Tropical convexity and its applications to zero-sum games

Minilecture, Part I

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Works with Akian, Allamigeon, Goubault, Guterman, Katz, Joswig, Meunier, Sergeev, Walsh; highlight: PhD of Benchimol and Qu.

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- “$2 + 3$” $=$

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- “2^3” $=$

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- “$\sqrt{-1}$” $= -0.5$

$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{”} =$$

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- “$\sqrt{-1}$” $=-0.5$

$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{”} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

The notation $a \oplus b := \max(a, b)$, $a \odot b := a + b$, $\mathbb{0} := -\infty$, $\mathbb{1} := 0$ is also used in the tropical/max-plus litterature

Max-plus semiring: $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$.

The sister algebra: min-plus

$$\text{“}a + b\text{”} = \min(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

- “$2 + 3$” $= 2$
- “2×3” $= 5$

Min-plus semiring: $\mathbb{R}_{\min}$.

Structures called idempotent

$$a + a = a$$

or of characteristic one. Compare with

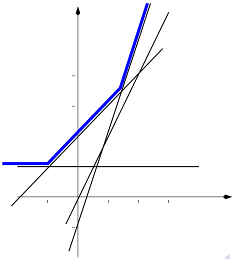
$$(p + 1)a = a \quad .$$

Exercises (cont)

- Find the roots of the max-plus polynomial
  $\text{“}1^{-1}X^3 + X^2 + 2X + 1^1\text{”}$.
  Nota bene: $\text{“}1^a\text{”} = 1 \times a = a$ is unambiguous,
  compare $\text{“}1\text{”} = 0$ with $\text{“}1^1\text{”} = 1$.

Exercises (cont)

- Find the roots of the max-plus polynomial "$1^{-1}X^3 + X^2 + 2X + 1^1$".
  Nota bene: "1^a" $= 1 \times a = a$ is unambiguous, compare "1" $= 0$ with "1^1" $= 1$.
- Answer: $\max(-1 + 3X, 2X, 2 + X, 1)$
  $= -1 + \max(X, -1) + 2\max(X, 1.5)$



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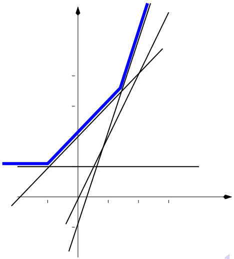
  $“1^{-1}X^3 + X^2 + 2X + 1^1”$.
  Nota bene: $“1^a” = 1 \times a = a$ is unambiguous,
  compare $“1” = 0$ with $“1^1” = 1$.
- Answer:
  $“1^{-1}X^3 + X^2 + 2X + 1^1 = 1^{-1}(X + 1^{-1})(X + 1^{3/2})^2”$



Theorem (Cuninghame-Green & Meijer, 80)

A max-plus polynomial function $p =$ "$a_n X^n + \cdots + a_0$" *can be factored uniquely as*

$$\begin{aligned} p &= \text{“}a_n (X + \alpha_1) \dots (X + \alpha_n)\text{”} \\ &= a_n + \max(X, \alpha_1) + \cdots + \max(X, \alpha_n) \enspace . \end{aligned}$$

The α_i *are the (tropical) roots.*

How to compute (tropical) roots?

Legendre-Fenchel = tropical Fourier transform

The map which sends coeffs: $(i \mapsto a_i)$ to the numerical function

$$X \mapsto p(X) = \max_{0 \leqslant i \leqslant n} a_i + i \times X$$

is a special case of Legendre-Fenchel transform

$$f : \mathbb{R}^n \to \mathbb{R}, \ f^\star : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\},$$
$$f^\star(p) = \sup_{x \in \mathbb{R}^n} \langle p, x \rangle - f(x) \ .$$

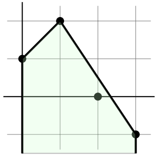
$f(i) = -a_i$ if $i \in \mathbb{N}$, $f(i) = +\infty$ otherwise
$f^\star = g^\star$ iff $\mathrm{lscvex}(f) = \mathrm{lscvex}(g)$

Newton polygon

$\Delta(p) =$ lower concave hull$\{(i, a_i) \mid 0 \leqslant i \leqslant n\}$.

Proposition

Let $p \in \mathbb{R}_{\max}[X]$ be a max-plus polynomial. Roots of $p =$ minus slopes of $\Delta(p)$. Multiplicity of root $\alpha =$ length of the interval of $\Delta(p)$ of slope $-\alpha$. *[Linear time]*



"$p = 1^{-1}X^3 + 1^0X^2 + 1^2X + 1^1$". The tropical roots are -1 (multiplicity 1) and 1.5 (multiplicity 2).

Exercises (cont.)

- Approximate essentially without computation the (usual) roots of

$$p = 2^{-2} + 2^2 X - 2^5 X^4 + 2X^6 \in \mathbb{C}[X]$$

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- Answer: -2^{-4}, $2^{-1}\{1, j, j^2\}$, $2^2\{1, -1\}$,

```
-0.0625   -0.25-0.433i  -0.25+0.433i    0.5    4.   - 4.
```

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-0.0625   -0.25-0.433i  -0.25+0.433i    0.5    4.   - 4.
```

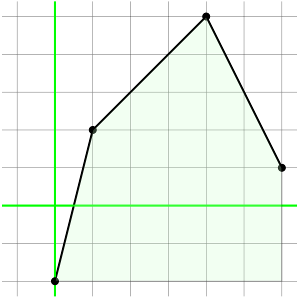
- Check in Scilab:

