# Periodic planar straight-frame graph drawings with polynomial resolution

Luca Castelli Aleardi, Eric Fusy, Anatolii Kostrygin

# Introduction

# Planar straight-line drawings

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Adjacency matrix of graph $G$

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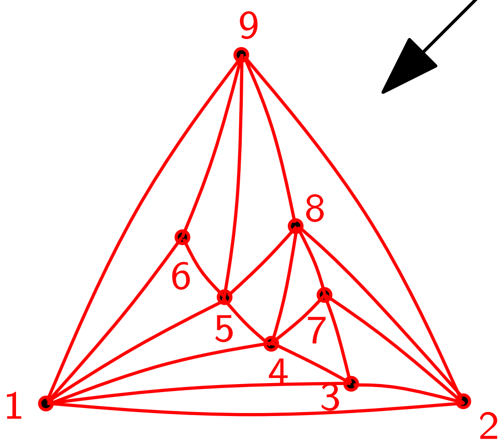


Combinatorial embedding
( = we know faces of $G$)

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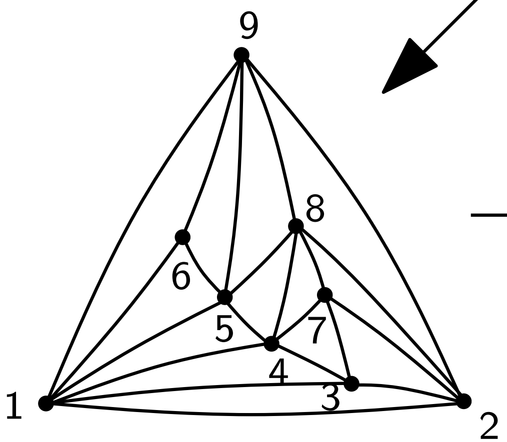


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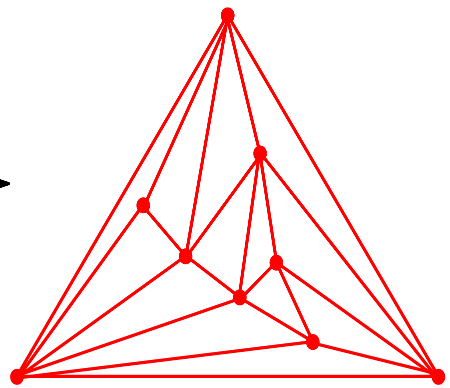

Planar straight-line drawing
$f : V(G) \to \mathbf{R}^2$

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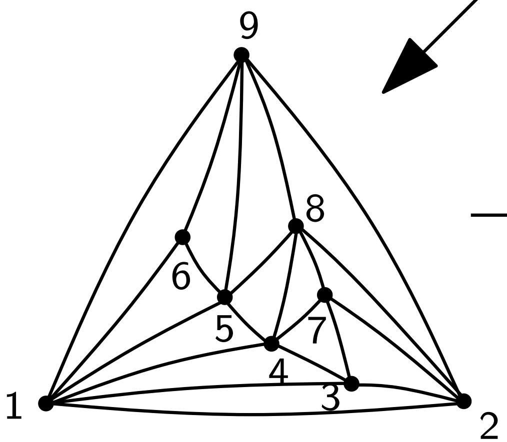


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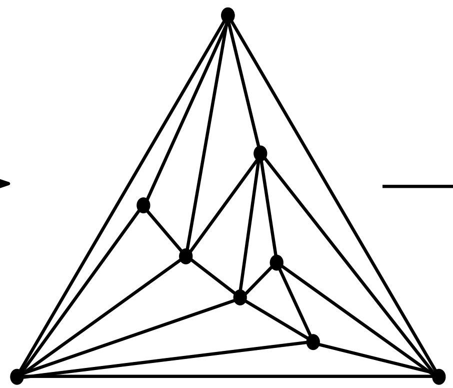

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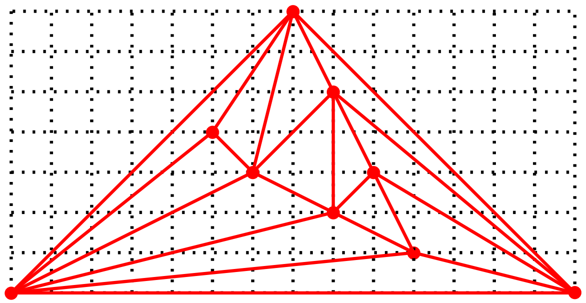

Drawing on a grid
$f : V(G) \to \mathbf{N}^2$

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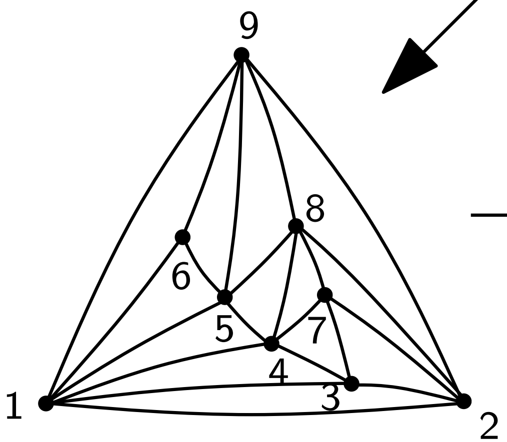
Adjacency matrix of graph $G$

Existence :

- Wagner '36, Fary '48

Algorithmes :

- Tutte '63
- Fraysseix et al. '89
- Schnyder'90

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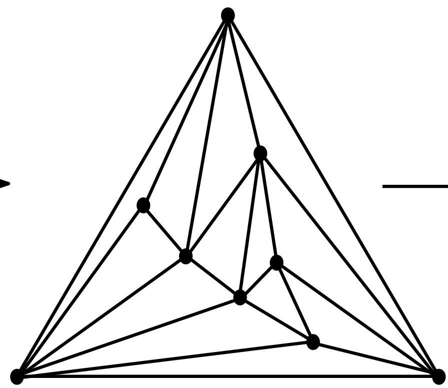

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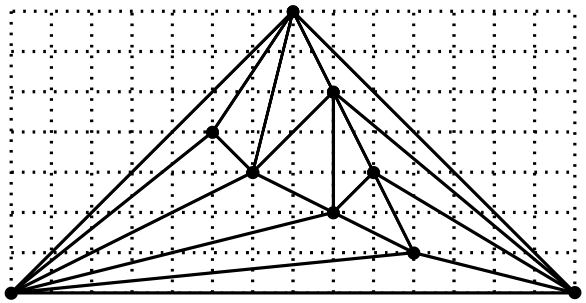

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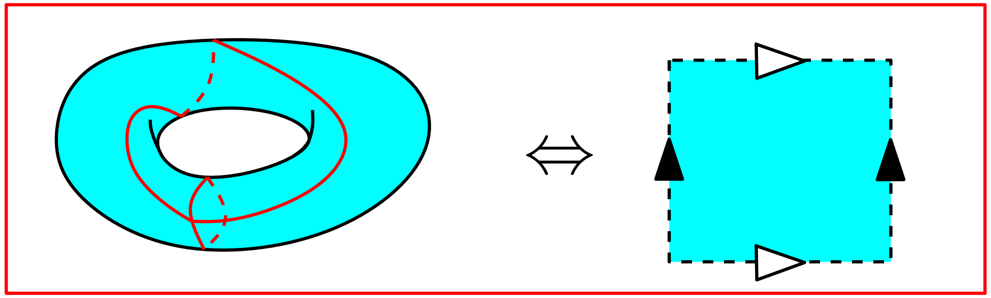
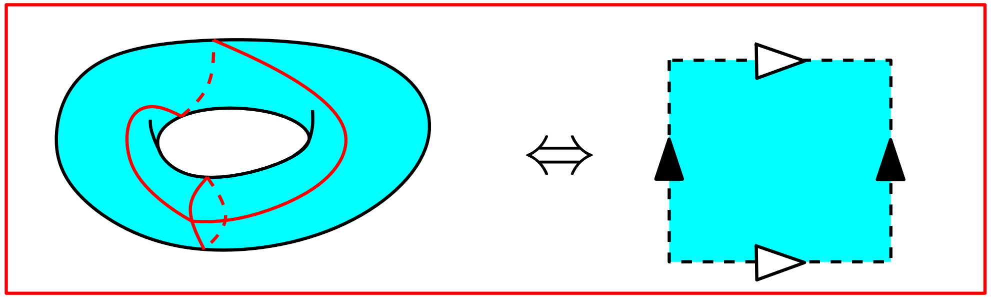
# Drawing on the torus



Cut the torus along 2 non-contractible cycles with a common point.

