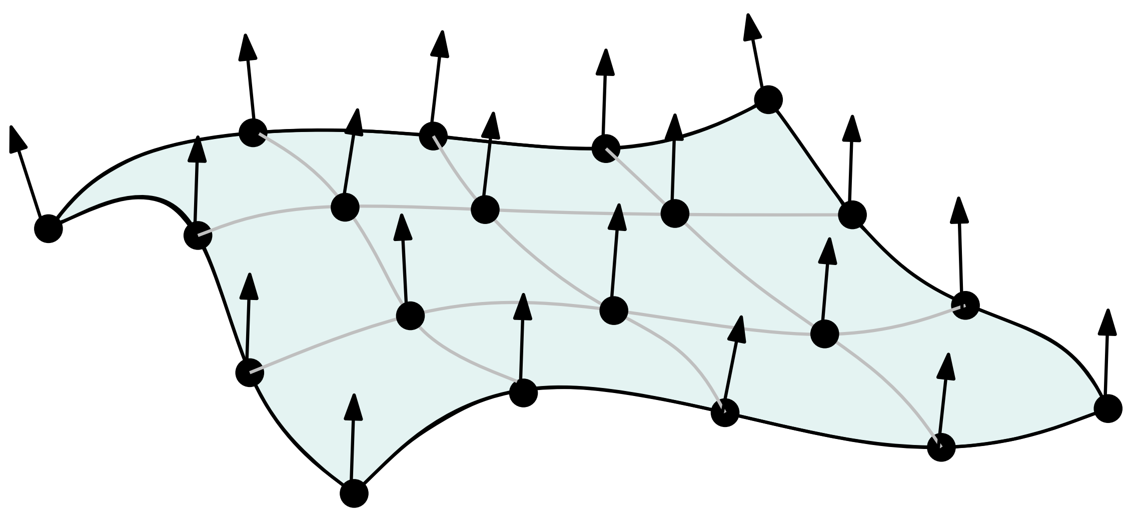

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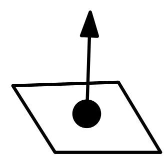

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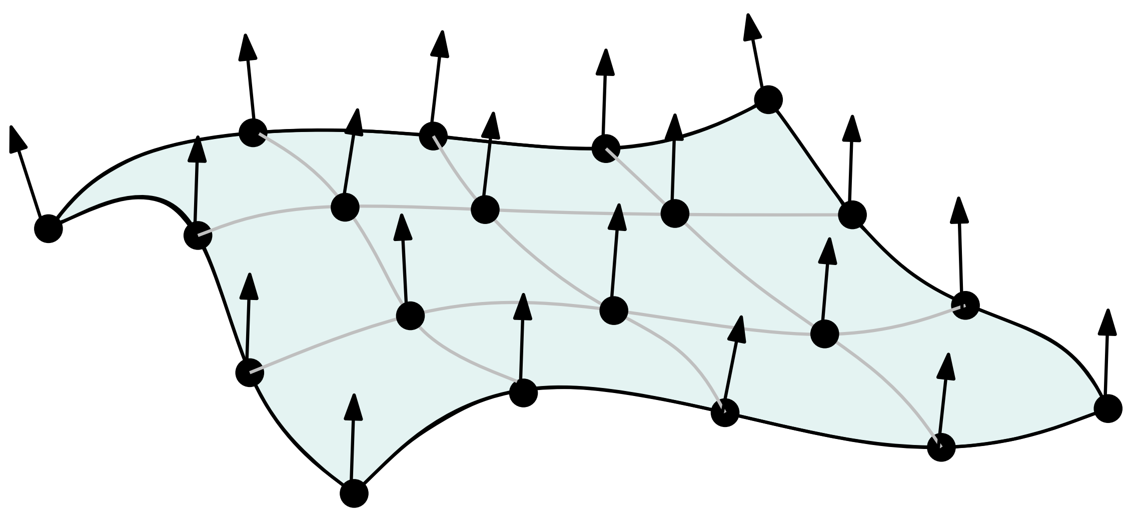

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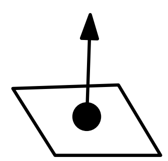

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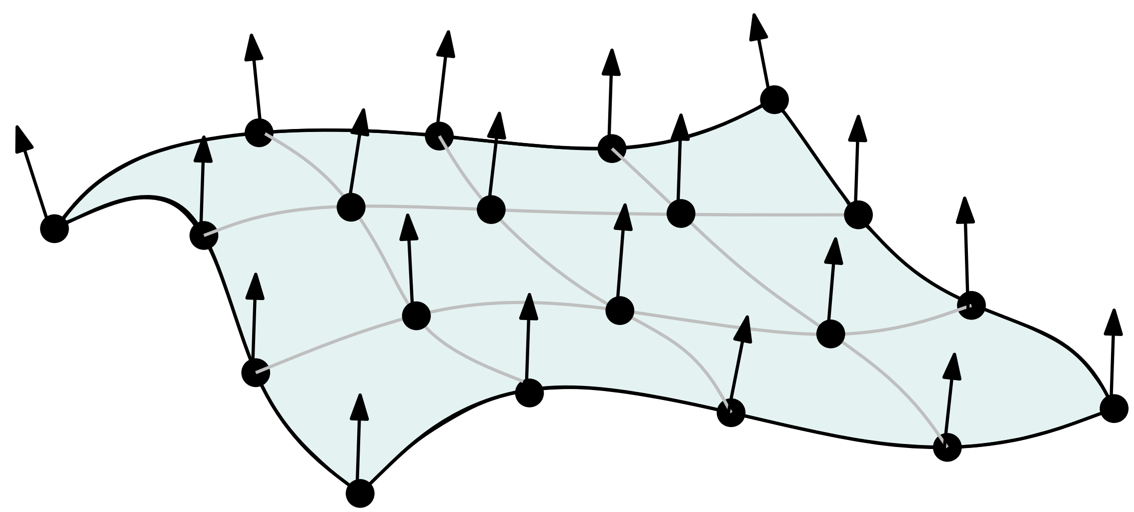

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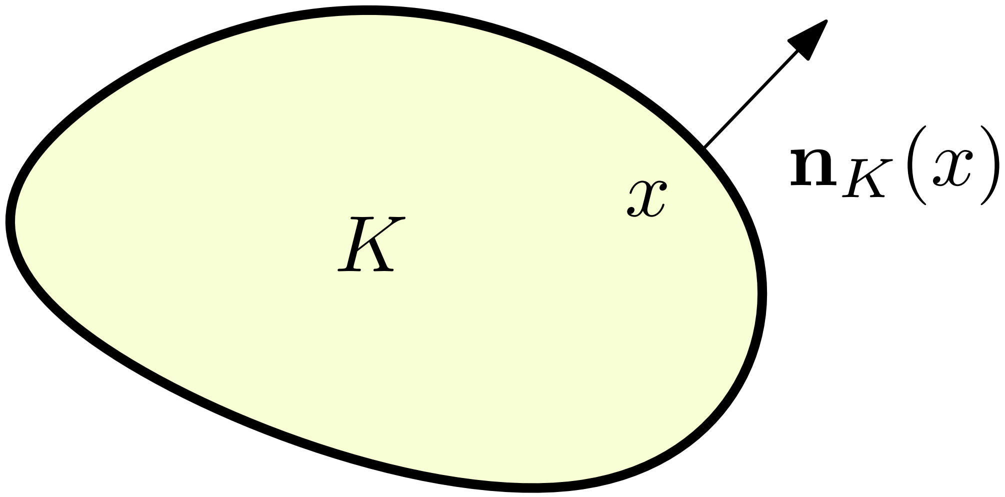
- Our goal: $S =$ boundary of a convex body, random sampling.

