



The Compressed Annotation Matrix: An Efficient Data Structure for Computing Persistent Cohomology

Jean-Daniel Boissonnat & Tamal Dey & Clément Maria

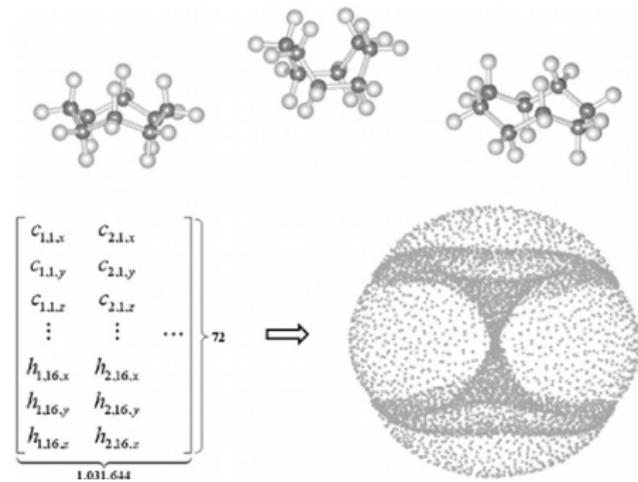
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Introduction

Motivation

Given a set of geometric points,
study the global features of the
underlying space: components,
holes, voids, etc.

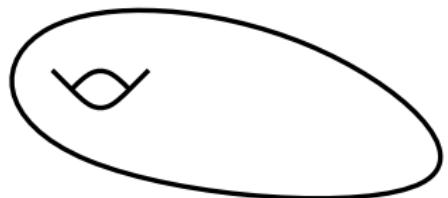
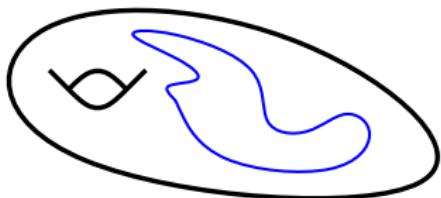
- ▶ Topological Data Analysis
- ▶ Machine Learning
- ▶ Vizualisation
- ▶ ...



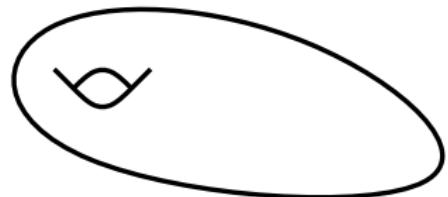
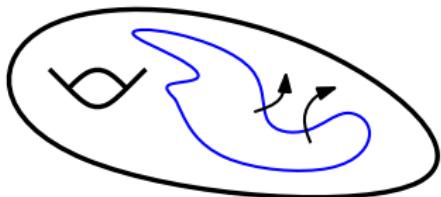
Conformation Space of the
Cyclo-octane Molecule

Image: [Martin, Thompson,
Coutsias, Watson]

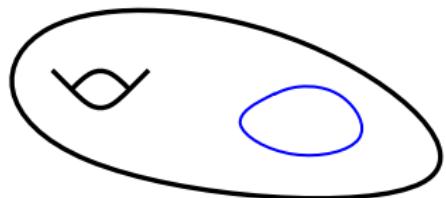
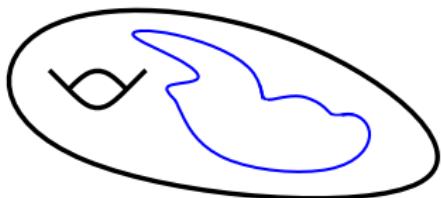
Introduction to Homology



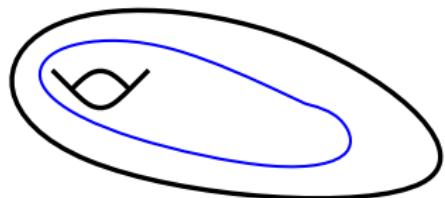
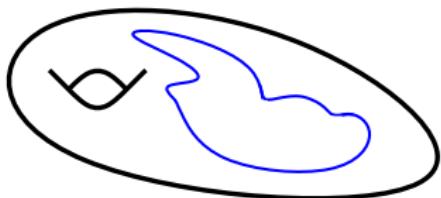
Introduction to Homology



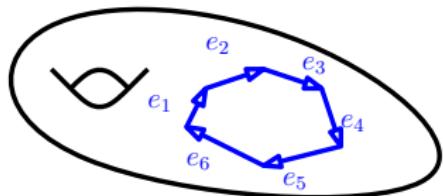
Introduction to Homology



Introduction to Homology



Introduction to Homology



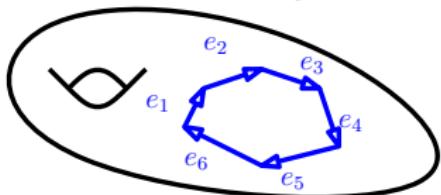
$\{e_1, e_2, e_3, e_4, e_5, e_6, \dots\}$