# Drawing on the torus



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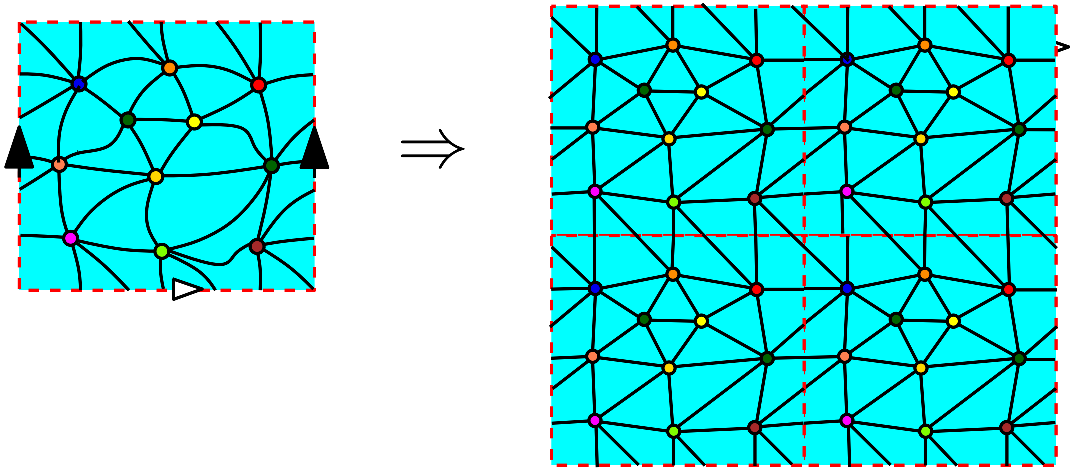
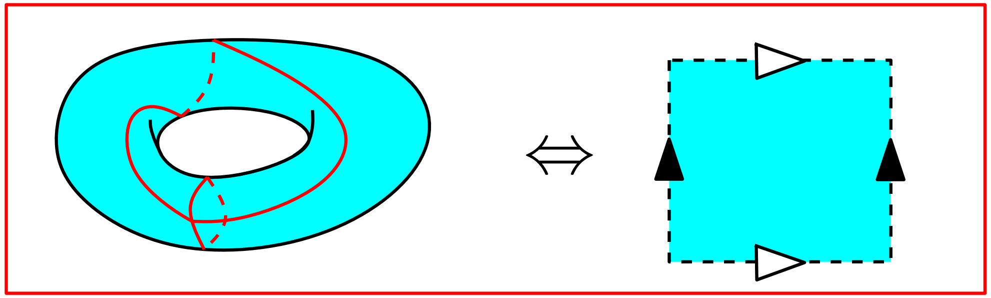


The drawing should be $x$- and $y$-periodic.

# Drawing on the torus



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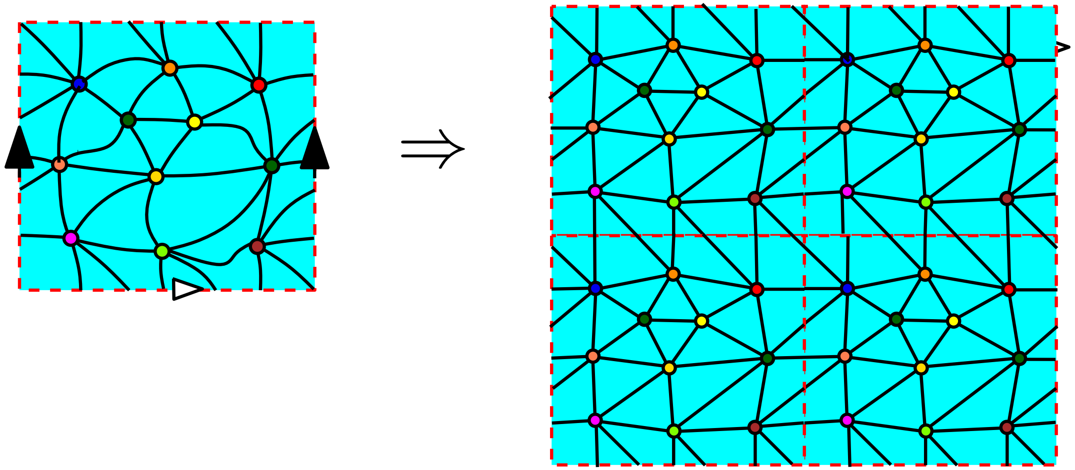

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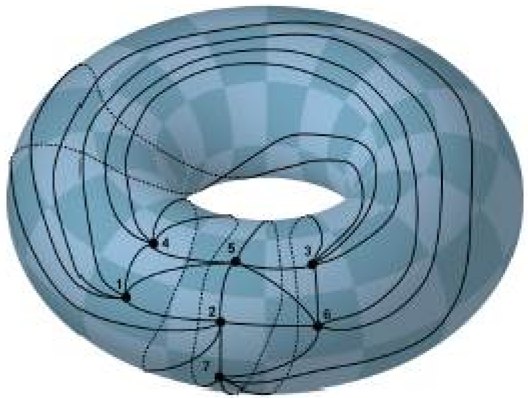
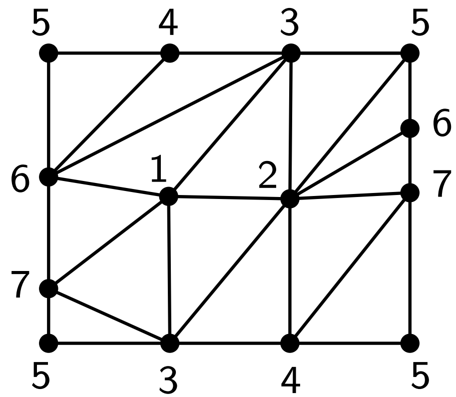



(a)




(b)

[Duncan et al., 2009]

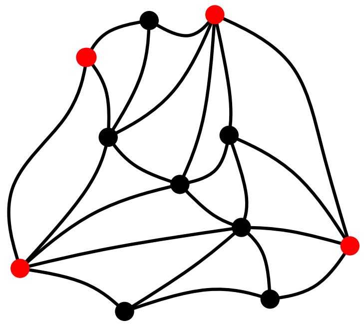
Bad example :

- not periodic;
- vertices on the sides are not identified.

# Some definitions

1) $k$-**scheme triangulation** is a quas-triangulation s.t.

- $k$ marked outer vertices are called **corners**;
- each path of the outer face contour between two consecutive corners is chordless.