1. Minkowski problem

Surface area measure

K = convex body in $\mathbb{R}^d$



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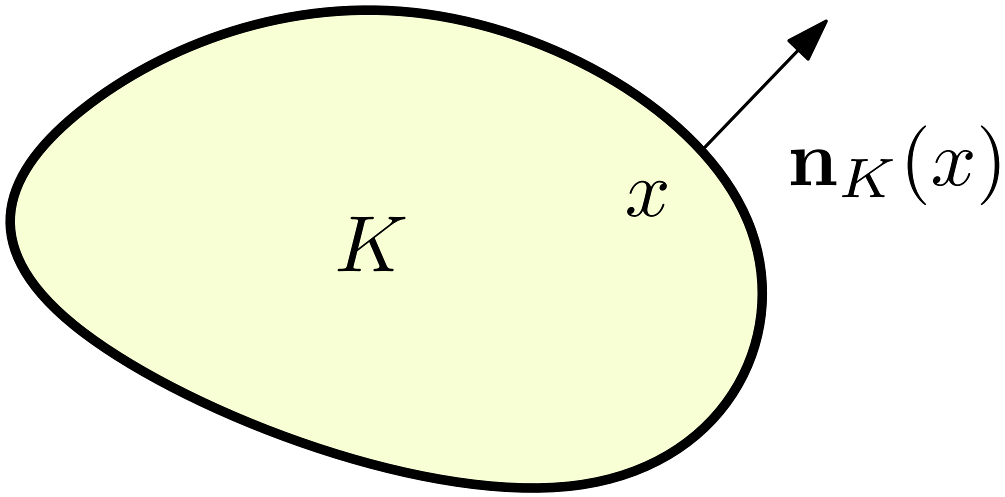

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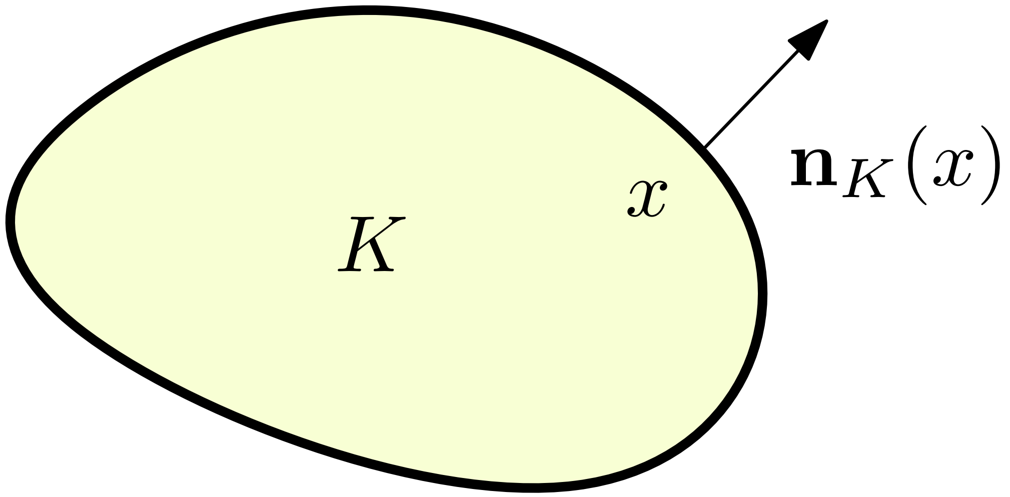

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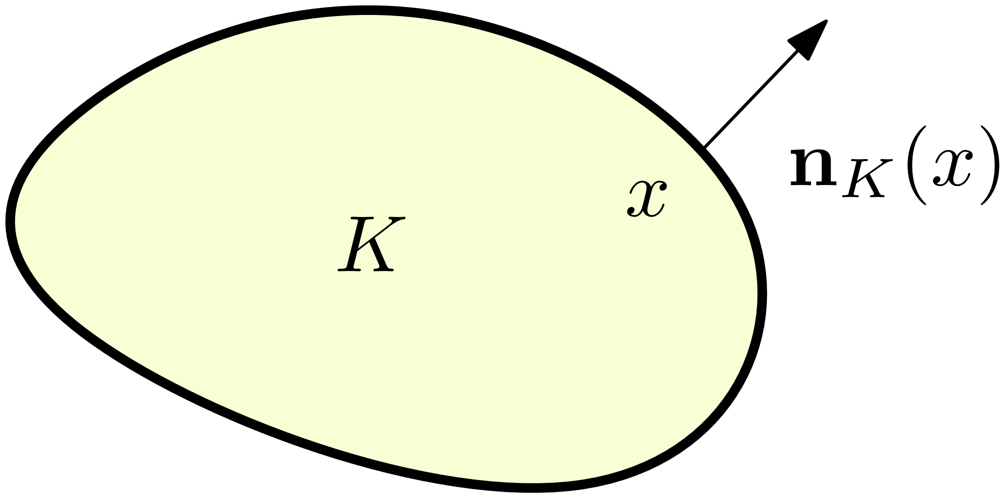

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F_i = ith face

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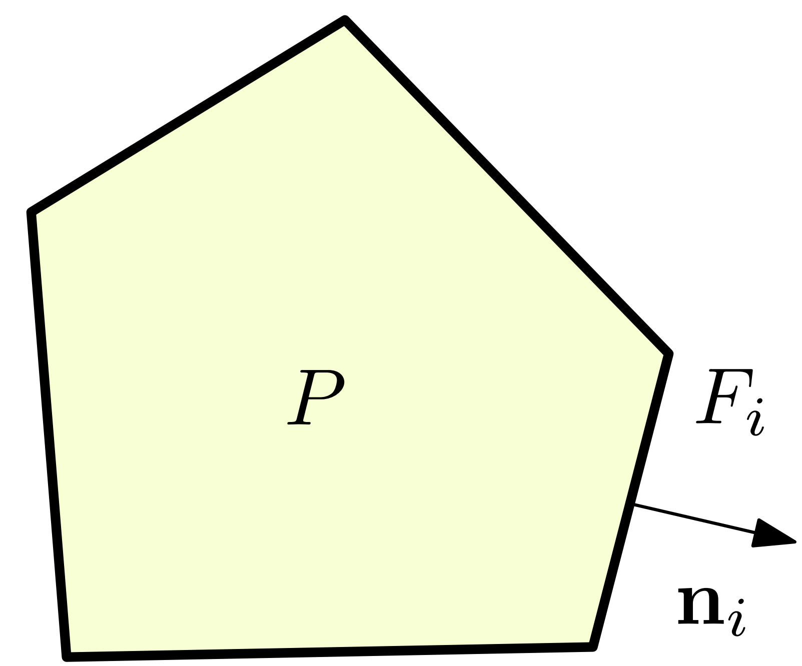