\mathbf{K}^p : set of p -simplices of \mathbf{K}

$\{ \nearrow, \rightarrow, \downarrow, \leftarrow, \nwarrow, \swarrow, \dots \}$

Introduction to Homology

$$e_1 + e_2 + e_3 + e_4 + e_5 + e_6$$



$$\{e_1, e_2, e_3, e_4, e_5, e_6, \dots\}$$

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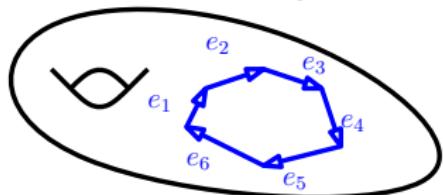
$$\{ \nearrow, \rightarrow, \downarrow, \leftarrow, \nwarrow, \swarrow, \dots \}$$

\mathbf{C}_p : Abelian group of formal sums
of p -simplices with \mathbb{Z} -coefficients

$$\mathbf{C}_1 = (\{\sum_i k_i e_i\}, +)$$

Introduction to Homology

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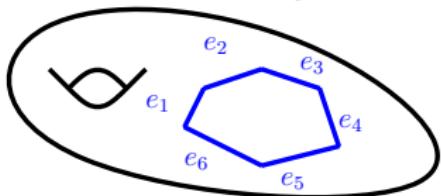
$\partial_p : \mathbf{C}_p \rightarrow \mathbf{C}_{p-1}$: Boundary operator

$$\partial_2 \begin{array}{c} \text{triangle} \\ \text{with curved arrow} \end{array} = \begin{array}{c} \diagup \\ - \\ \diagdown \end{array} + \begin{array}{c} \diagdown \\ + \\ \diagup \end{array}$$

$$\partial_2 [a, b, c] = [ab] - [bc] + [ca]$$

Introduction to Homology

$$e_1 + e_2 + e_3 + e_4 + e_5 + e_6$$



$$\{e_1, e_2, e_3, e_4, e_5, e_6, \dots\}$$

\mathbf{K}^p : set of p -simplices of \mathbf{K}

$$\{ / , \backslash , \diagup , \diagdown , \dots \}$$

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$$\mathbb{Z}_2$$

$$\mathbf{C}_1 = (\{\sum_i k_i e_i\}, +)$$

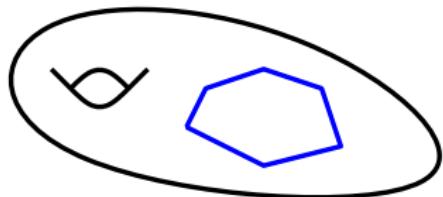
$k_i \in \{0, 1\}$

$\partial_p : \mathbf{C}_p \rightarrow \mathbf{C}_{p-1}$: Boundary operator

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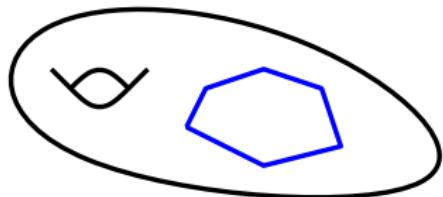
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Introduction to Homology



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Introduction to Homology

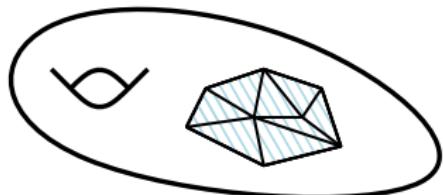


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$\mathbf{Z}_p = \ker \partial_p$: the p -cycles

$$\partial_1 = \begin{array}{c} \text{a blue hexagon} \\ = \end{array} \begin{array}{c} \text{a blue hexagon with arrows} \\ \bullet \times 2 \quad \bullet \times 2 \\ \bullet \times 2 \quad \bullet \times 2 \\ \bullet \times 2 \quad \bullet \times 2 \end{array} = 0$$

Introduction to Homology



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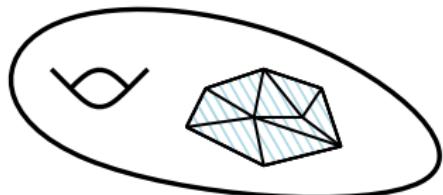
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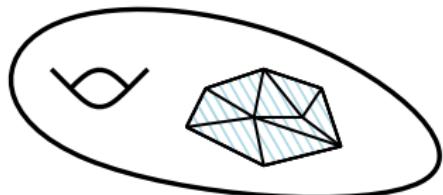
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Fundamental property: $\partial_p \circ \partial_{p+1} = 0$

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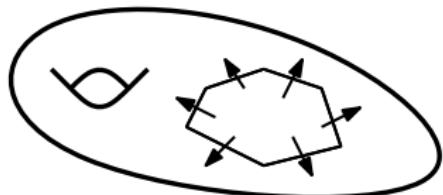
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$\mathbf{H}_p = \mathbf{Z}_p / \mathbf{B}_p$: p^{th} homology group