```
-0.0624  -0.226-0.434i -0.226+0.434i  0.522  4.00  -4.00
```

Solution: Associate to $\sum_k a_k X^k \in \mathbb{C}[X]$ the max-plus polynomial

$$X \mapsto \max_k \log_2 |a_k| + kX \quad .$$

$p = 2^{-2} + 2^2 X - 2^5 X^4 + 2^1 X^6$, tropical roots are -4 (mult. 1), -1 (mult. 3), 2 (mult. 2)



Theorem (Hadamard, Ostrowski, Polyá)

Let $p = \sum_k a_k X^k$ *with roots* $\zeta_i \in \mathbb{C}$, $|\zeta_1| \geqslant \ldots \geqslant |\zeta_n|$, $\alpha_1 \geqslant \ldots \geqslant \alpha_n$ *tropical roots of* $\max_k \log |a_k| + kX$.

$$\frac{1}{C_n^k} \exp(\alpha_1 + \cdots + \alpha_k) \leqslant |\zeta_1 \cdots \zeta_k| \leqslant cst_k \exp(\alpha_1 + \cdots + \alpha_k)$$

Corollary

$$cst''_{n,k} \exp(\alpha_k) \leqslant |\zeta_k| \leqslant cst'_{n,k} \exp(\alpha_k)$$

Hadamard: $\mathrm{cst}_k \leqslant k+1$ (1891, memoir on Zeta function)
Ostrowski: lower bound, $\mathrm{cst}_k \leqslant 2k+1$ (1940, Graeffe method)
Polyá $\mathrm{cst}_k < e\sqrt{k+1}$ (reproduced by Ostrowski).

Proof = variation on Jensen formula

$$\frac{|a_0|R^k}{|\zeta_n \cdots \zeta_{n-k+1}|} \leqslant \exp\Big(\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta\Big), \quad \forall R>0$$

Akian, SG, Sharify arXiv:1304.2967; more bounds, matrix extension;

decomposition results in Bini, Noferini, Sharify arXiv:1206.3632

One more exercise

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} ,$$

Eigenvalues ? $\epsilon \to 0$

One more exercise

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$$\mathcal{L}^1_\varepsilon \sim \varepsilon^{-1/3},\ \mathcal{L}^2_\varepsilon \sim j\varepsilon^{-1/3},\ \mathcal{L}^3_\varepsilon \sim j^2\varepsilon^{-1/3}.$$

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Answer without computation using tropical algebra, solution later in this lecture

A partial history of max-plus / tropical algebra

In the late 80's in France, the term "algèbres exotiques" was used

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SÉMINAIRE SUR LES ALGÈBRES EXOTIQUES
ET LES SYSTÈMES A ÉVÉNEMENTS DISCRETS

3 et 4 juin 1987

Organisé par :

le Centre National de la Recherche Scientifique
le Centre National d'Etudes des Télécommunications
l'Institut National de la Recherche en Informatique et Automatique au :

Centre National d'Etudes des Télécommunications
38-40, rue du Général Leclerc
92131 Issy-les-Moulineaux

Pour la modélisation des processus continus, on dispose aujourd'hui de théories ayant atteint une certaine maturité. Il n'en va pas de même pour ce qu'il est désormais convenu d'appeler « systèmes à événements discrets » et que l'on rencontre dans l'étude des ateliers flexibles, des réseaux d'ordinateurs ou de télécommunications, des circuits VLSI spécialisés en traitement du signal, pour ne citer que quelques exemples. Diverses approches et théories de ces systèmes s'appuyant sur des outils mathématiques variés ont néanmoins émergé.

Ce séminaire à caractère didactique, organisé dans le cadre de l'ATP-CNRS « Méthodologie de l'Automatique et de l'Analyse des Systèmes », avec le concours du CNET et de l'INRIA, a pour objectifs d'une part d'initier les participants à certaines de ces théories et aux outils correspondants, et d'autre part de constituer un lieu de rencontre et de confrontation de ces approches.