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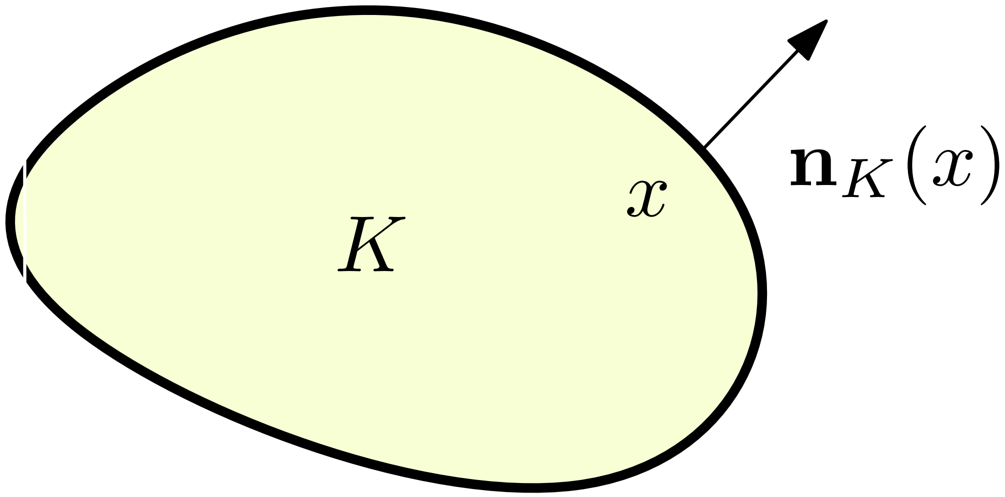

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$$\boxed{\mu_P = \sum_{i=1}^N \mathrm{area}(F_i)\delta_{\mathbf{n}_i}}$$

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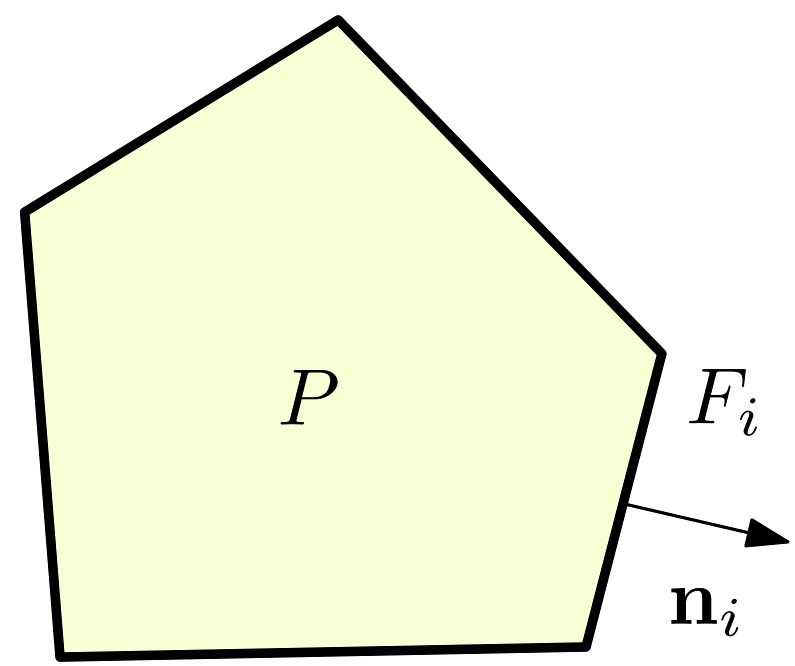

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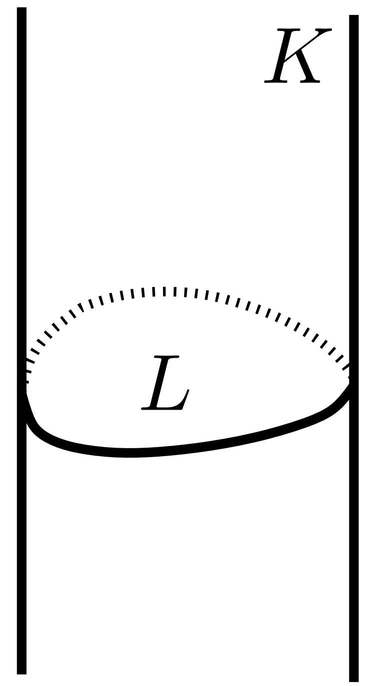
Avoids this situation:

If $L \subseteq \mathbb{R}^{d-1}$, and $K = L \times \mathbb{R}$, then $\mathbf{n}_K(\partial K) \subseteq \mathcal{S}^{d-2} \times \{0\}$



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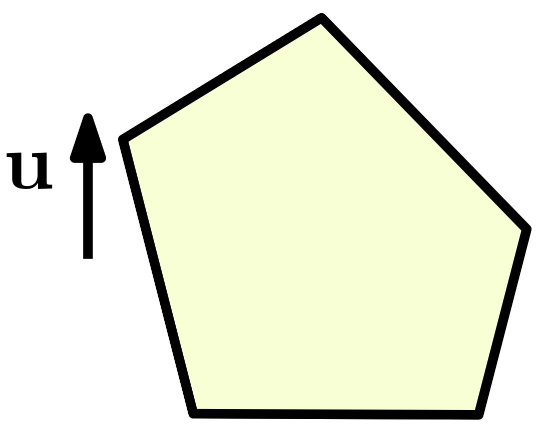

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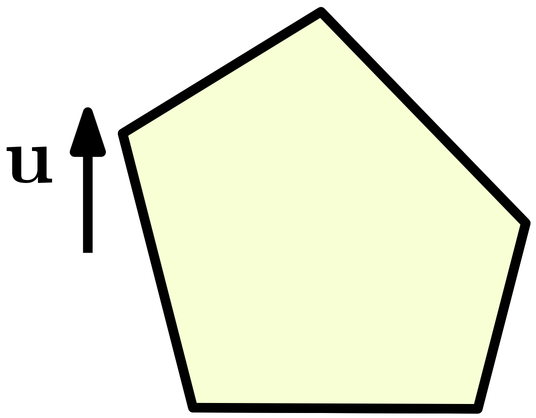