Introduction to Homology



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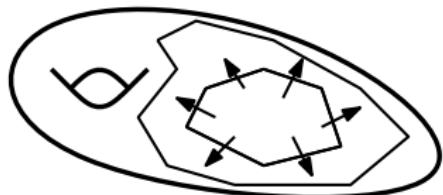
$\mathbf{B}_p = \text{im } \partial_{p+1}$: the p -boundaries

$$\begin{array}{c} \text{empty hexagon} \\ \text{with vertices labeled} \\ \bullet \times 2 \quad \bullet \times 2 \\ \bullet \times 2 \quad \bullet \times 2 \\ \bullet \times 2 \end{array} = \partial_2 \begin{array}{c} \text{hexagon with diagonal} \\ \text{and vertical edges shaded} \end{array}$$

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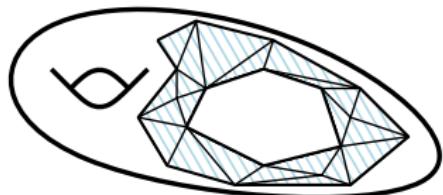
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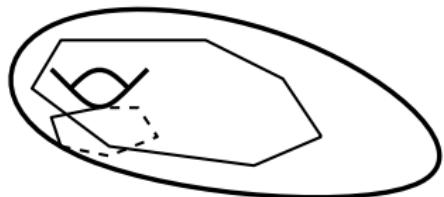
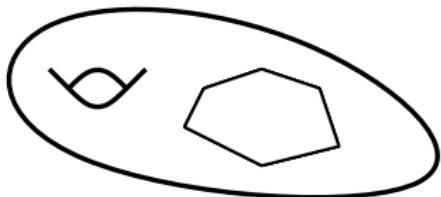
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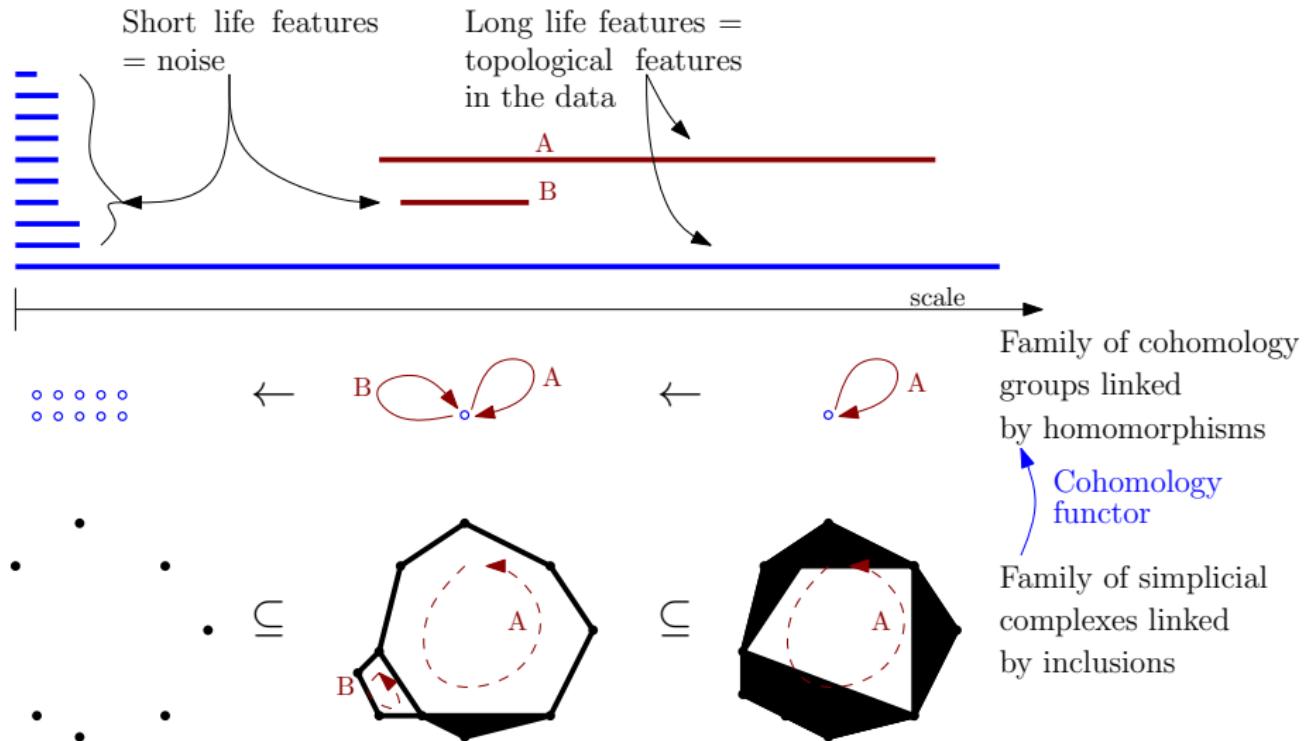
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Cohomology and Persistence