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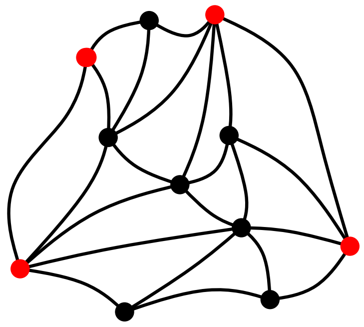
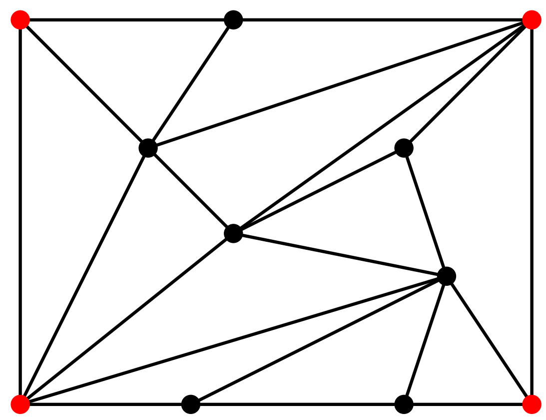


2) $G$ is a 4-scheme triangulation.
A **straight-frame drawing** of G is

- a planar straight-line drawing of G;
- the outer face is an axis-aligned rectangle;
- its corners are the corners of G.



# Previous result. Straight-frame drawing.

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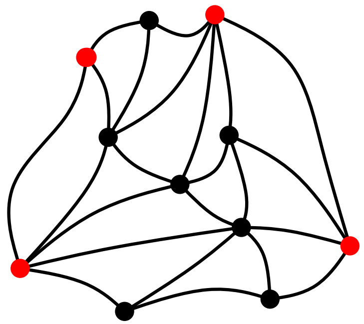

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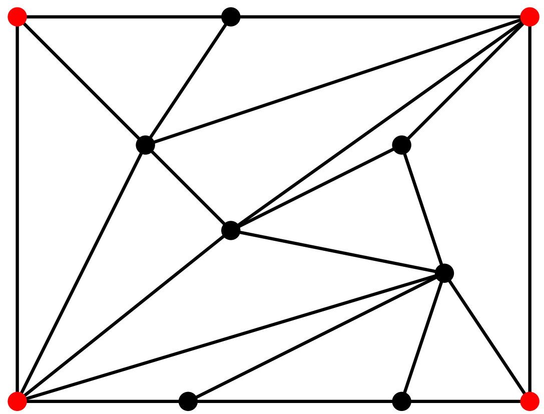

- a planar straight-line drawing of G;
- the outer face is an axis-aligned rectangle;
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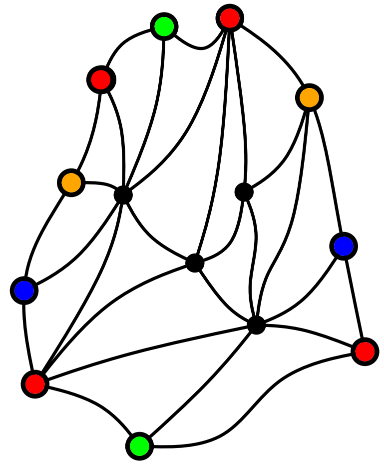
**Theorem 1 [Duncan et al.]** Each 4-scheme triangulation with $n$ vertices admits a straight-frame drawing on a grid of size $O(n^2 \times n)$.

# Some definitions. Periodic case.



1) Denote the paths between consecutive corners by $S_1, \ldots, S_k$. Then a 4-scheme triangulation satisfying $|S_1| = |S_3|$ and $|S_2| = |S_4|$ is called **balanced**.

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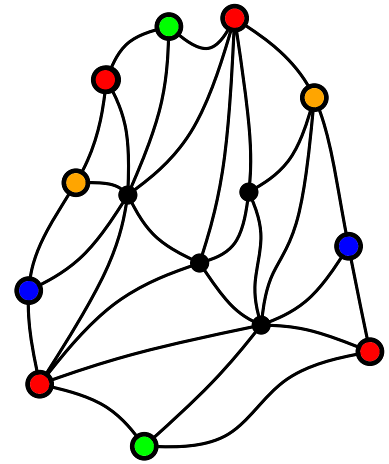

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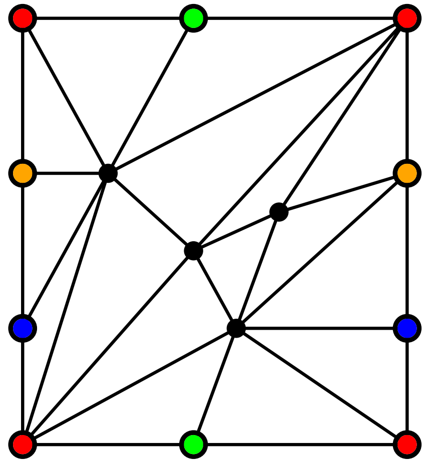
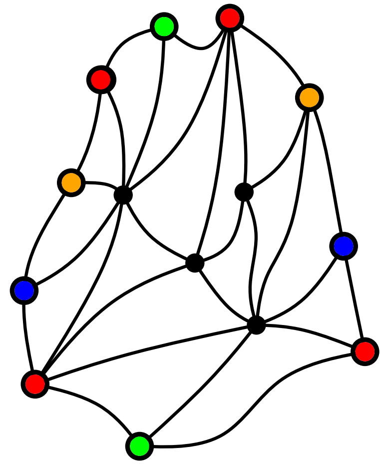


2) Its straight-frame drawing is **periodic** if

- the abscissas of vertices of the same rank along $S_1$ and $S_3$ coincide;
- the ordinates of vertices of the same rank along $S_2$ and $S_4$ coincide.

# Our result. Periodic case.



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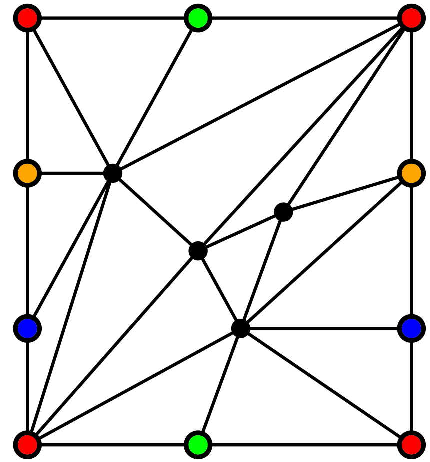

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**Theorem 2** Each balanced 4-scheme-triangulation admits a periodic straight- frame drawing on a (regular) grid of size $O(n^4 \times n^4)$.