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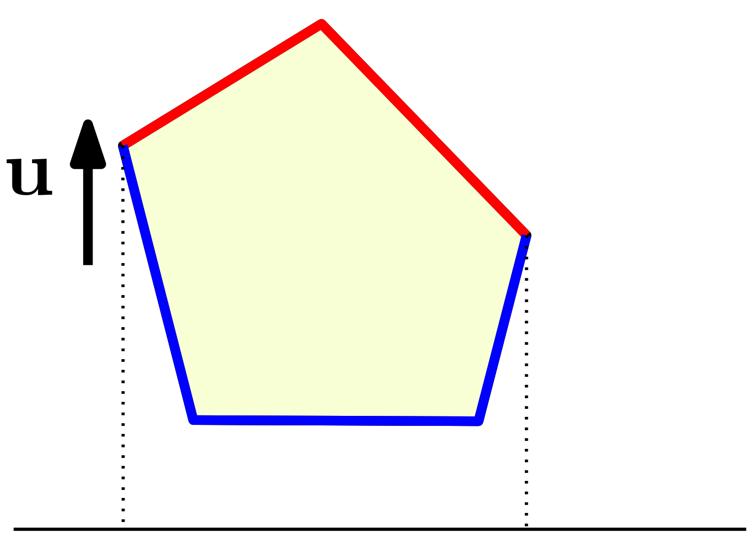
$$= \int_{\partial K} \langle \mathbf{n}_K(x) | \mathbf{u} \rangle \, \mathrm{d}\, x$$

$$= \mathrm{area}(\pi_H(\color{red}{\partial K})) - \mathrm{area}(\pi_H(\color{blue}{\partial K}))$$



$H = \{\mathbf{u}\}^{\perp}$

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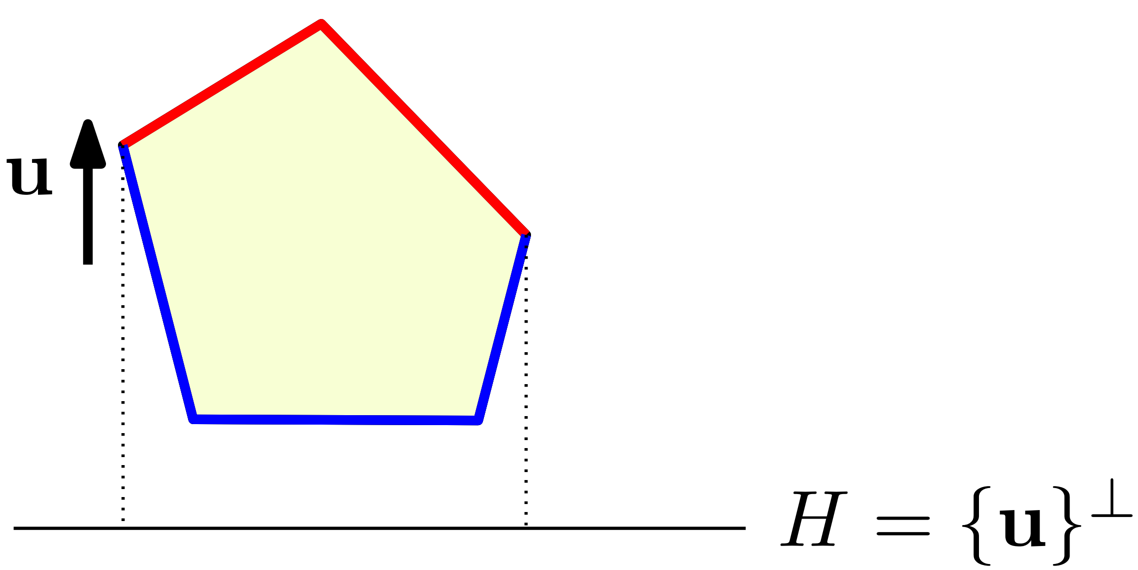

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Theorem (Minkowski-Alexandrov): Given any measure μ on $\mathcal{S}^{d-1}$, which satisfies **(non-degeneracy)** and **(zero-mean)**, there exists a convex body K with $\mu = \mu_K$. This body is unique up to translation.

Stability in Minkowski theorem

Definition: $\mathrm{d}_{\mathrm{bL}}(\mu, \nu) = \max_{f \in \mathrm{BL}_1} |\int_{\mathcal{S}^{d-1}} f \,\mathrm{d}\,\mu - \int_{\mathcal{S}^{d-1}} f \,\mathrm{d}\,\nu|$, where

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$\mathrm{d_{bL}}(\mu,\nu) = \mathrm{Wass}_1(\mu,\nu)$ when $\mu(\mathcal{S}^{d-1}) = \nu(\mathcal{S}^{d-1})$.

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$$R \geq \max(\mathrm{circumrad}(K), \mathrm{circumrad}(L))$$

the following inequality holds:

$$\min_{x \in \mathbb{R}^d} \mathrm{d}_H(K, L) \leq \mathrm{const}(r, R, d) \mathrm{d}_{\mathrm{bL}}(\mu_K, \mu_L)^{1/d}$$

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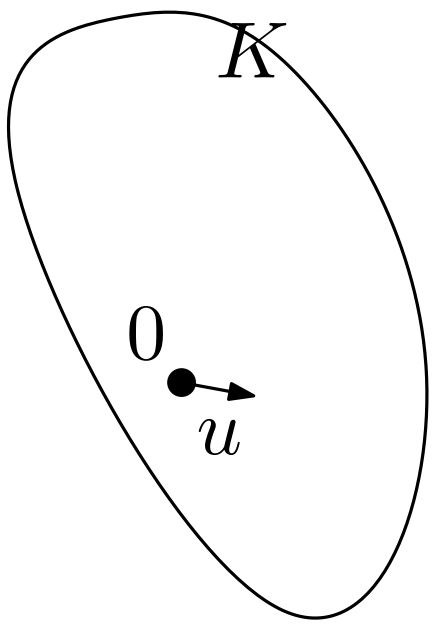
- Reconstruction result under random sampling

2. Improved stability theorem

A new distance between surface area measures



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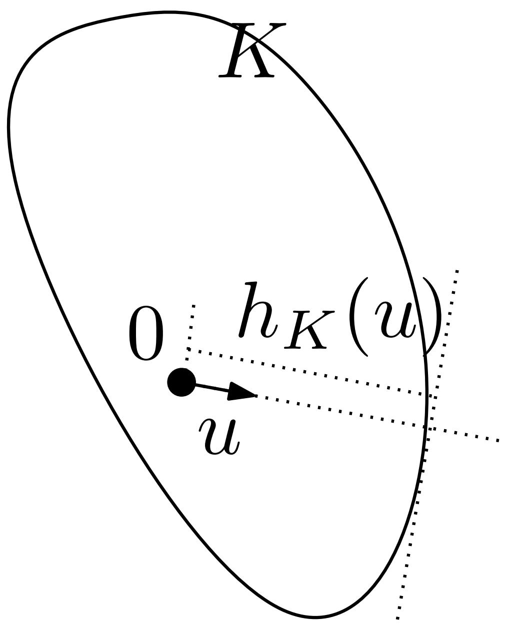