Conférenciers invités (liste provisoire) : P. Caspi, IMAG Grenoble ; P. Chretienne, Univ. de Paris VI ; R. A. Cuninghame Green, Univ. de Birmingham, UK ; G. Cohen, Ecole des Mines Fontainebleau ; N. Halbwachs, IMAG Grenoble ; M. Minoux, STEI Issy-les-Moulineaux ; P. Moller, IIASA Vienne, AUT ; G. J. Olsder, Univ. Delft, Pays-Bas ; J. P. Quadrat, INRIA Rocquencourt ; Ch. Reutenauer, Univ. Paris VI ; M. Viot, CNRS et Ecole Polytechnique Palaiseau.

Comité d'organisation : P. Chemouil, CNET Issy-les-Moulineaux ; G. Cohen, Ecole des Mines Fontainebleau ; J. P. Quadrat, INRIA Rocquencourt ; M. Viot, CNRS et Ecole Polytechnique Palaiseau.

Toutes les personnes intéressées sont invitées à contacter le plus vite possible :

Monsieur G. Cohen
CAI-ENSMP
35, rue Saint-Honoré
77305 Fontainebleau Cedex
Tél. (1) 64.22.48.21

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of its properties are given there. See also [104]. Its "magical properties" can be seen as related to the Lax formula for the solution of

$$\frac{\partial w}{\partial t} - \frac{1}{2}|\nabla w|^2 = 0 \quad \text{for } x \in \mathbb{R}^N,\ t \geq 0,\ w|_{t=0} = v \text{ on } \mathbb{R}^N,$$

which is

$$w(x,t) = \sup_y \left\{ v(y) - \frac{1}{2t}|x-y|^2 \right\}.$$

Indeed, the coincidence of this solution formula and solutions produced by the method of charteristics leads to the properties used. Of course, this is a heuristic connection, since characteristic methods require too much regularity to be rigorous here.

The inf convolution can also be seen as a nonlinear analogue of the standard mollification when replacing the "linear structure of L^2 and its duality" by the "nonlinear structure of L^∞ or C." One can also interpret this analogy in terms of the so-called exotic algebra $(\mathbb{R}, \max, +)$.

The term "tropical" is in the honor of Imre Simon, 1943 - 2009

who lived in Sao Paulo (south tropic).

These algebras were invented by various schools in the world

- Cuninghame-Green 1960- OR (scheduling, optimization)
- Vorobyev $\sim$65 ... Zimmerman, Butkovic; Optimization
- Maslov $\sim$ 80'- ... Kolokoltsov, Litvinov, Samborskii, Shpiz... Quasi-classic analysis, variations calculus
- Simon $\sim$ 78- ... Hashiguchi, Leung, Pin, Krob, ... Automata theory
- Gondran, Minoux $\sim$ 77 Operations research
- Cohen, Quadrat, Viot $\sim$ 83- ... Olsder, Baccelli, S.G., Akian discrete event systems, optimal control, idempotent probabilities, linear algebra
- Nussbaum 86- Nonlinear analysis, dynamical systems, also related work in linear algebra, Friedland 88, Bapat ˜94
- Kim, Roush 84 Incline algebras
- Fleming, McEneaney $\sim$00- max-plus approximation of HJB
- Del Moral $\sim$95 Puhalskii $\sim$99, idempotent probabilities.

Since 2000' in pure maths, tropical geometry: Viro, Mikhalkin, Passare, Sturmfels ..., recent work by Connes, Consani

Menu: applied tropical geometry, connections between...

- tropical convexity
- dynamic programming / zero-sum games
- Perron-Frobenius theory
- metric geometry

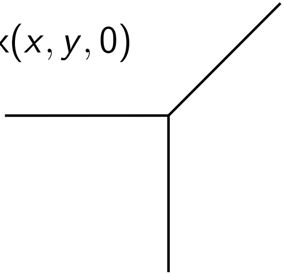