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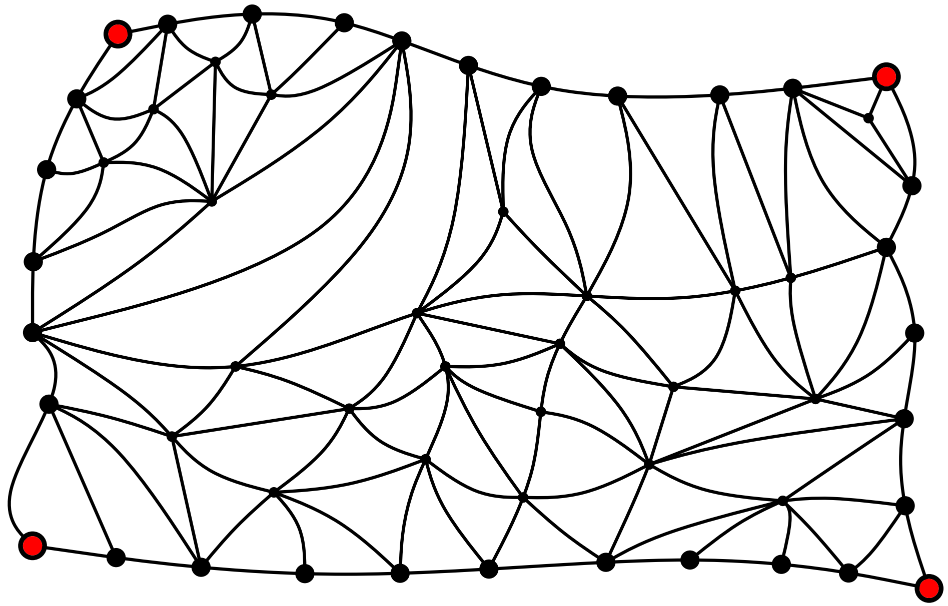
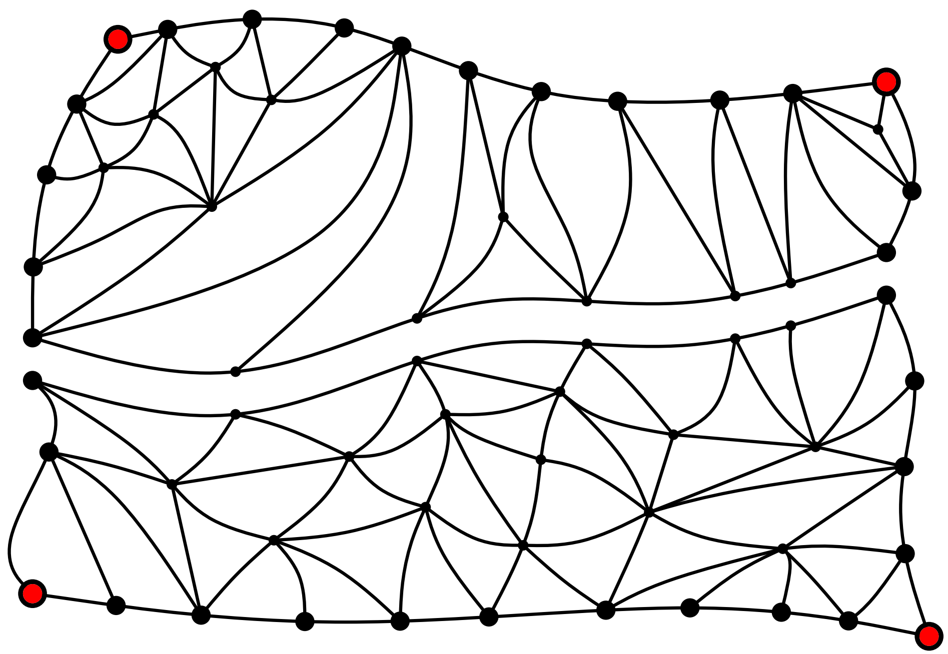
# Sketch of the proof of the theorem 2

**Let's draw this balanced 4-scheme triangulation . . .**



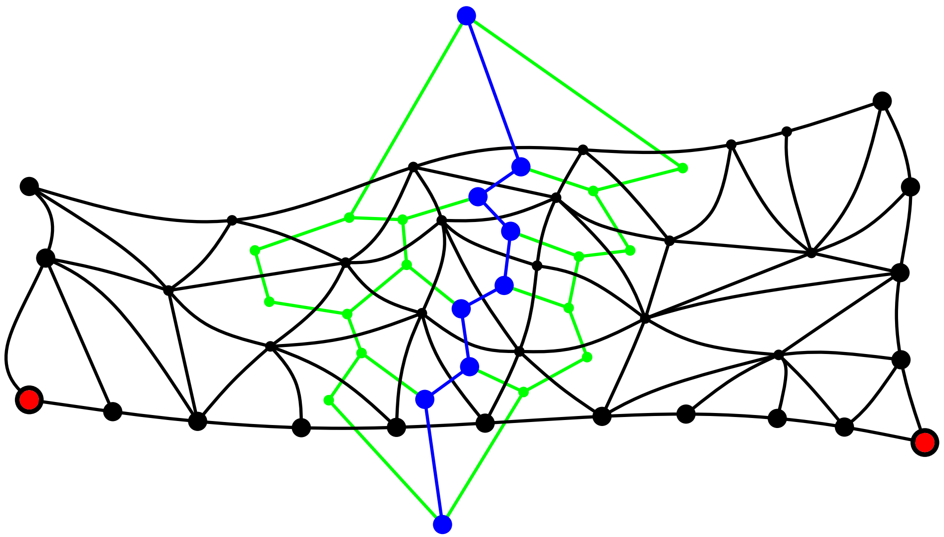
# Step 1. Analysis of cords

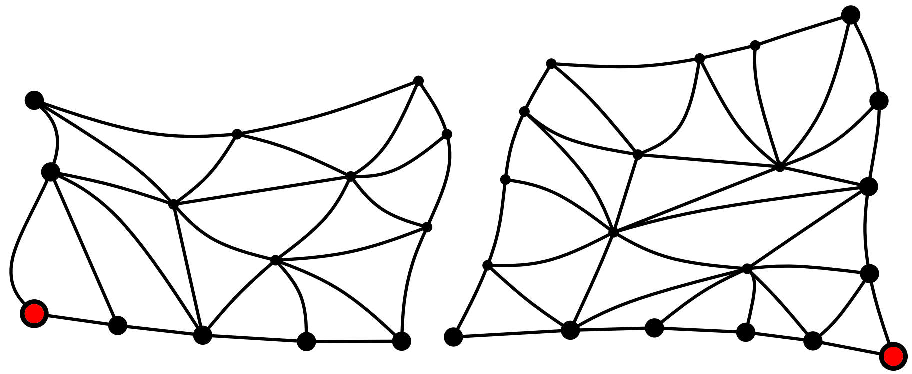
- Suppose that there is no "vertical" cord.
- Then there exists a closest to the upper-side cordless path.
- Each vertex of the path is on the dist. 1 from the upper-side.
- Let's cut the graph along this path.



# Step 2. Bottom part

- Need to take care only about left and right sides.
- Upper side should not be straight.
- Can find a **river** from upper to bottom side.
- Let's cut along this river.



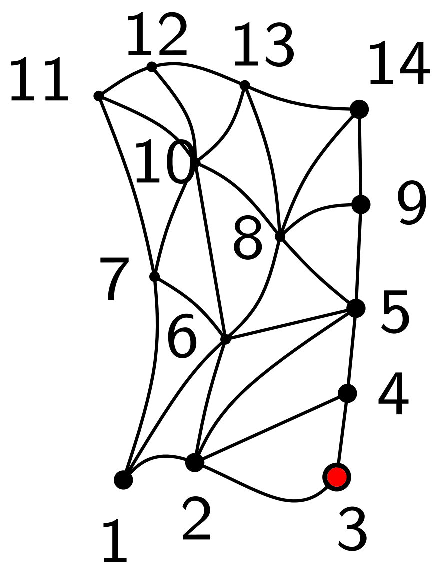


Bottom-left

Bottom-right

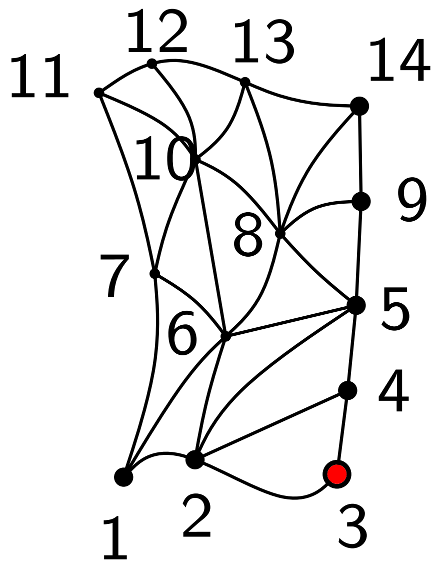
# Step 3. Bottom-left part

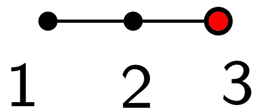
- Turn by 90.
- Find a canonical order.