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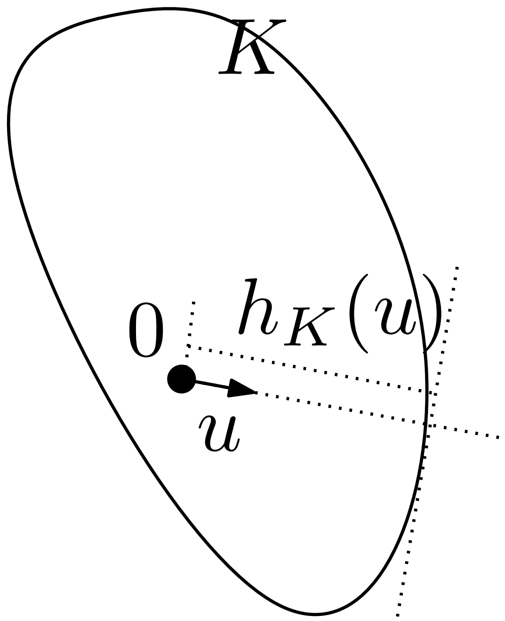

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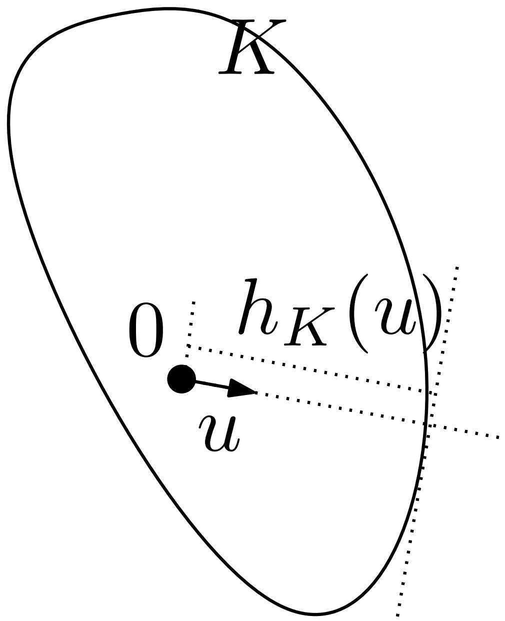

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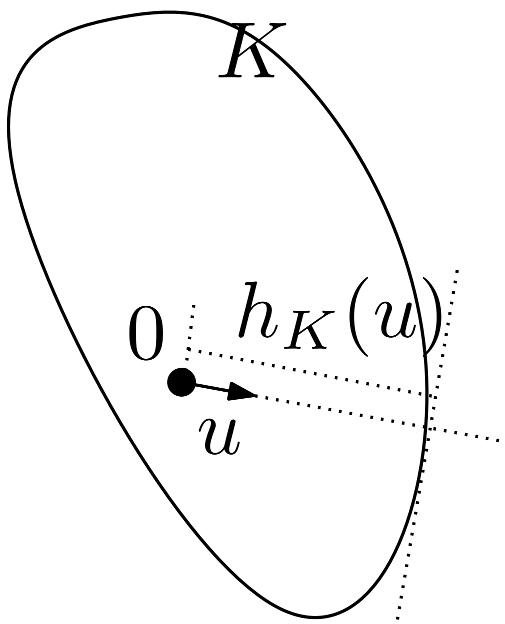

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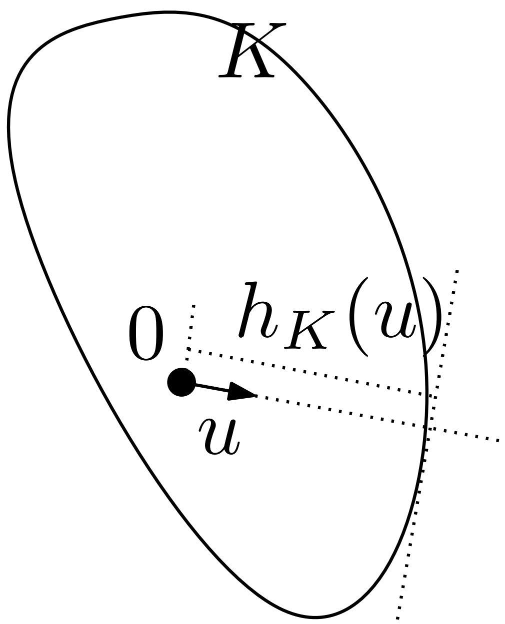

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- Since $\mathrm{C}_1 \subseteq \mathrm{BL}_1$, $\mathrm{d}_\mathrm{C} \leq \mathrm{d}_\mathrm{bL}$; a stability result for d_C is stronger.

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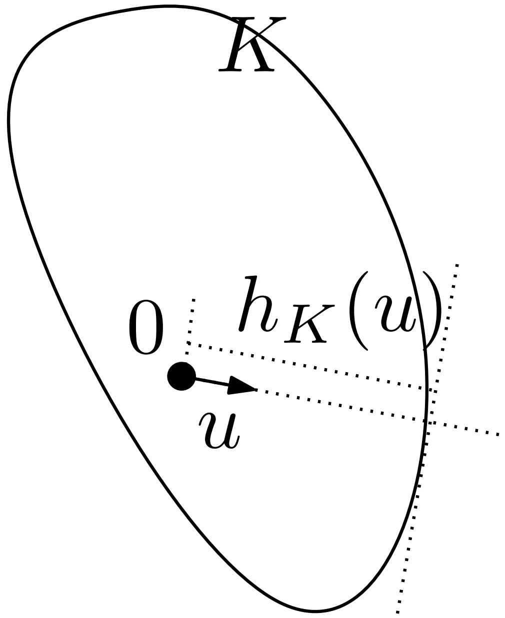

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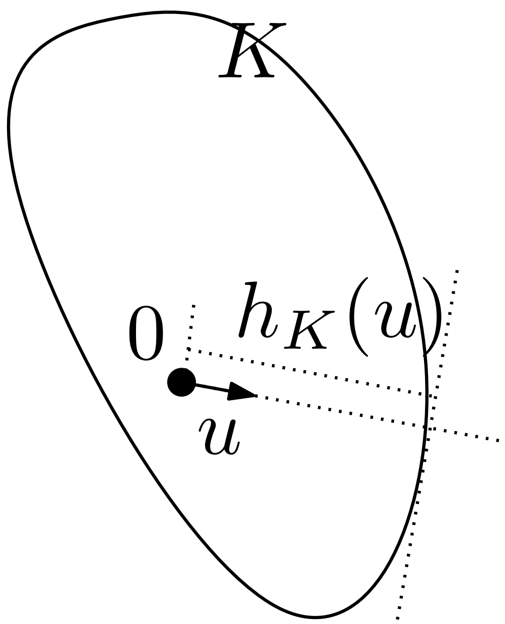
A new distance between surface area measures



Definition: Support function of K, $\mathrm{h}_K : \mathcal{S}^{d-1} \to \mathbb{R}$

$$\mathrm{h}_K(u) := \max_{p \in K} \langle u | p \rangle$$

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- d_{C} satisfies the triangle inequality, but $\mathrm{d}_{\mathrm{C}}(\mu, \nu) = 0 \not\Rightarrow \mu = \nu$.
- However, for surface area measures, $\mathrm{d}_{\mathrm{C}}(\mu_K, \mu_L) = 0 \Rightarrow \mu_K = \mu_L$.