Some elementary tropical geometry

A tropical line in the plane is the set of (x, y) such that the max in

$$\text{“}ax + by + c\text{”}$$

is attained at least twice.

$\max(x, y, 0)$



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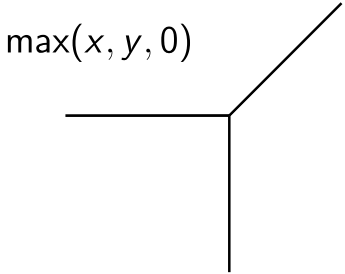
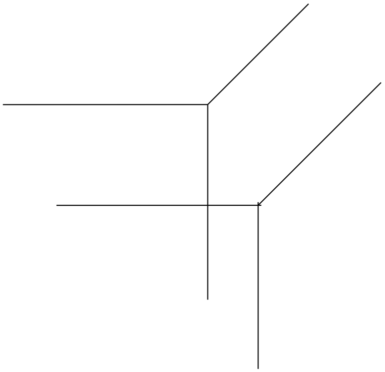
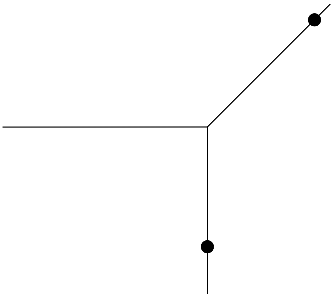


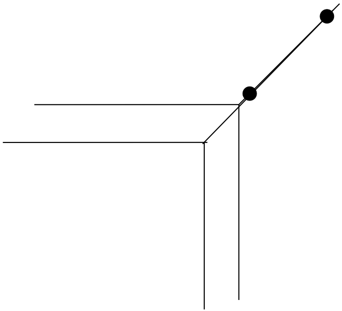
Two generic tropical lines meet at a unique point



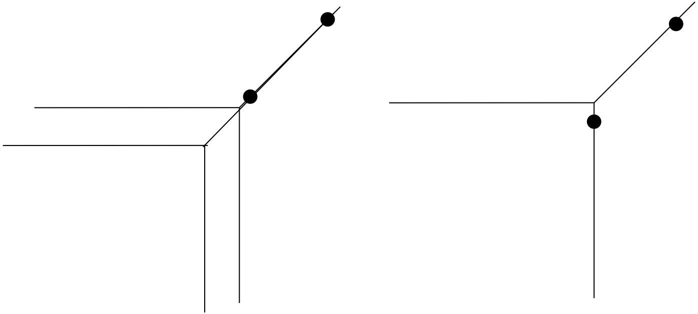
By two generic points passes a unique tropical line



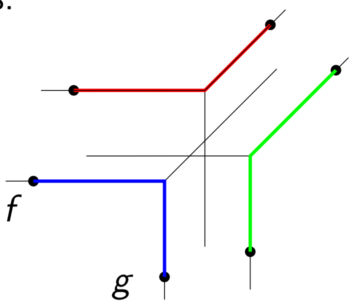
non generic case



non generic case resolved by perturbation



Tropical segments:



$$[f, g] := \{\text{“}\lambda f + \mu g\text{”} \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\},\ \text{“}\lambda + \mu = 1\text{”}\}.$$

(The condition “$\lambda, \mu \geqslant 0$” is automatic.)

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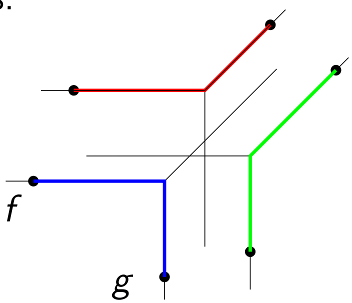
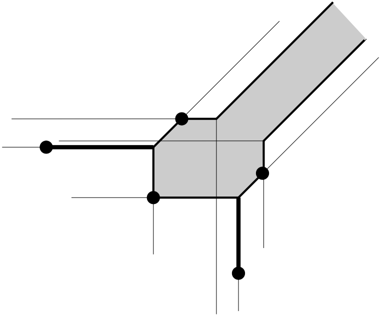


$[f, g] := \{ \sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\},\ \max(\lambda, \mu) = 0\}$.

(The condition $\lambda, \mu \geqslant -\infty$ is automatic.)

Tropical convex set: $f, g \in C \implies [f, g] \in C$



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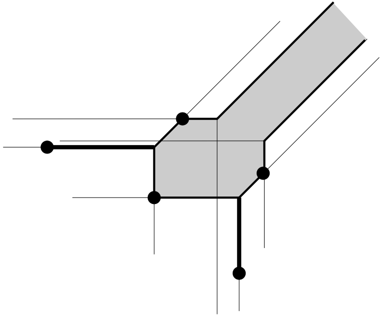


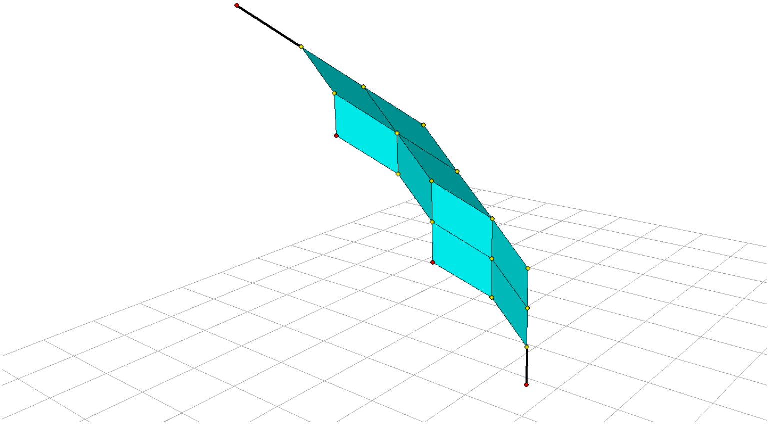
Tropical convex cone: ommit "$\lambda + \mu = 1$", i.e., replace $[f, g]$ by $\{\sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\}\}$

Homogeneization

A convex set C in $\mathbb{R}^n_{\max}$ is a cross section of a convex cone $\hat{C}$ in $\mathbb{R}^{n+1}_{\max}$,

$$\hat{C} := \{(\lambda + u, \lambda) \mid u \in C, \lambda \in \mathbb{R}_{\max}\}$$

A tropical polytope with four vertices



Structure of the polyhedral complex: Develin, Sturmfels

The previous drawing was generated by POLYMAKE of Gawrilow and Joswig, in which an extension allows one to handle easily tropical polyhedra. They were drawn with JAVAVIEW.

See Joswig arXiv:0809.4694 for more information.

Tropical polyhedra handled by ocaml TPLib, Allamigeon

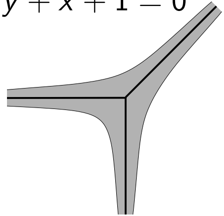
Motivation ?

The tropical point of view arises with log glasses

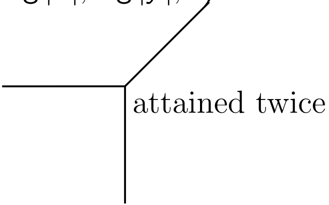
Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the "log-log plot"

$$A(V) := \{(\log|z_1|, \dots, \log|z_n|) \mid (z_1, \dots, z_n) \in V\} \ .$$

$y + x + 1 = 0$



$\max(\log|x|, \log|y|, 0)$



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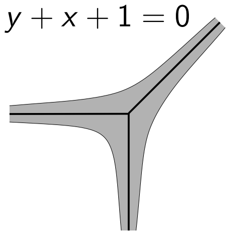
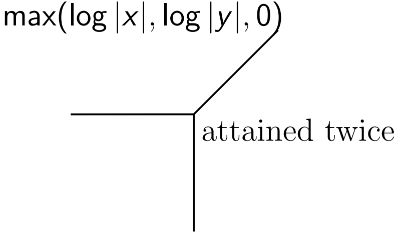


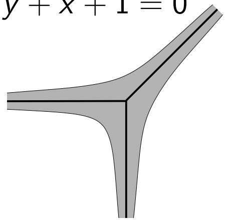


$$|y| \leqslant |x| + 1,\ |x| \leqslant |y| + 1,\ 1 \leqslant |x| + |y|$$

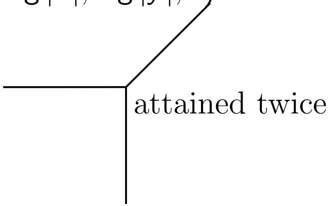
Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the "log-log plot"

$$A(V) := \{(\log|z_1|, \ldots, \log|z_n|) \mid (z_1, \ldots, z_n) \in V\} \ .$$

$y + x + 1 = 0$



$\max(\log|x|, \log|y|, 0)$

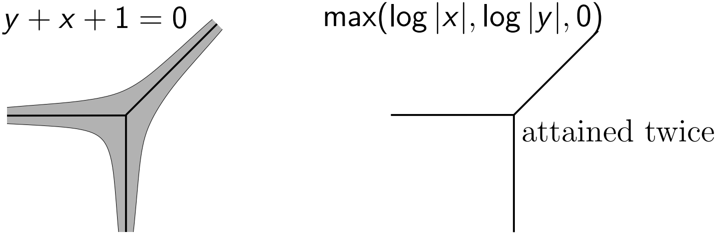


$X := \log|x|$, $Y := \log|y|$
$Y \leqslant \log(e^X + 1)$, $X \leqslant \log(e^Y + 1)$, $0 \leqslant \log(e^X + e^Y)$