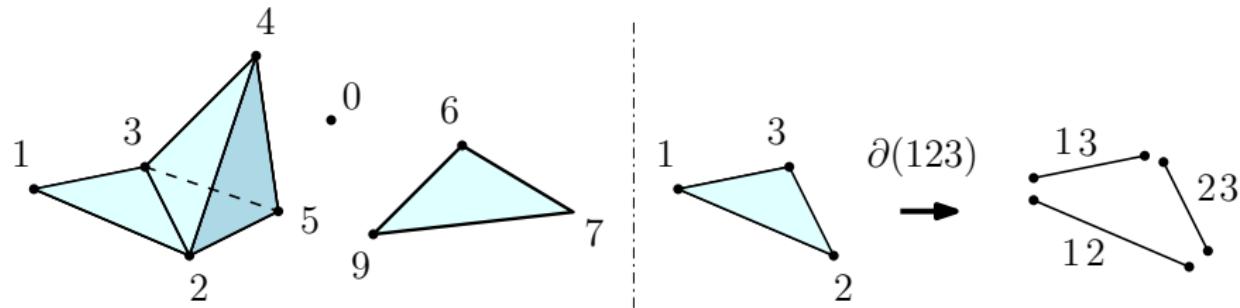
Simplicial Complex

Given a set $V = \{1 \cdots n\}$ of vertices, an **abstract simplicial complex** \mathcal{K} on V is a family of subsets of vertices s.t.:

$$\forall \sigma \in \mathcal{K} : \tau \subseteq \sigma \Rightarrow \tau \in \mathcal{K}$$

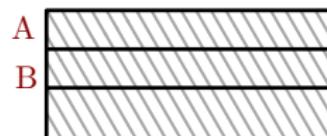
Such σ is called a simplex

Boundary of a simplex σ : $\partial\sigma = \{\tau \subset \sigma | \dim(\sigma) = \dim(\tau) + 1\}$



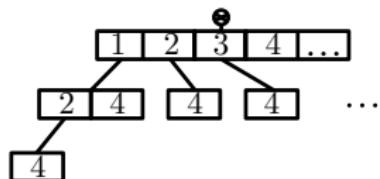
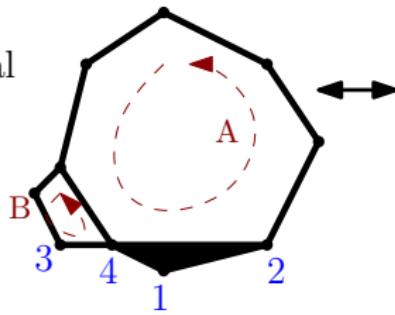
Representation

Cohomology group



Matrix representation

Simplicial complex

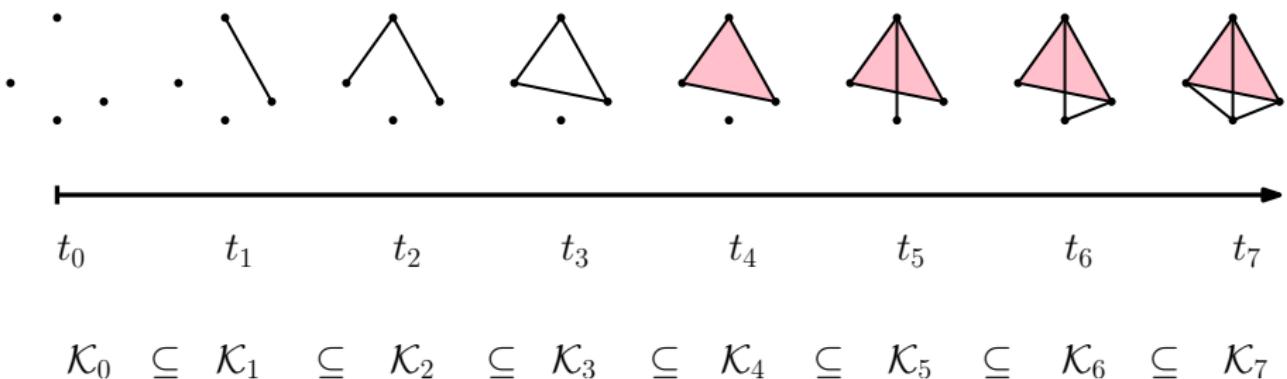


◦ Simplex tree
[Boissonnat, M. '12]

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Persistent Cohomology Algorithm

Cohomology Algorithm



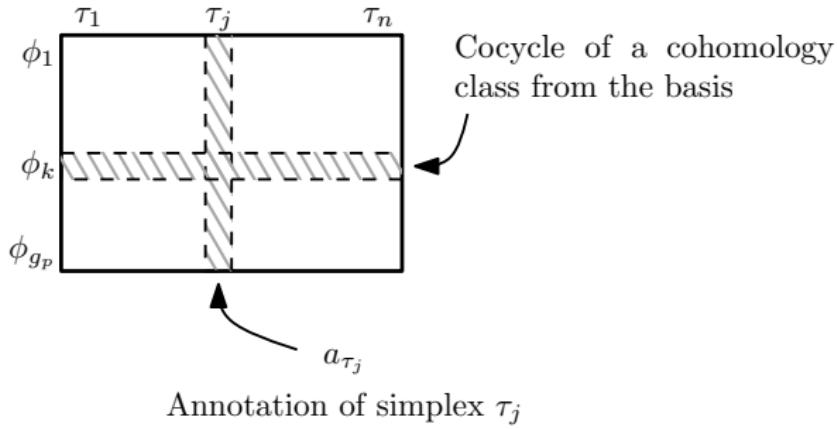
- ▶ Insert the simplices in the order of the filtration
- ▶ Update the cohomology groups accordingly