# Step 3. Bottom-left part

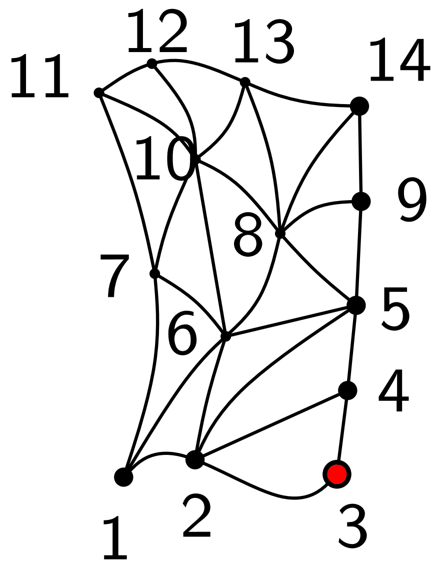
- Turn by 90.
- Find a canonical order.
- Draw with incremental algorithm.

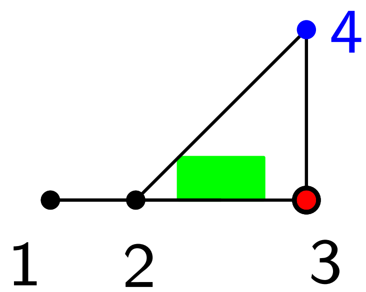


# Step 3. Bottom-left part

- Turn by 90.
- Find a canonical order.
- Draw with incremental algorithm.
- Remember the final distances between bottom vertices.



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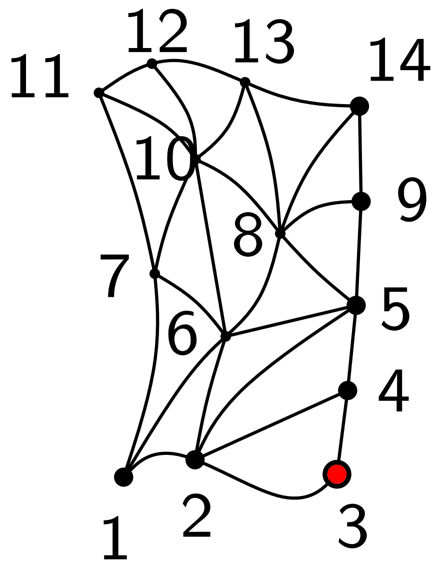


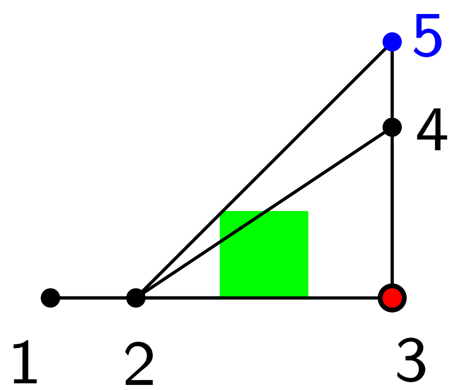

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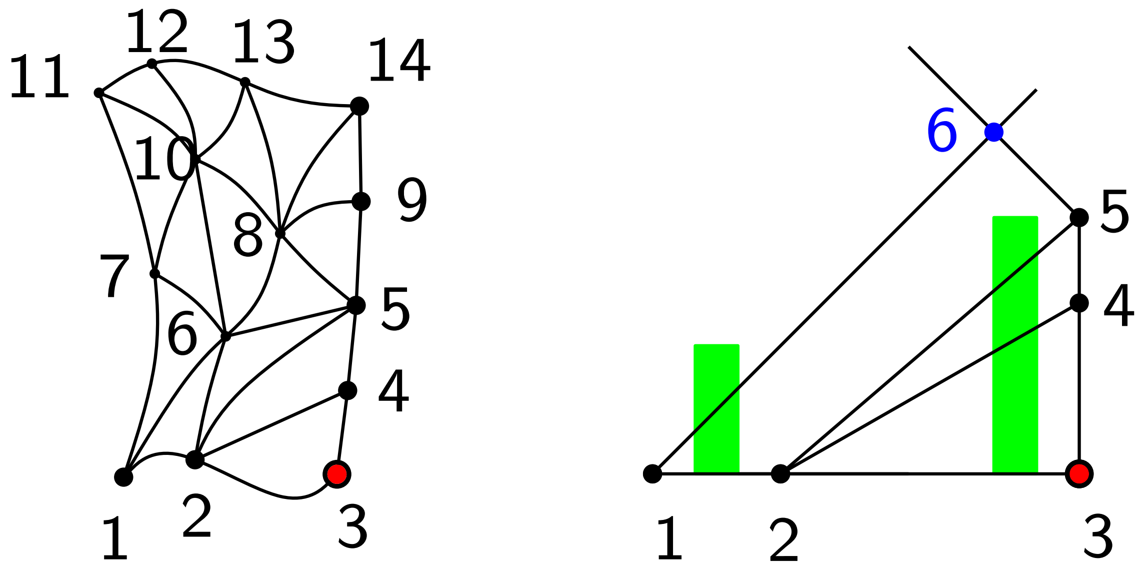

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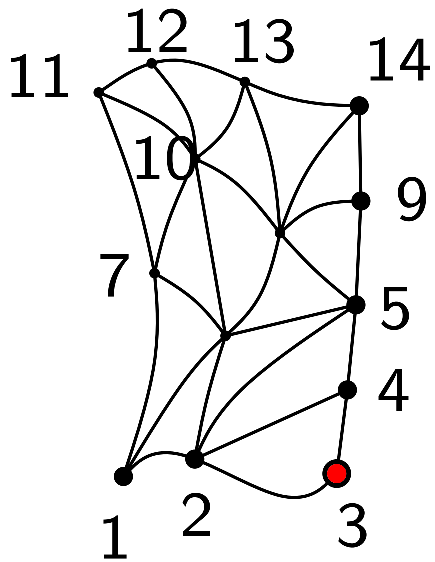
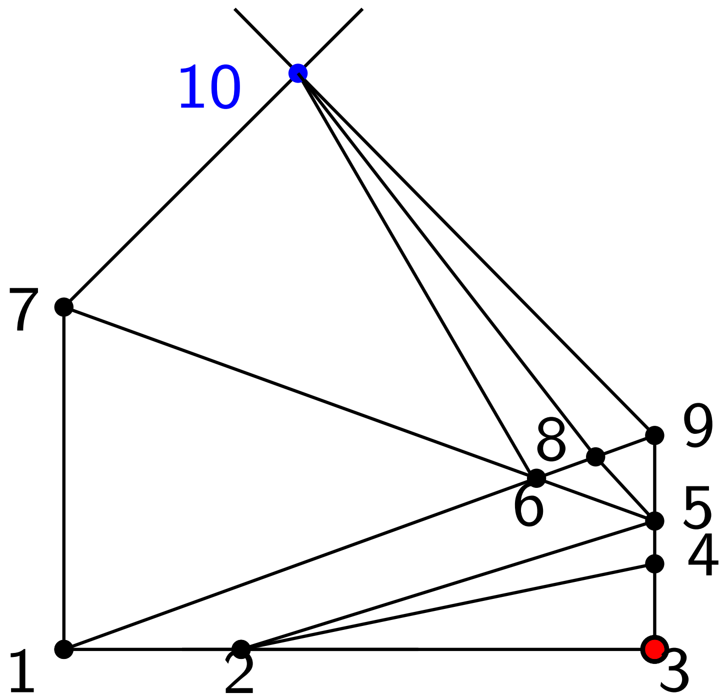
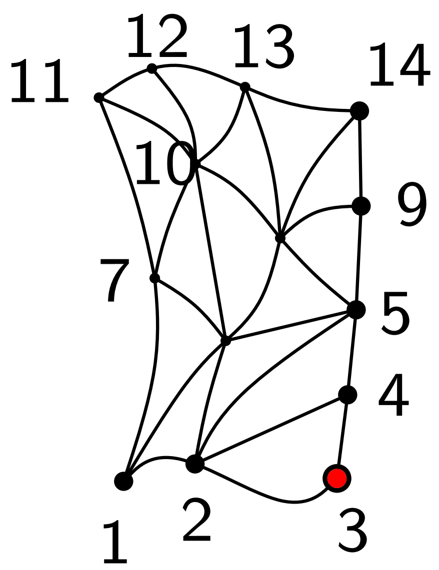
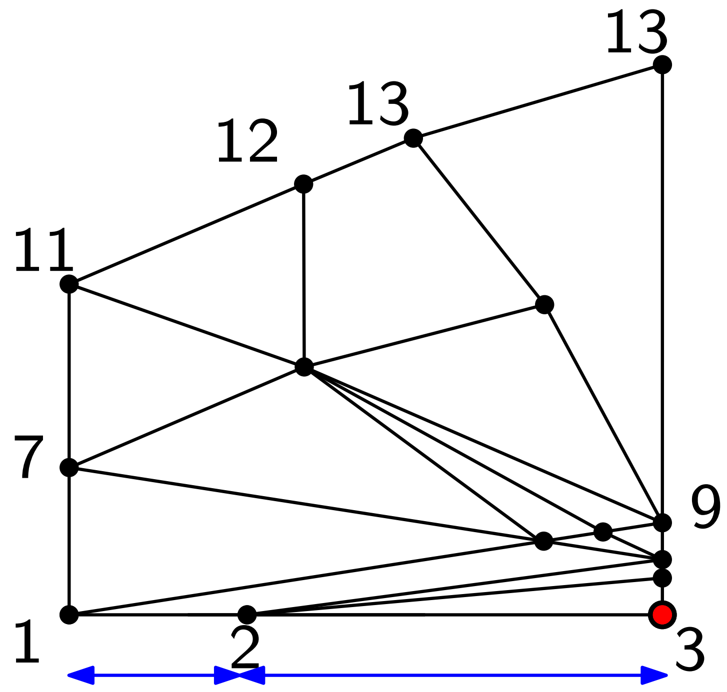