Viro's log-glasses, related to Maslov's **dequantization**

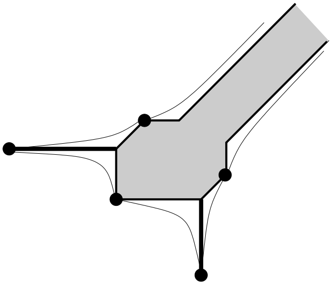
$$a +_h b := h \log(e^{a/h} + e^{b/h}), \qquad h \to 0^+$$

With h-log glasses, the amoeba of the line retracts to the tropical line as $h \to 0^+$



$$\max(a, b) \leqslant a +_h b \leqslant h \log 2 + \max(a, b)$$

Tropical convex sets are deformations of classical convex sets



Briec and Horvath 04

$$[a, b] := \{\lambda a +_p \mu b,\ \lambda, \mu \geqslant 0, \lambda +_p \mu = 1\}$$

$$a +_p b = (a^p + b^p)^{1/p}$$

See Passare & Rullgard, Duke Math. 04

Introduction to amoebas: lecture notes by Alain Yger.

Metric estimates: Avendaño, Kogan, Nisse, Rojas, arXiv:1307.3681

Nonarchimedean valuation point of view

Alternatively [Sturmfels' point of view], the tropical line "$\max(X, Y, 0)$ attained twice" can be seen as the image by the valuation of the line $x + y + 1$ over the field of complex Puiseux series, $\mathbb{C}\{\{t\}\}$, equipped with the valuation $\operatorname{val} s = -$ smallest exponent of s.

E.g., $\operatorname{val}(t^{-1/2} - t + 7t^{3/2} + \dots) = 1/2$

$\operatorname{val}(z_1 + z_2) \leqslant \max(\operatorname{val}(z_1), \operatorname{val}(z_2))$, with equality when $\operatorname{val}(z_1) \neq \operatorname{val}(z_2)$.

$\operatorname{val}(z_1 z_2) = \operatorname{val}(z_1) + \operatorname{val}(z_2)$

tropical hyperplanes (complex version)

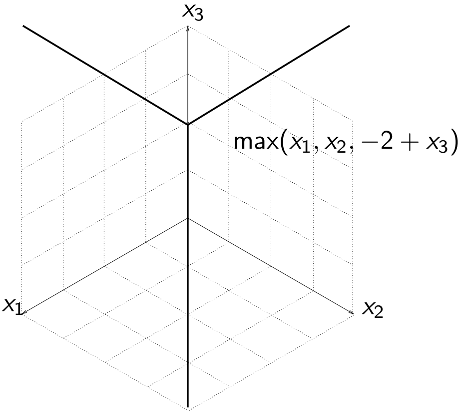
Given $a \in \mathbb{R}^n_{\max}$, $a \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \text{“}ax = 0\text{”}\}$$

tropical hyperplanes (complex version)

Given $a \in \mathbb{R}^n_{\max}$, $a \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} a_i + x_i \text{ attained twice}\}$$



The rational points of tropical hyperplanes are images of hyperplanes of $(\mathbb{C}\{\{t\}\})^n$ by the valuation, see:

Theorem (Kapranov)

Given $p = \sum_\alpha p_\alpha z^\alpha \in \mathbb{C}\{\{t\}\}[z_1, \ldots, z_n]$, *and* $Z \in \mathbb{Q}^n$,

$$\exists z \in (\mathbb{C}\{\{t\}\})^n, \qquad p(z) = 0, \qquad Z = \operatorname{val} z$$

iff

$$\max_\alpha \operatorname{val} p_\alpha + \langle \alpha, Z \rangle \textit{ attained twice}$$

Restriction to $\mathbb{Q}$ can be avoided by working with Puiseux series with real exponents (Markwig) or Hahn series (well ordered support), cvg issues: van den Dries $\mathbb{R}_{an*}$ o-minimal model.

real tropical hyperplanes

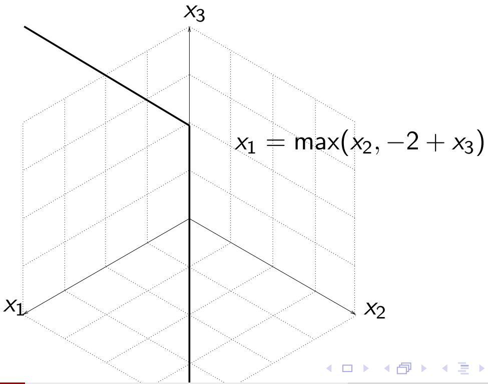
Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$, $a_i = -\infty$ or $b_i = -\infty$, $\forall i$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \text{“}ax = bx\text{”}\}$$

real tropical hyperplanes

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$, $a_i = -\infty$ or $b_i = -\infty$, $\forall i$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} a_i + x_i = \max_{1 \leqslant i \leqslant n} b_i + x_i\}$$



real tropical hyperplanes

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$, $a_i = -\infty$ or $b_i = -\infty$, $\forall i$,

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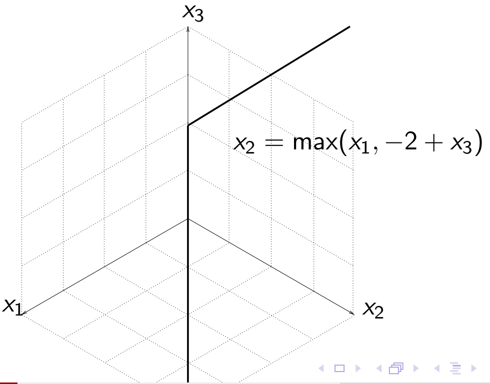

real tropical hyperplanes