Cohomology Algorithm

[de Silva, Vejdemo-Johansson, Morozov '11],[Dey, Fan, Wang]

\mathcal{K}_i

$$H^p(\mathcal{K}_i) = ([\phi_1], \dots, [\phi_{g_p}])$$

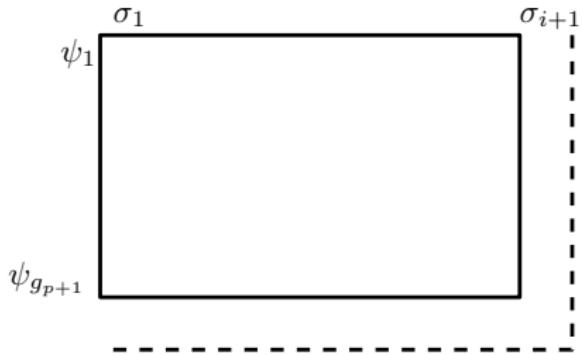
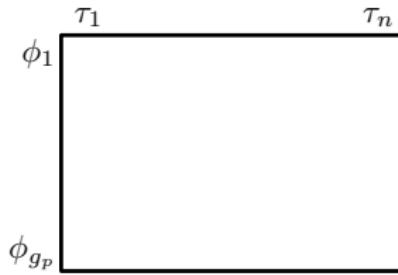


Cohomology Algorithm

[de Silva, Vejdemo-Johansson, Morozov '11],[Dey, Fan, Wang]

$$\mathcal{K}_i \hookrightarrow \mathcal{K}_i \cup \sigma_{i+1} = \mathcal{K}_{i+1} \quad H^*(\mathcal{K}_i) \rightarrow H^*(\mathcal{K}_{i+1})$$

$$H^p(\mathcal{K}_i) \quad H^{p+1}(\mathcal{K}_i)$$

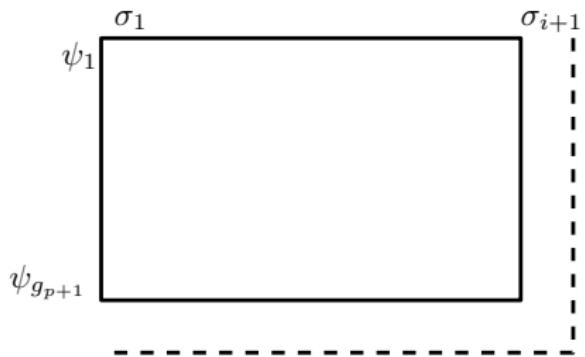
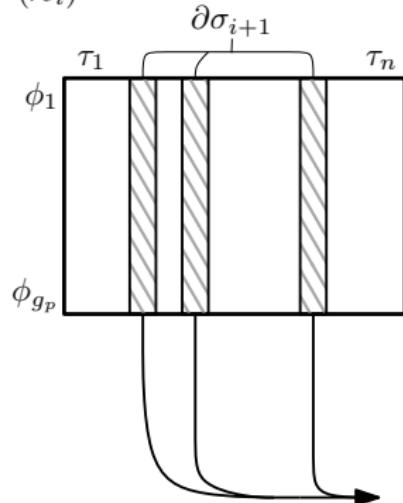


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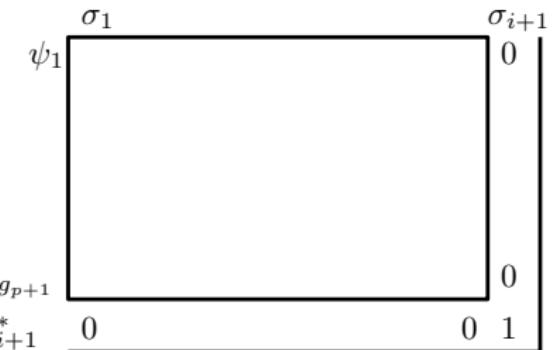
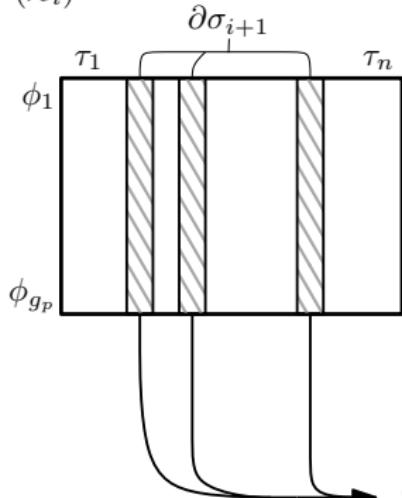
$$a_{\partial\sigma_{i+1}} = \sum_{\tau \in \partial\sigma_{i+1}} (\pm 1) a_\tau$$

Cohomology Algorithm

[de Silva, Vejdemo-Johansson, Morozov '11], [Dey, Fan, Wang]

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$$a_{\partial\sigma_{i+1}} = \sum_{\tau \in \partial\sigma_{i+1}} (\pm 1) a_\tau$$