# Step 3. Bottom-left part

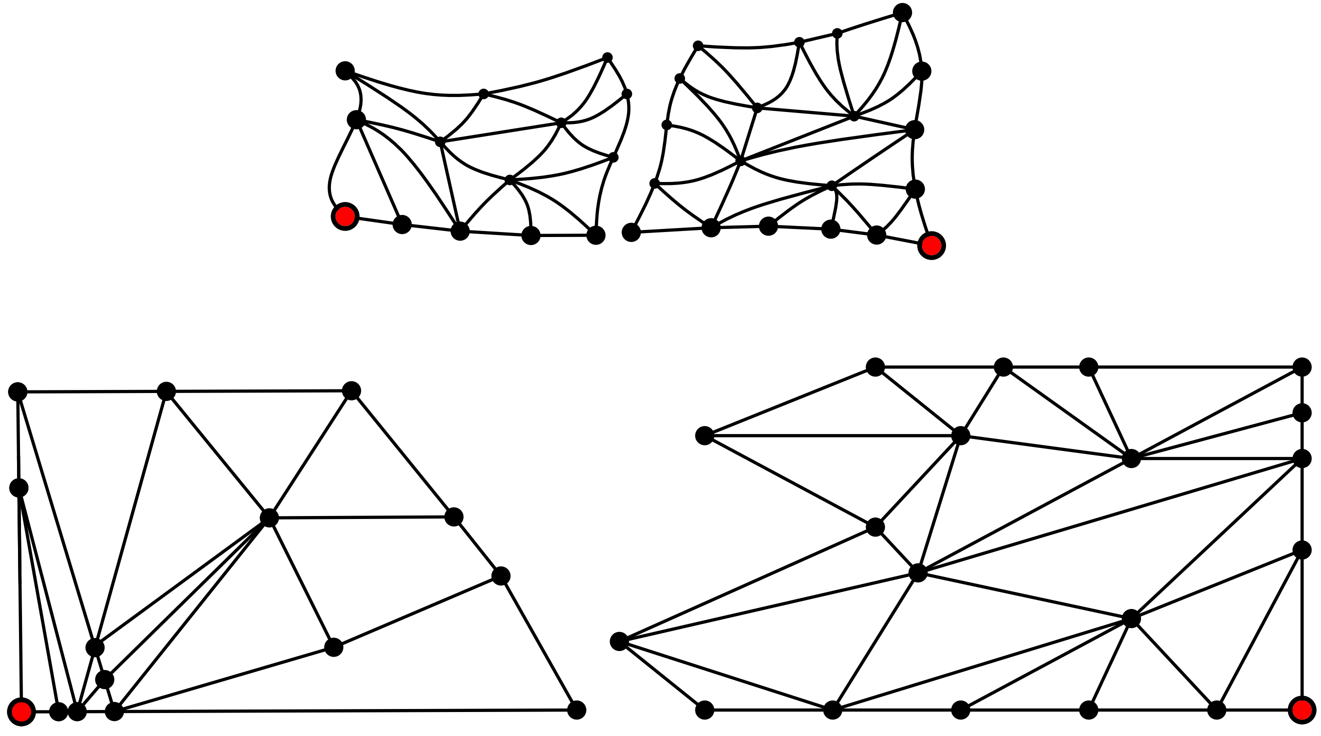
- Turn by 90.
- Find a canonical order.
- Draw with incremental algorithm.
- Remember the distanced between vertices on the bottom side.