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$, $a_i = -\infty$ or $b_i = -\infty$, $\forall i$,

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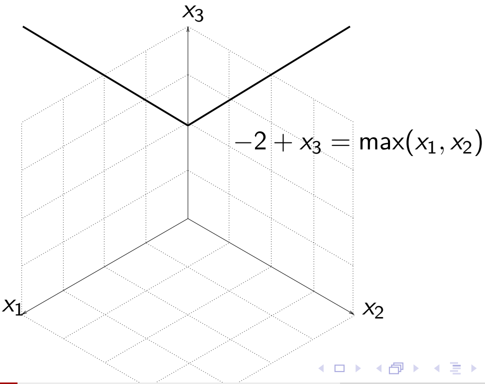
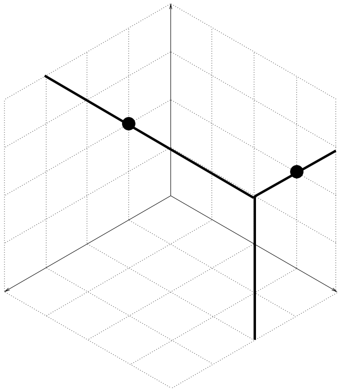
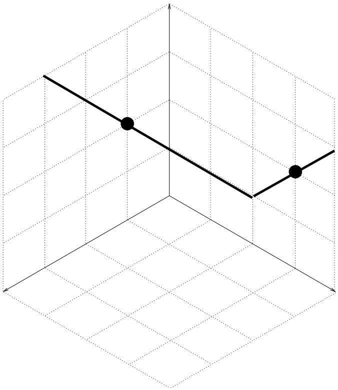


Real tropical hyperplanes are images of hyperplanes of $\mathbb{R}\{\{t\}\}$ by the valuation.

- Given any n points in $\mathbb{R}^n_{\max}$ in general position, there is a unique (complex) affine tropical hyperplane passing through them. Richter-Gebert, Sturmfels, Theobalt, 05,
- Given any n points in $\mathbb{R}^n_{\max}$ in general position, there is a unique real affine tropical hyperplane passing through them. Max Plus, 90, see also Akian, SG, Guterman 09

Let these points be given by the columns of a $(n+1) \times n$ matrix M, in projective coordinates. The vector a such that $H = \{x \mid \text{“}a \cdot x = 0\text{”}\}$ contains the points is solution of “$aM = 0$”. Hence, $a_i = \text{“}(-1)^i D_i\text{”}$, where D_i is the ith Cramer determinant (delete row i of M).





How are Cramer det defined?

For the complex (RGST 05) version, ignore sign

$$\det A = \text{``}\sum_{\sigma} \operatorname{sgn} \prod_{i} A_{i\sigma(i)}\text{''} = \max_{\sigma} \sum_{i} A_{i\sigma(i)}$$

This is an optimal assignment problem

For the real (Max Plus 90) version, the signs of the maximising permutations tells on which side of the equality "$ax = bx$" the coefficients should be put.

general position: only one opt assignment.

Proofs

- Extensions of tropical semiring, Maxplus 90, Akian, SG, Guterman, 09, 13, Izhakian, Rowen 09, related formalism : Krasner hyperfield 57 (different axioms but comparable expressivity)
- Coherent matching fields Richter-Gebert, Sturmfels, Theobalt, 05, building on Sturmfels and Zelevinsky, The Newton polytope of the product of maximal minors of a $(n+1) \times n$ matrix is a transportation polytope.

The symmetrized tropical semiring

Recall that $\mathbb{Z} = \mathbb{N}^2/\nabla$, where $(a', a'')\nabla(b', b'')$ if $a' + b'' = a'' + b'$, $-(a', a'') = (a'', a')$.
Replace $\mathbb{N}$ by $\mathbb{R}_{\max}$, $\mathbb{S}_{\max} = \mathbb{R}^2_{\max}/\sim$

$a \sim b$ if $a = b$ or $(a \nabla b$ and $a, b \not\nabla \mathbb{0})$.

$\mathbb{S}_{\max} = \mathbb{R}_{\max} \cup \ominus\mathbb{R}_{\max} \cup \mathbb{R}^{\bullet}_{\max}$; $u = (u, \mathbb{0})$, $\ominus u = (\mathbb{0}, u)$, $u^{\bullet} := u \ominus u = (u, u)$ with $u \in \mathbb{R}_{\max}$.

$\mathbb{S}^{\vee}_{\max} := \mathbb{R}_{\max} \cup \ominus\mathbb{R}_{\max}$: signed elements.

E.g., $2 \oplus (\ominus 3) = \ominus 3$, but $3 \ominus 3 = 3^{\bullet}$. Think of u as $\Theta(t^u)$, $u^{\bullet} = O(t^u)$, $\Theta(t^2) - \Theta(t^3) = -\Theta(t^3)$ but $\Theta(t^3) - \Theta(t^3) = O(t^3)$.

$$x_1 = \max(2 + x_2, 7 + x_3) \iff x_1 \ominus 2x_2 \ominus 7x_3 \nabla \mathbb{0}$$

Need to solve $aM\nabla\mathbb{0}$, where M is of $(n+1)\times n$, $a \in \mathbb{S}_{\max}^{n+1}$.

Theorem (Transfer theorem)

Any polynomial identity valid in rings is valid in the extensions of semirings.

Akian, SG, Guterman 09, using an idea of Reutenauer and Straubing 86.