(i) $a_{\partial\sigma_{i+1}} = 0 \Rightarrow \sigma_{i+1}^*$ independent cocycle

Cohomology Algorithm

[de Silva, Vejdemo-Johansson, Morozov '11], [Dey, Fan, Wang]

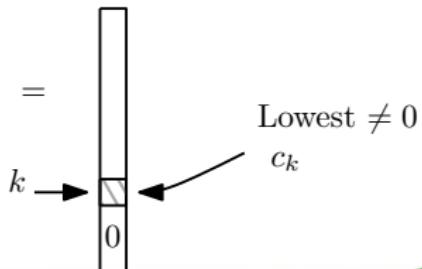
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$$H^p(\mathcal{K}_i) \quad H^{p+1}(\mathcal{K}_i)$$

	τ_1	τ_j	τ_n	
ϕ_1				
ϕ_k	0 … 0	0 … 0	0 … 0	0
ϕ_{g_p}				

	σ_1	σ_{i+1}	0
ψ_1			0
ψ_k			⋮
$\psi_{g_{p+1}}$			0

$$a_{\partial\sigma_{i+1}} = \sum_{\tau \in \partial\sigma_{i+1}} (\pm 1) a_\tau =$$



(ii) $a_{\partial\sigma_{i+1}} \neq 0 \Rightarrow$ Reduction

Cohomology Algorithm

[de Silva, Vejdemo-Johansson, Morozov '11], [Dey, Fan, Wang]

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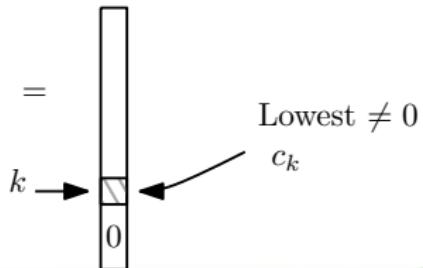
$$H^p(\mathcal{K}_i) \quad H^{p+1}(\mathcal{K}_i)$$

	τ_1	τ_j	τ_n	
ϕ_1				
ϕ_k	0 ... 0	0 ... 0	0 ... 0	0
ϕ_{g_p}				

	σ_1	σ_{i+1}	
ψ_1		0	
ψ_k		\vdots	
$\psi_{g_{p+1}}$		0	

$$a_{\tau_j} \leftarrow a_{\tau_j} - \frac{a_{\tau_j}[k]}{c_k} a_{\partial\sigma_{i+1}}$$

$$a_{\partial\sigma_{i+1}} = \sum_{\tau \in \partial\sigma_{i+1}} (\pm 1) a_\tau =$$



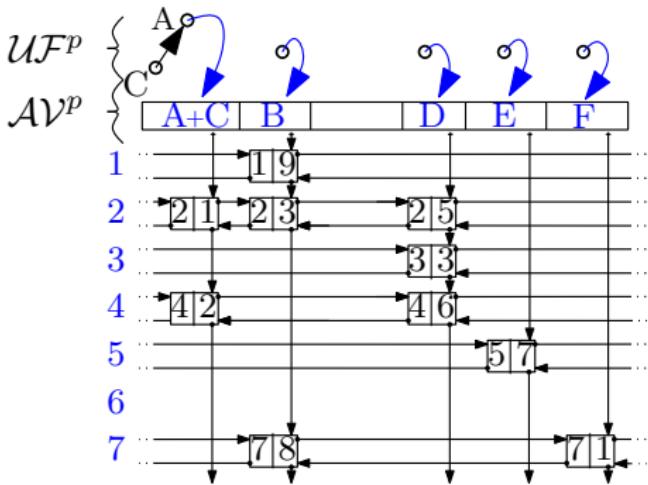
(ii) $a_{\partial\sigma_{i+1}} \neq 0 \Rightarrow$ Reduction

3

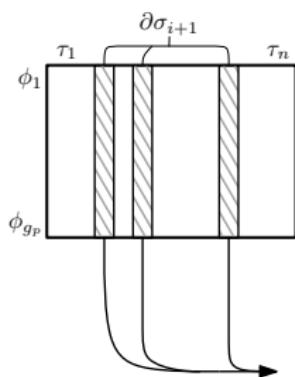
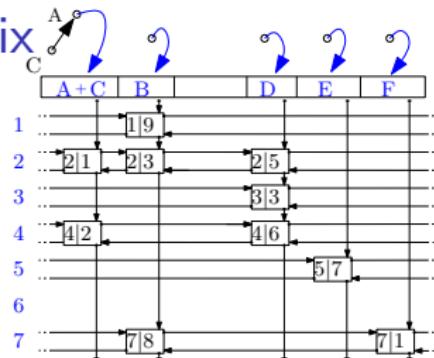
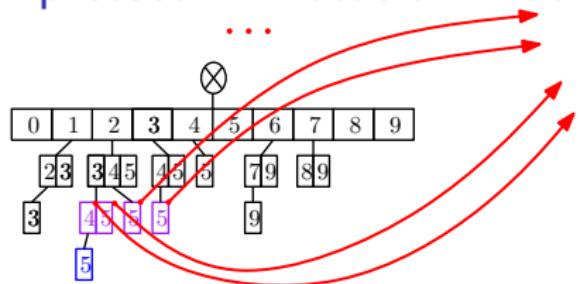
Compressed Annotation Matrix