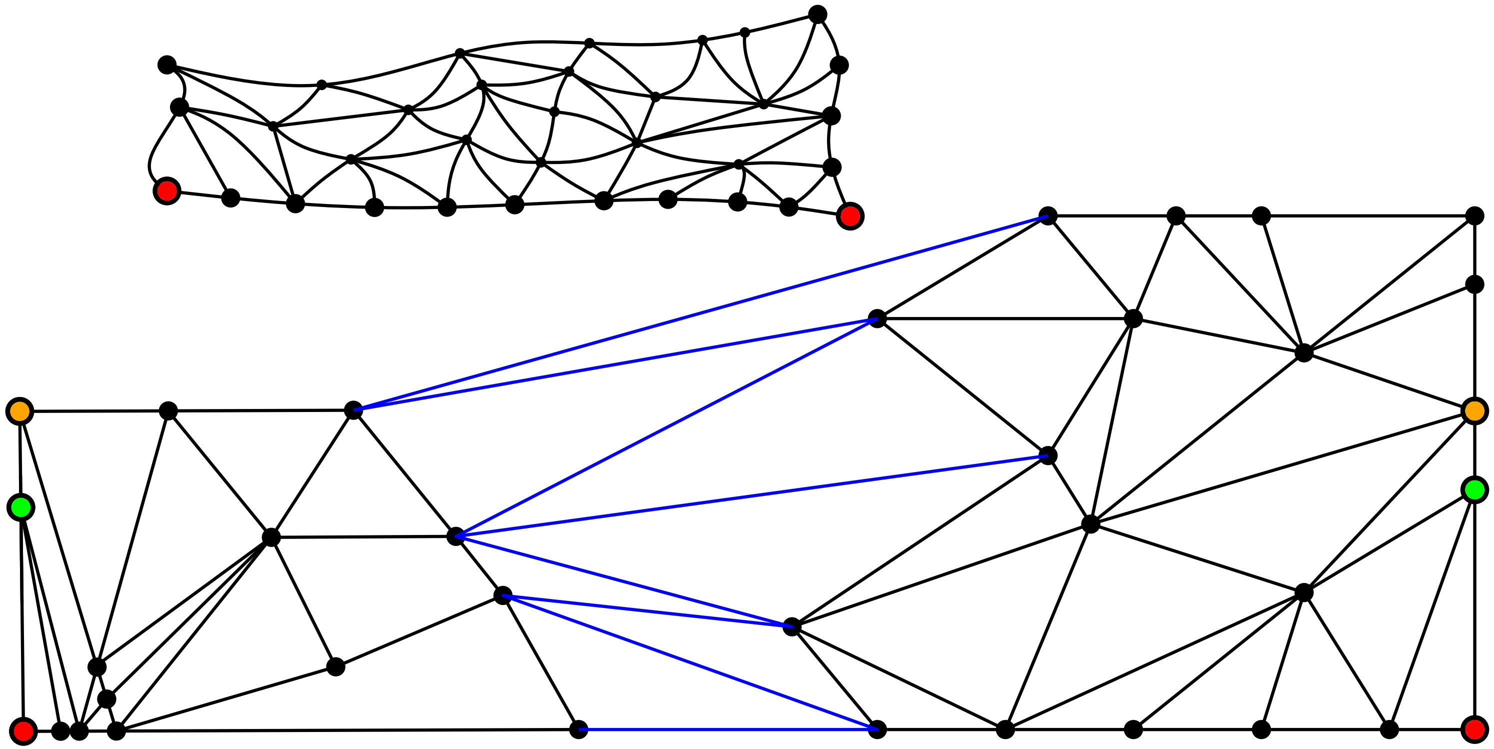
# Step 4. Whole bottom part

- Repeat the step 3 for right part.
- Place the parts one opposite other.



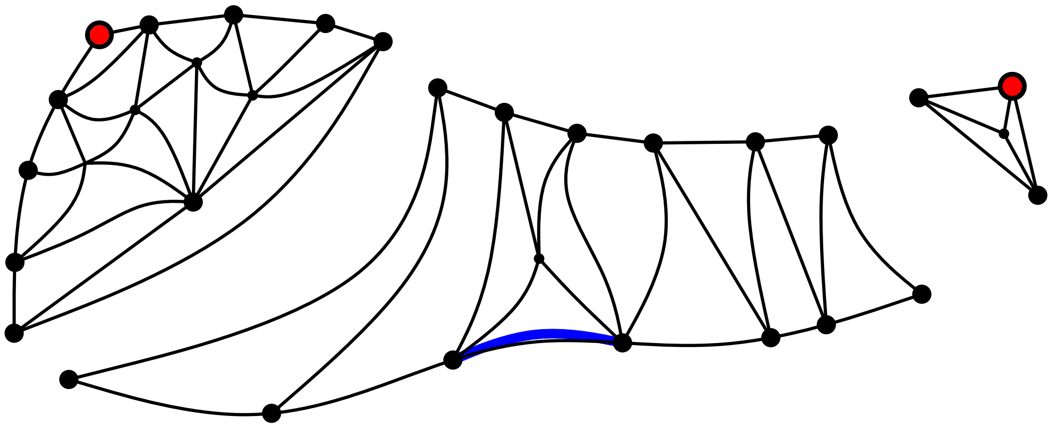
# Step 4. Whole bottom part

- Repeat the step 3 for right part.
- Place the parts one opposite other.
- Adjust sizes.
- Fill a draw with edges of the river.