Stability theorem for the convex-dual distance

Theorem (Abdallah-M. '13): Given a convex body K and μ a measure on $\mathcal{S}^{d-1}$ such that $\mathrm{d}_{\mathrm{C}}(\mu_K, \mu) \leq \varepsilon_0(K)$, there exists a convex body L such that $\mu = \mu_L$ and moreover

$$\min_{x \in \mathbb{R}^d} \mathrm{d}_H(K, L) \leq \mathrm{const}(K) \mathrm{d}_{\mathrm{C}}(\mu_K, \mu_L)^{1/d}$$

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- exponent $\frac{1}{d}$ is most likely non-optimal (best possible: $\frac{1}{d-1}$)
- to drop the requirement $r \leq \mathrm{inrad}(\mu_L)$, $R \geq \mathrm{circumrad}(\mu_L)$, we introduce the **weak rotundity** and exploit a lemma of Cheng and Yau.

Weak rotundity

Definition: The weak rotundity of a measure μ on $\mathcal{S}^{d-1}$ is:

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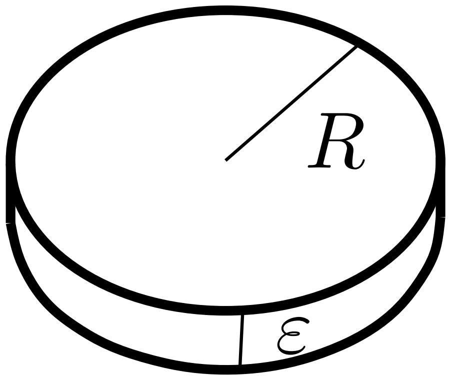

$K = \mathrm{B}^{d-1}(0,R) \times [0,\varepsilon]$

$\mathrm{circumrad}(K) \simeq R$ $\quad$ $\mathrm{area}(\partial K) \simeq R^{d-1}$

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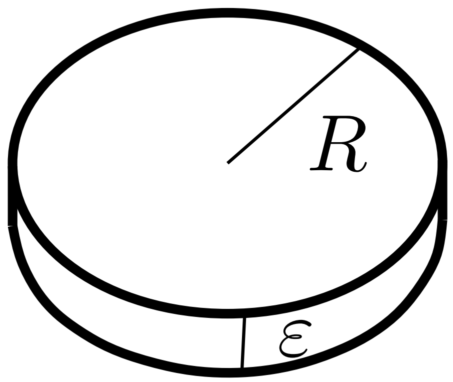

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- **Cheng-Yau Lemma:**

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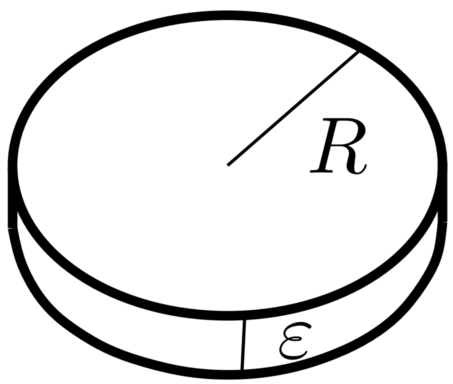

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- Is $\mathrm{rotund}(\mu)$ stable under perturbations of μ ?

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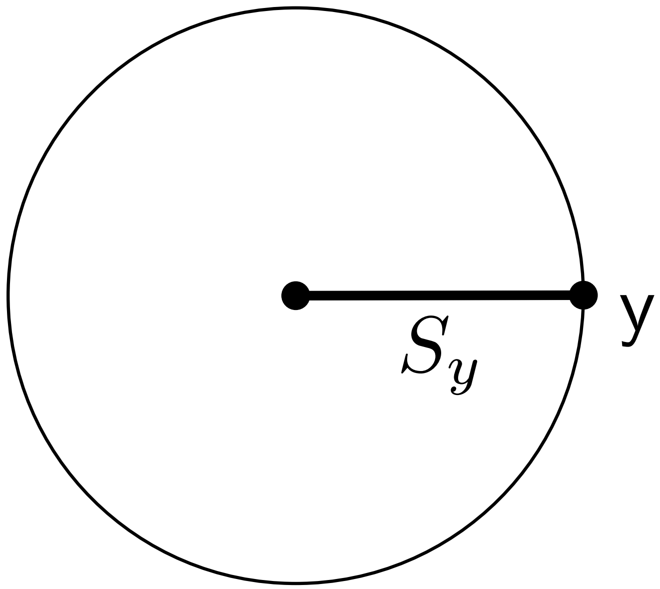

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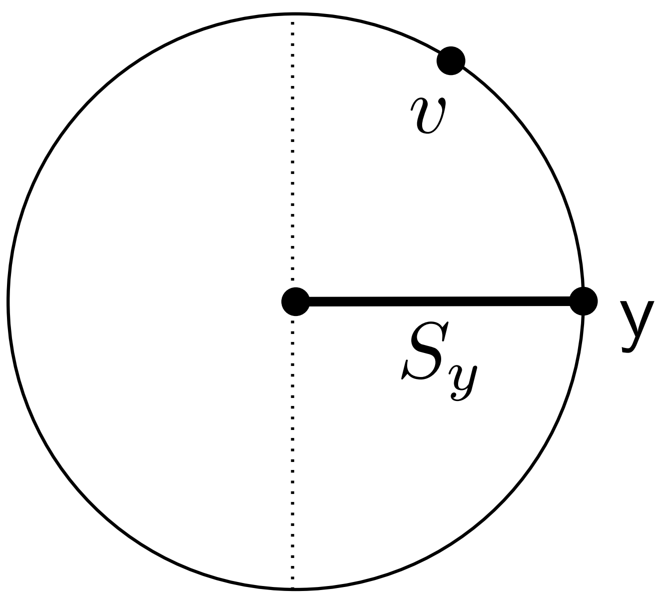

$$h_{S_y}(v) = \max_{x \in S_y} \langle x|v\rangle = \begin{cases} \langle v|y\rangle \text{ if } \langle v|y\rangle \geq 0 \\ 0 \text{ if } \langle v|y\rangle \leq 0 \end{cases}$$