Compressed Annotation Matrix

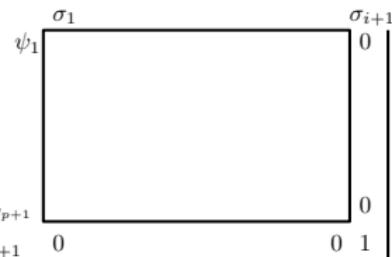
$$\begin{array}{c}
 \text{A} \quad \text{B} \quad \text{C} \quad \text{D} \quad \text{E} \quad \text{F} \\
 \begin{matrix}
 1 & \left(\begin{array}{cccccc} 0 & 9 & 0 & 0 & 0 & 0 \\ 2 & 1 & 3 & 1 & 5 & 0 & 0 \\ 3 & 0 & 0 & 0 & 3 & 0 & 0 \\ 4 & 2 & 0 & 2 & 6 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 7 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 8 & 0 & 0 & 0 & 1 \end{array} \right) \\
 \end{matrix}
 \end{array}$$



Compressed Annotation Matrix



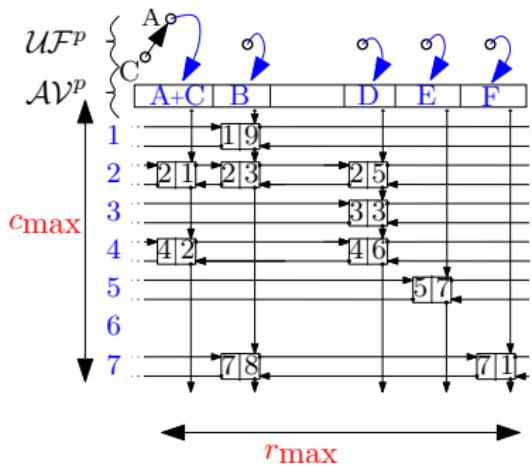
$$a_{\partial\sigma_{i+1}} = \sum_{\tau \in \partial\sigma_{i+1}} (\pm 1) a_\tau \neq 0$$



$$a_{\tau_j} \leftarrow a_{\tau_j} - \frac{a_{\tau_j}[k]}{c_k} a_{\partial\sigma_{i+1}}$$

Complexity Analysis

- ▶ m number of simplices
- ▶ k dimension of the complex
- ▶ c_{\max} longest column
- ▶ r_{\max} longest row
- ▶ $\mathcal{C}_{\mathcal{AV}}$: operation in \mathcal{AV}
- ▶ \mathcal{C}_∂ : boundary computation of σ
- ▶ $\alpha(\cdot)$ inverse Ackermann function



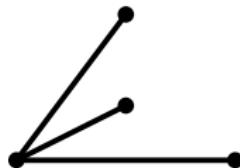
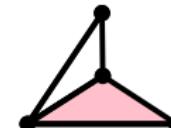
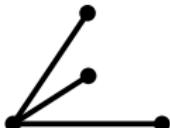
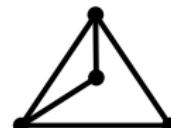
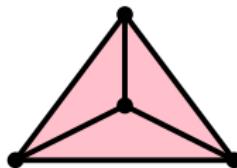
$$O(m \times [\mathcal{C}_\partial + k(\alpha(m) + c_{\max}) + r_{\max}(c_{\max} + \mathcal{C}_{\mathcal{AV}} + \alpha(m))])$$