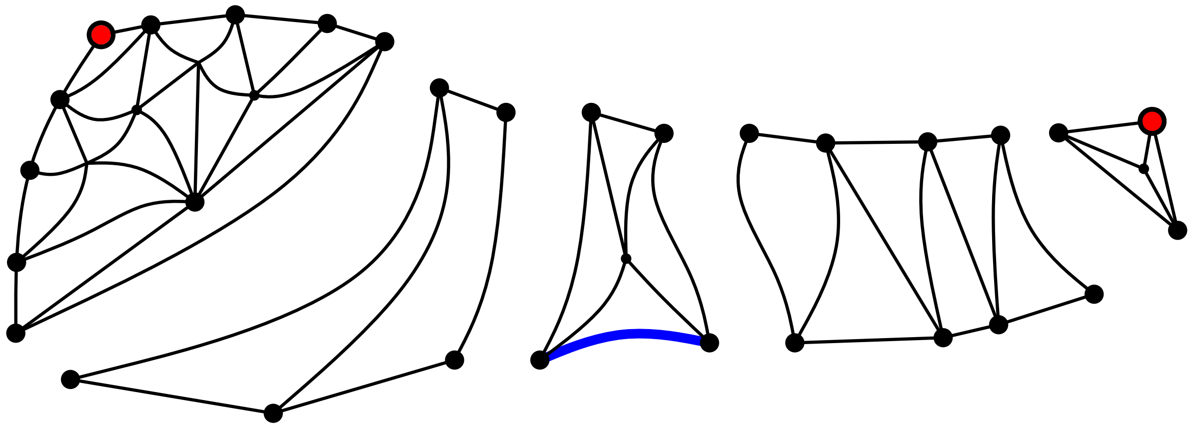
# Step 5. Decomposition of upper part

- Cut to triangles
- Find an edge adjacent to the bottom river



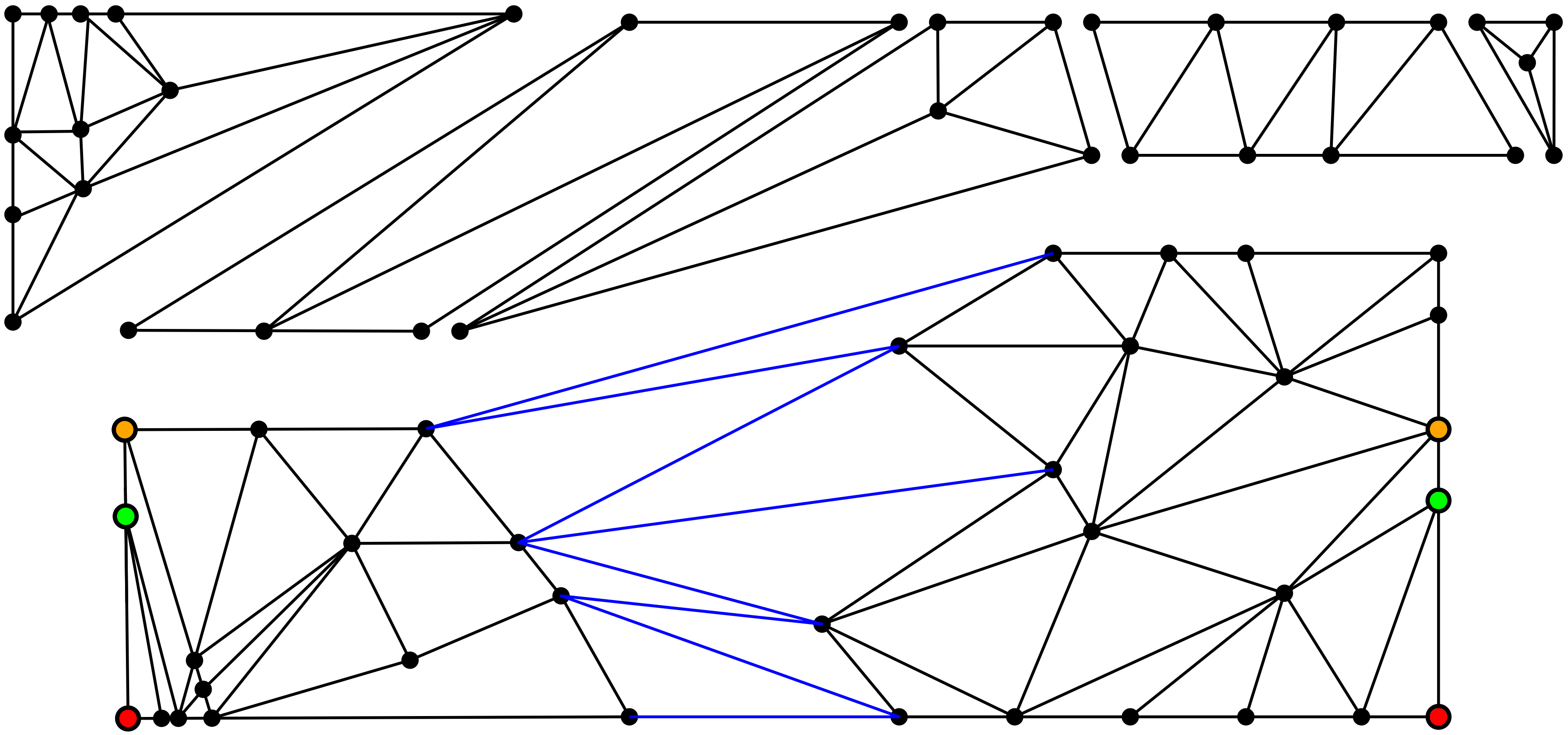
# Step 5. Decomposition of upper part

- Cut to triangles
- Find an edge adjacent to the bottom river
- Decompose the rest into 3 parts



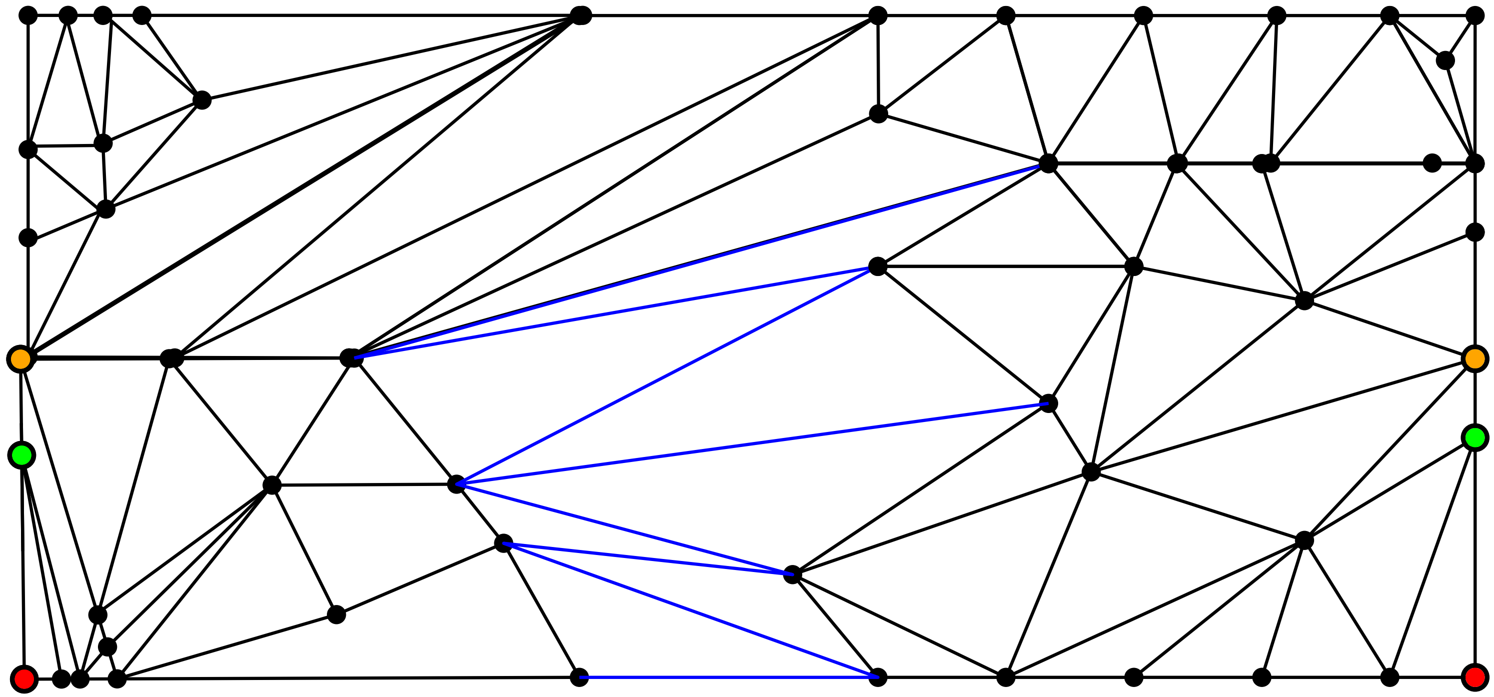
# Step 6. All together

- Draw every sub-part of upper part.



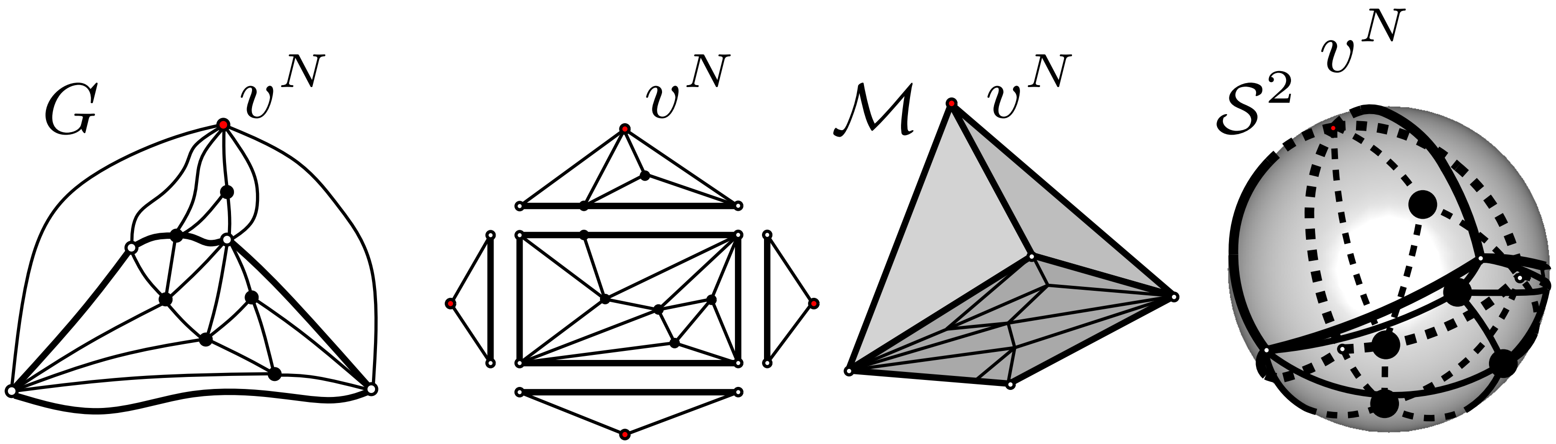
# Step 6. All together

- Draw every sub-part of upper part.
- Paste all together.



# Other applications

# Geodesic spherical drawing




**Algorithm**:

- partition the faces of the initial graph;
- dessiner draw every rectangle according their lateral sides;
- construct a pyramid from the rectangles;
- place a small copy in the center of sphere;
- project its edges on the sphere.

# An arbitrary polygon

- Suppose we can draw an arbitrary quadrangle, it's size of grid is $P(n)$
- Using divide and conquer strategy we can draw any $k$-gon
- Size of grid will be proportional to $O(P(n)^{\log k})$.