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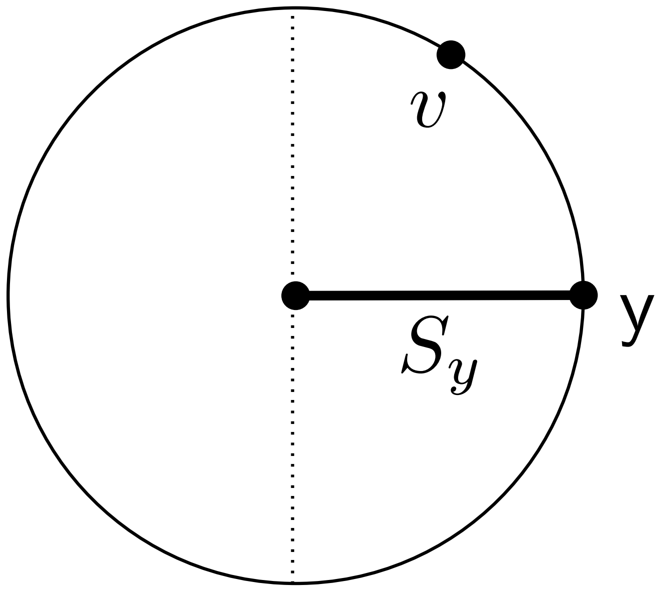

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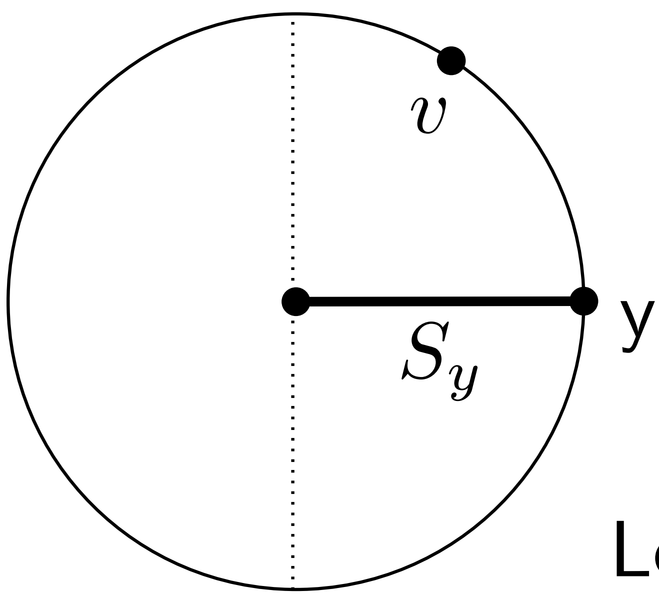

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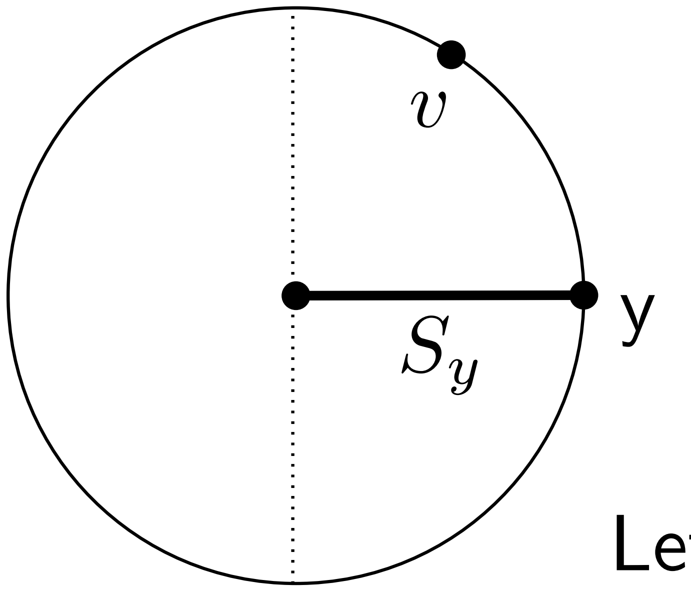

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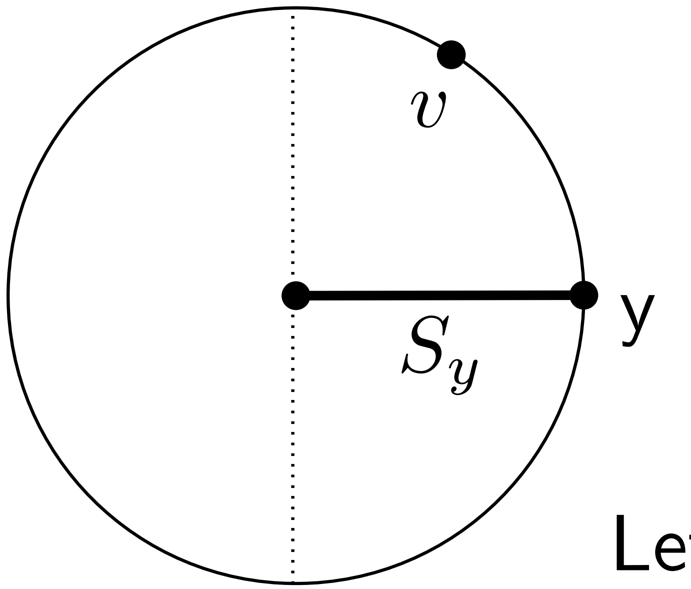

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One concludes using $\mathrm{rotund}(\mu) = \min_{y\in\mathcal{S}^{d-1}} f_\mu(y)$.

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Therefore μ satisfies **(non-degeneracy)** and **(zero-mean)**.

The result follows from Minkowski's theorem.

3. Reconstruction under random sampling

Reconstruction theorem

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Theorem (Abdallah-M. '13): Let K be a convex body w. $\mathrm{area}(\partial K) = 1$. Then for $\eta > 0$, $p \in (0,1)$, and N random normal measurements with

$$N \geq \mathrm{const}(K,d)\eta^{-\frac{1}{2}(3d+d^2)}\log(1/p),$$

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- Analysis using the bound involving d_{bL} would give η^{-6} and η^{-12}.