and simplifying a bit: $O(m c_{\max}(k + r_{\max}))$

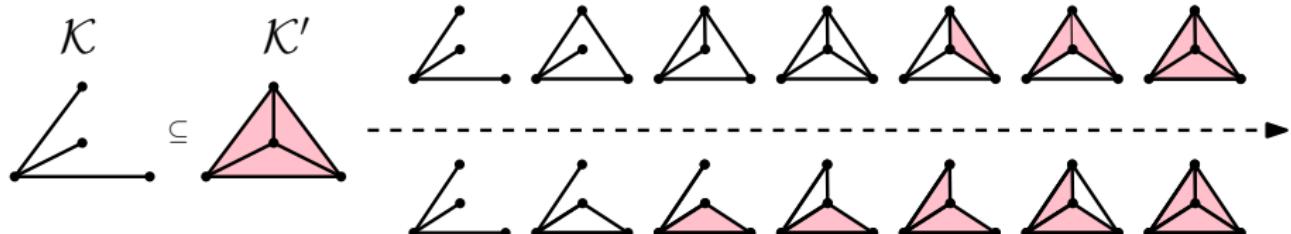
4

Reordering the Simplices

Reordering Iso-Simplices

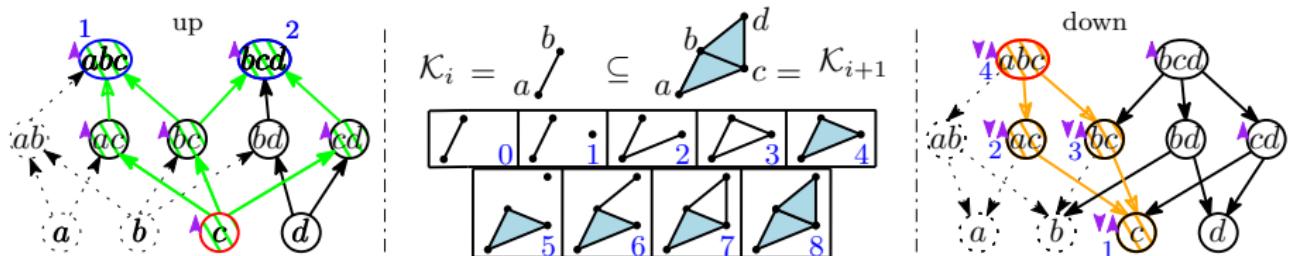
 \mathcal{K}  \subseteq \mathcal{K}' 

Reordering Iso-Simplices



Given a simplex σ

- ▶ Order its maximal cofaces “depth-first”
- ▶ Insert the maximal faces and their subfaces



5 Experiments

Timings

Data	Cpx	D	d	$ \mathcal{K} $	DioCoH		PHAT^\perp		PHAT		CAM	
					\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}
Cy8	Rips	24	2	21×10^6	420	4822	44	—	5.3	—	6.4	6.5
S4	Rips	5	4	72×10^6	943	1026	95	—	3591	—	22.5	23.2
L57	Rips	—	3	34×10^6	239	240	35.2	—	972	—	9.3	9.5
Bro	Wit	25	?	3.2×10^6	807	T_∞	6.3	—	0.88	—	2.7	2.9
KI	Wit	5	2	74×10^6	569	662	101	—	1785	—	19.7	19.9
L35	Wit	—	3	18×10^6	109	110	17.5	—	869	—	5.1	5.1
Bud	α Sh	3	2	1.4×10^6	30.0	30.9	2.6	—	0.32	—	0.7	0.7
Nep	α Sh	3	2	57×10^6	T_∞	T_∞	163	—	33	—	39.5	40.2

- ▶ DioCoH: Dionysus (persistent cohomology)
- ▶ PHAT $^\perp$: Phat (persistent cohomology)
- ▶ PHAT: Phat (persistent homology)

Timings

Data	Cpx	D	d	$ \mathcal{K} $	DioCoH		PHAT $^\perp$		PHAT		CAM	
					\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}
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Timings

Data	Cpx	D	d	$ \mathcal{K} $	DioCoH		PHAT^\perp		PHAT		CAM	
					\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}
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- ▶ PHAT $^\perp$: Phat (persistent cohomology)
- ▶ PHAT: Phat (persistent homology)

Timings

Data	Cpx	D	d	$ \mathcal{K} $	DioCoH		PHAT [⊥]		PHAT		CAM	
					\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}
Cy8	Rips	24	2	21×10^6	420	4822	44	—	5.3	—	6.4	6.5
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Bud	α Sh	3	2	1.4×10^6	30.0	30.9	2.6	—	0.32	—	0.7	0.7
Nep	α Sh	3	2	57×10^6	T_∞	T_∞	163	—	33	—	39.5	40.2

- ▶ DioCoH: Dionysus (persistent cohomology)
- ▶ PHAT[⊥]: Phat (persistent cohomology)
- ▶ PHAT: Phat (persistent homology)

Timings

Data	Cpx	D	d	$ \mathcal{K} $	DioCoH		PHAT [†]		PHAT		CAM	
					\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}	\mathbb{Z}_2	\mathbb{Z}_{11}
Cy8	Rips	24	2	21×10^6	420	4822	44	—	5.3	—	6.4	6.5
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Nep	α Sh	3	2	57×10^6	T_∞	T_∞	163	—	33	—	39.5	40.2

CAM: Computational time = $2.7 \times 10^{-7} \leq \cdot \leq 9.1 \times 10^{-7}$ seconds
 per simplex on all examples / all field coefficients

Statistics

Nep	$ M $	#Fop.	Nep	average	maximum			
Compression	126057	84×10^6	$c_{\text{av}}, c_{\text{max}}$	0.79	18			
\neg Compression	574426	3860×10^6	$r_{\text{av}}, r_{\text{max}}$	1.02	18			
Bro	time	Bro	\mathbb{Z}_{11}	\mathbb{Q}				
Reordering	2.9 s.		M_{DS}	$a_{\partial\sigma}$	M_{op}	M_{DS}	$a_{\partial\sigma}$	M_{op}
\neg Reordering	14.2 s.		71%	19%	10%	67%	21%	12%

- ▶ Compression: matrix 4.5 times smaller
- ▶ Compression: 46 times less operations: ≤ 1.5 op. per simplex
- ▶ Complexity: $O(m c_{\text{max}}(k + r_{\text{max}}))$
- ▶ Reordering: 4.9 times faster
- ▶ $\mathbb{Z}_{11} \rightarrow \mathbb{Q}$: only 8% slower.

Question?

Thank you!