Eg $P_A(A) = 0$ becomes $P_A^+(A) = P_A^-(A)$, or $P_A(A)\nabla\mathbb{0}$, where P_A characteristic polynomial.

Lemma (Elimination)

$x \nabla b$, $cx \nabla d$, $x \in \mathbb{S}_{\max}^{\vee}$ implies $cb \nabla d$.

Cramer theorem is proved by Gaussian elimination

Izhakian introduced the bi-valued tropical semiring, $2 \oplus 2 = 2^{\bullet}$, to remind that max is attained twice. The "complex" tropical Cramer theorem is proved along the same lines.

Tropical half-spaces

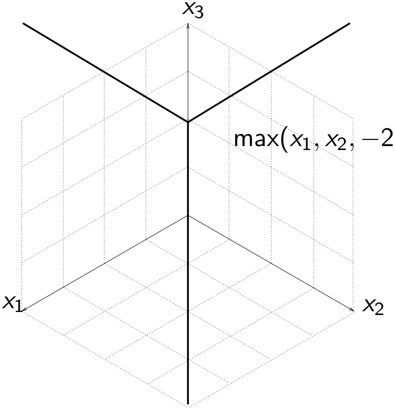
Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \text{“}ax \leqslant bx\text{”}\}$$

Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} a_i + x_i \leqslant \max_{1 \leqslant i \leqslant n} b_i + x_i\}$$



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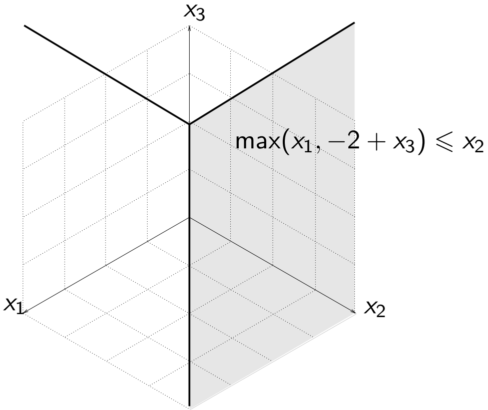

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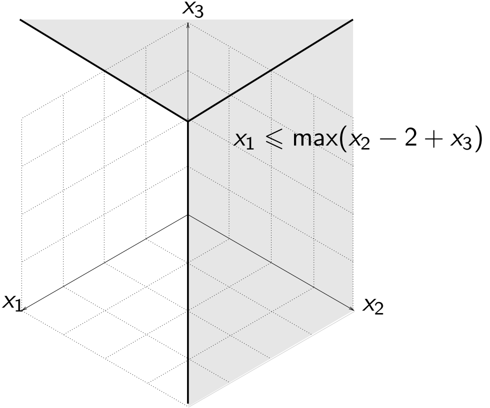

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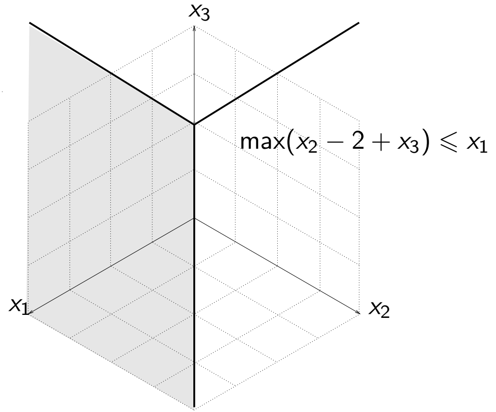


Tropical sesquilinear form and Hilbert's metric

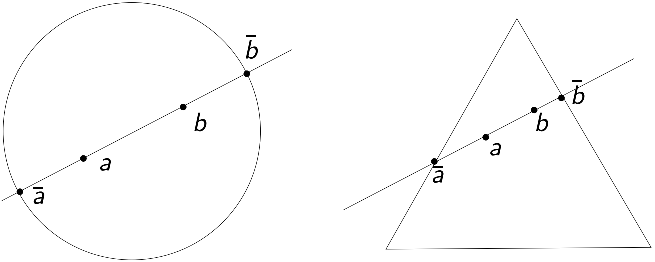
$$
\begin{aligned}
x/v &:= \max\{\lambda \mid \text{“}\lambda v\text{”} \leqslant x\} \\
&= \min_i (x_i - v_i) \quad \text{if } x, v \in \mathbb{R}^n \enspace .
\end{aligned}
$$

$$
\delta(x,y) = \text{“}(x/y)(y/x)\text{”} = \min_i (x_i - y_i) + \min_j (y_j - x_j)
$$

$d = -\delta$ is the (additive) Hilbert's projective metric

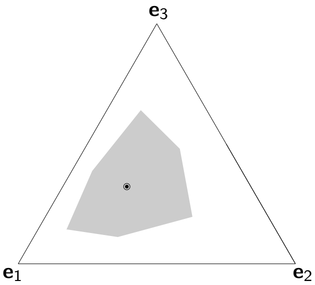
$$
d(x,y) = \|x-y\|_H , \quad \|z\|_H := \max_{1\leqslant i\leqslant d} z_i - \min_{1\leqslant i\leqslant d} z_i \enspace .
$$

Hilbert's metric on an open convex set



$$d_H(a,b) = \log \frac{|b-\bar{a}||a-\bar{b}|}{|a-\bar{a}||b-\bar{b}|} \quad .$$

disc: Klein model of the hyperbolic space;
simplex: d_H conjugate to the additive (tropical) Hilbert metric (take $\exp : \mathbb{R}^n \to \mathbb{R}^n_+$ and a cross section of $\mathbb{R}^n_+$).



A ball in Hilbert's metric is classically and tropically convex.

Projection on a tropical cone

If the tropical convex cone $C \subset \mathbb{R}^n_{\max}$ generated by U is stable by arbitrary sups (closed in Scott topology -non-Haussdorf-):

$$P_C(x) = \max\{v \in C \mid v \leqslant x\}$$
$$= \max_{u \in U} (x/u) + u \enspace .$$

Similar to $$P_C(x) = \sum_{u \in U} \langle x, u \rangle u$$

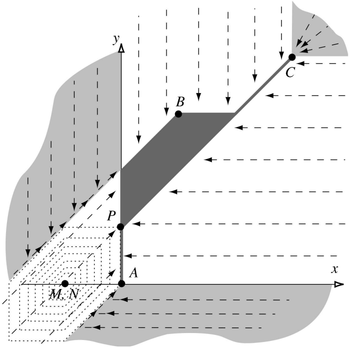
$$C = \mathrm{Col}(A), \quad [P_C(x)]_i = \max_{k \in [p]} \min_{j \in [n]} (A_{ik} - A_{jk} + x_j), \quad i \in [n]$$

Cuninghame-Green; Gondran, Minoux; Cohen, SG, Quadrat; Ardila; Joswig, Sturmfels, Yu

Best approximation in Hilbert's projective metric

Prop.(Cohen, SG, Quadrat, in Bensoussan Festschrift 01)

$$d(x, P_{\mathcal{V}}(x)) = \min_{y \in \mathcal{V}} d(x, y) \ .$$

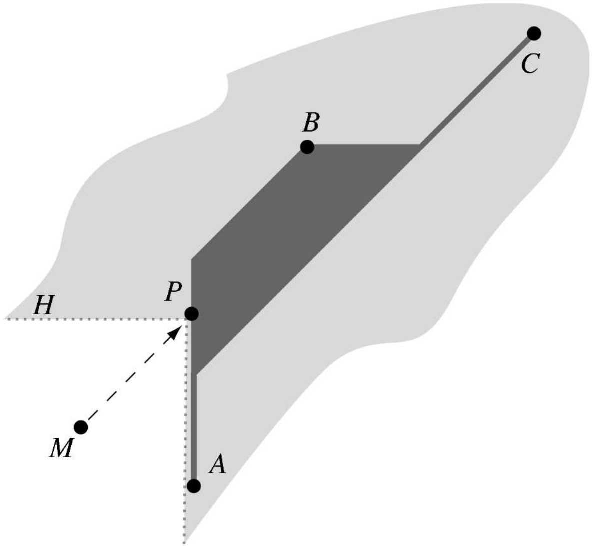

y
B
C
P
A
x
M, N

Separation

Goes back to Zimmermann 77, simple geometric construction in Cohen, SG, Quadrat in Ben01, LAA04.
C closed linear cone of $\mathbb{R}^d_{\max}$, or complete semimodule
If $y \notin C$, then, the tropical half-space

$$\mathcal{H} := \{v \mid y/v \leqslant P_C(y)/v\}$$

contains C and not y.