Reconstruction theorem: sketch of proof

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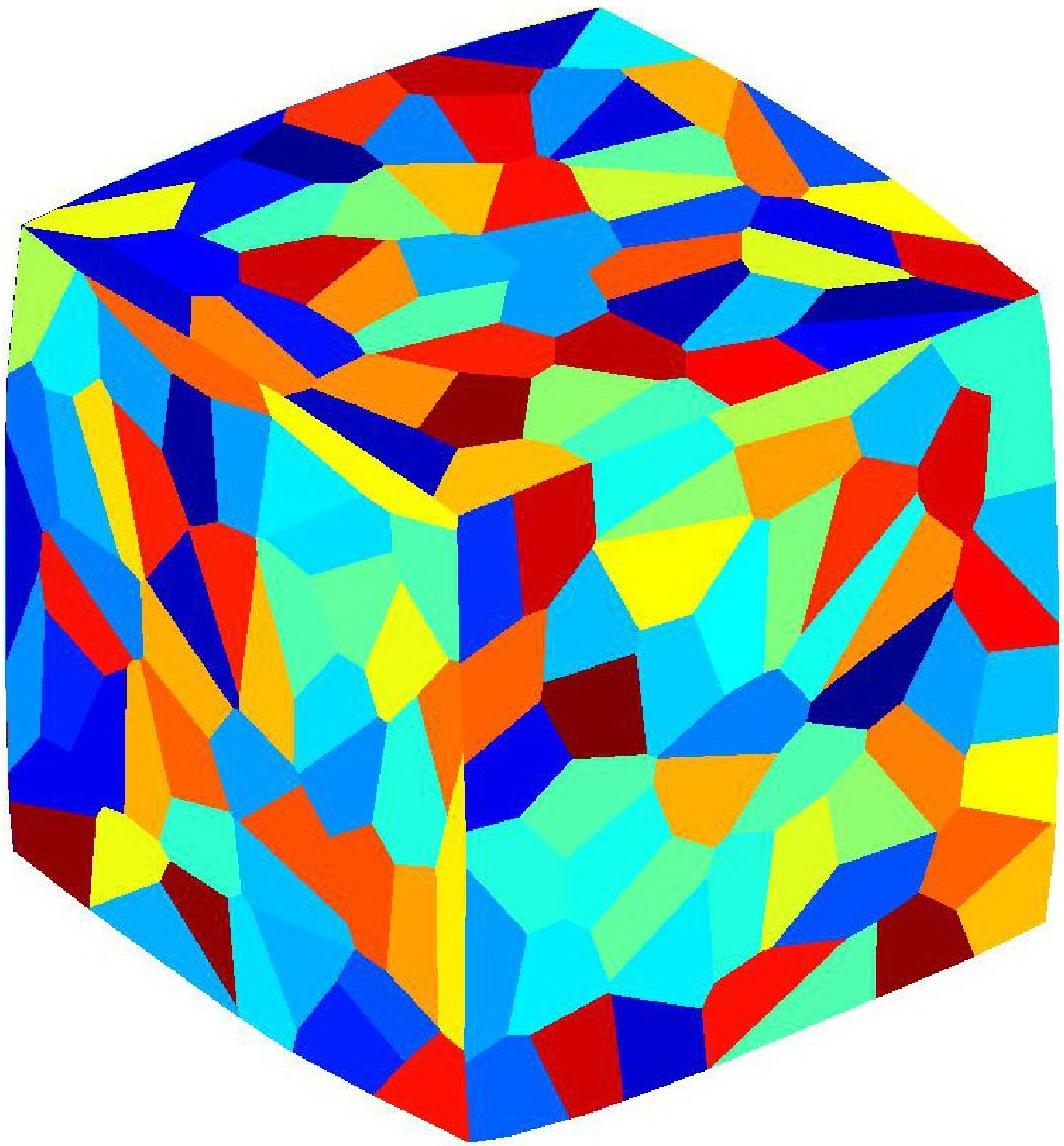
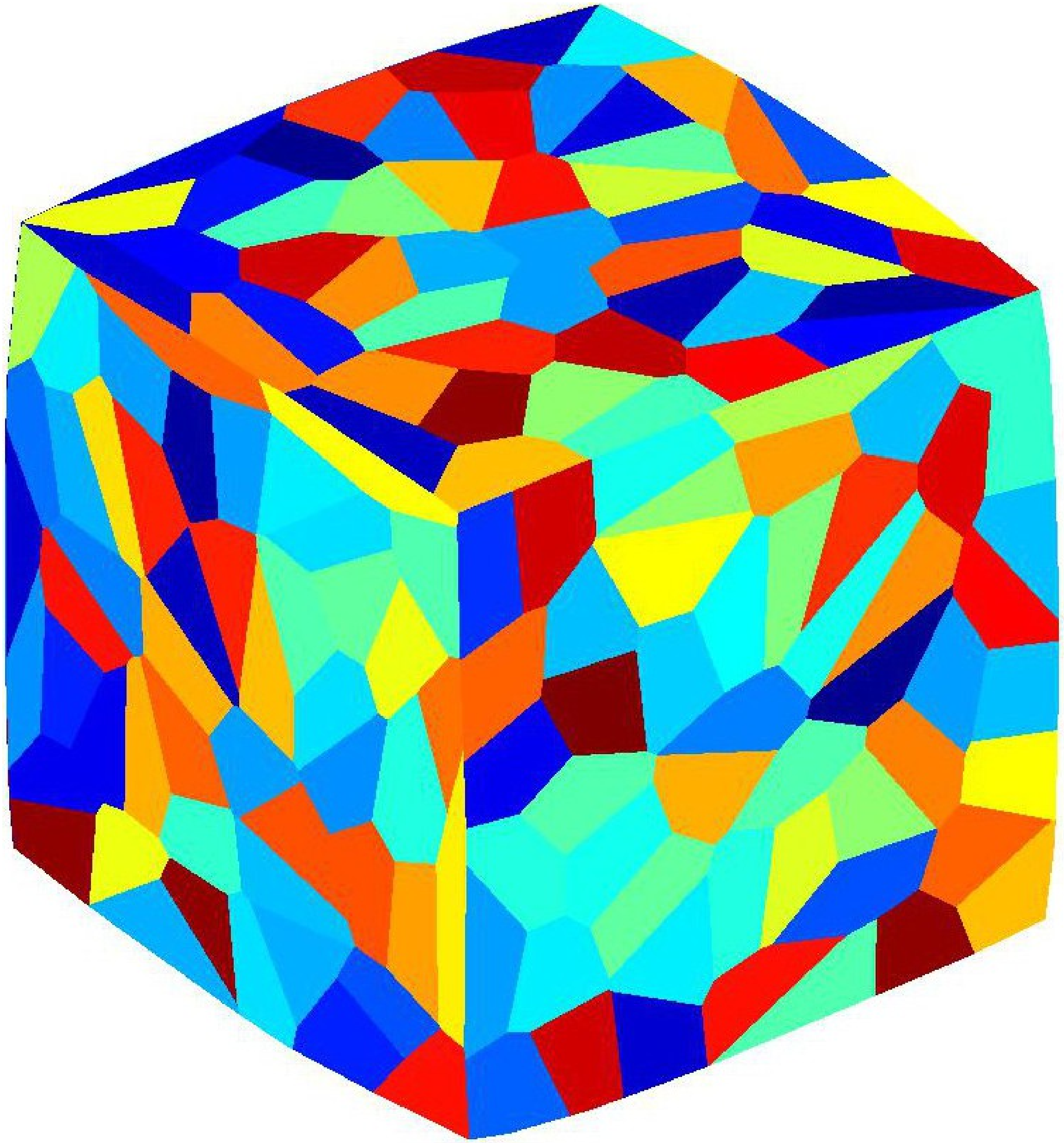
- Conclusion uses the $\mathrm{d_C}$-stability of the weak rotundity and mean, and the stability theorem using the convex-dual distance.

Reconstruction theorem: an example



- Reconstruction of a unit cube from $N = 300$ random normal measurements, with a uniform noise of 0.05 on the normals.

Reconstruction theorem: an example



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- Computation: variational characterization of solutions to Minkowski problem.

Conclusion

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Thank you!