Compare with the optimality condition for the projection on a convex cone C: $\langle y - P_C(y), v\rangle \leqslant 0, \forall v \in C$


C
B
P
H
M
A

Corollary (Zimmermann; Samborski, Shpiz; Cohen, SG, Quadrat, Singer; Develin, Sturmfels; Joswig...)

A tropical convex cone closed (in the Euclidean topology) is the intersection of tropical half-spaces.

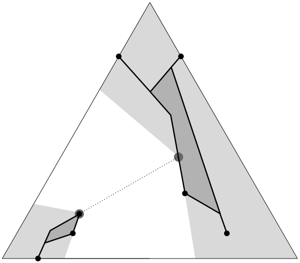
$\mathbb{R}_{\max}$ is equipped with the topology of the metric $(x, y) \mapsto \max_i |e^{x_i} - e^{y_i}|$ inherited from the Euclidean topology by log-glasses.

The apex $-P_C(y)$ of the algebraic separating half-space $\mathcal{H}$ above may have some $+\infty$ coordinates, and therefore may not be closed in the Euclidean topology (always Scott closed). The proof needs a perturbation argument, this is where the assumption that C is closed (and not only stable by arbitrary sups = Scott closed) is needed.

Separation of several convex sets / cyclic projections SG, Sergeev, Fund. i priklad. mat. 07

If $\mathcal{V}_1 \cap \cdots \cap \mathcal{V}_k = \{\text{“}0\text{”}\}$, we can find half-spaces $\mathcal{H}_i$ such that $\mathcal{H}_i \supset \mathcal{V}_i$ and $\mathcal{H}_1 \cap \cdots \cap \mathcal{H}_k = \{\text{“}0\text{”}\}$. The apices of these half-spaces are obtained from an eigenvector u of the cylic projector

$$\text{“}P_{\mathcal{V}_1} \cdots P_{\mathcal{V}_k}(u) = \lambda u\text{”}$$



The existence of such an eigenvector is obtained by a technique from nonlinear analysis, case of $-\infty$ entries dealt with by perturbation (Collatz-Wielandt theorem), more on this next lecture.

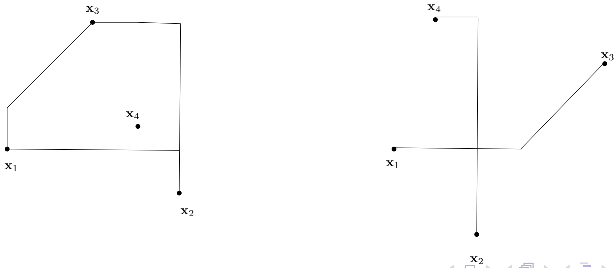
Theorem (tropical Helly theorem, Briec and Horvath, 04)

If a finite collection of cones of $\mathbb{R}^d_{\max}$ has a "zero" intersection, then a subfamily of at most d of them also has a "zero" intersection.

- Proved by "dequantization" from classical Helly (passing to the limit).
- Alt proof by SG and Sergeev 07 by cyclic projection (at each projection one coordinate decreases)

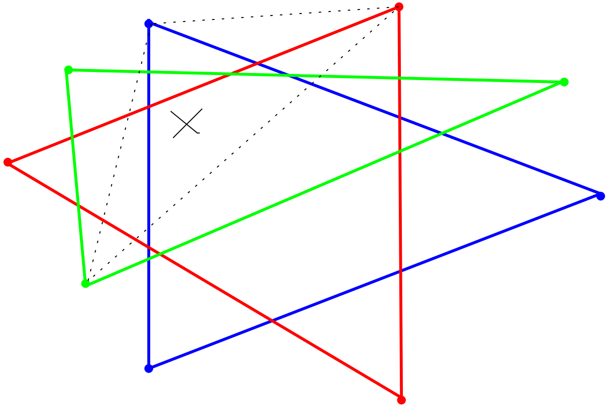
Tropical Radon

In SG and Meunier, DCG09, Helly deduced from the tropical Radon's theorem: a subset of $d+1$ vectors in dimension d can be partitioned in two subsets generating cones with a "non-zero" intersection.
Radon theorem follows from tropical Cramer theory (signs provide partition).



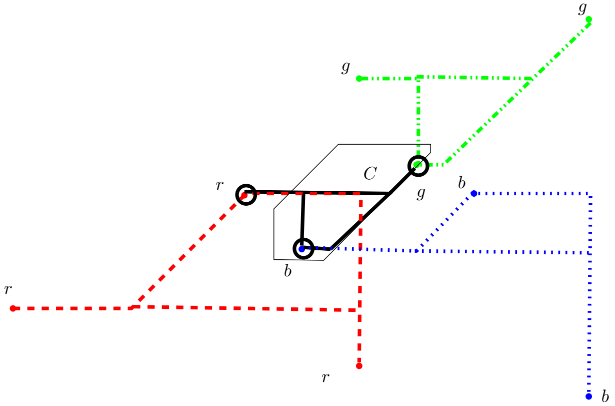
More advanced results of tropical convex geometry, SG and Meunier, DCG09

Barany's Colorful Caratheodory Theorem



More advanced results of tropical convex geometry, SG and Meunier, DCG09

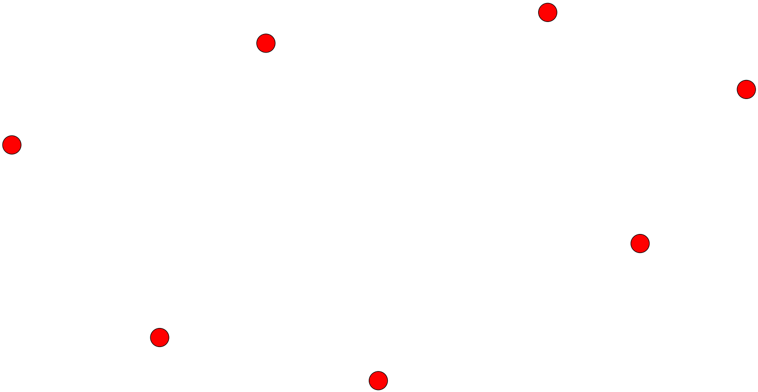
Barany's Colorful Caratheodory Theorem . . . Tropical



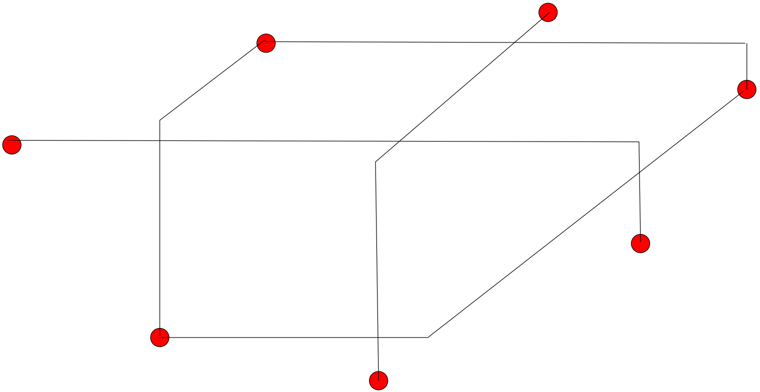
Tropical Tverberg's theorem, SG and Meunier, DCG09.
Let X be a set of $(d+1)(q-1)+1$ points in $\mathbb{R}^d_{\max}$.
Then there are q pairwise disjoint subsets $X_1, X_2, \ldots, X_q$ of X whose tropical convex hulls have a common point.

Deduced from the classical one by a limit argument, no direct combinatorial proof known.

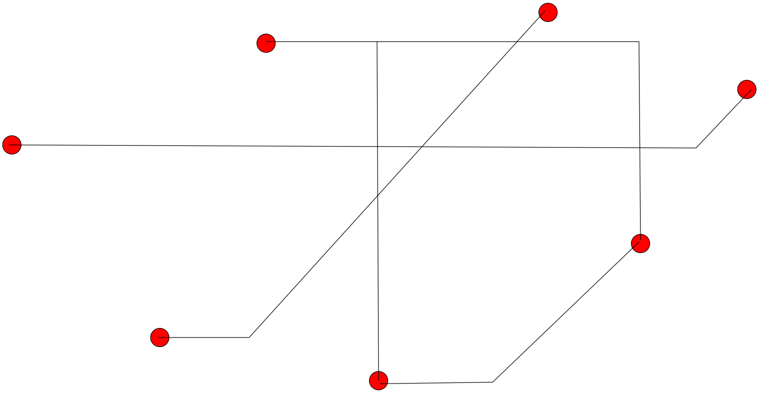
Tropical Tverberg



Tropical Tverberg



Tropical Tverberg



Dutch cheese conjecture

Let $q \geqslant 2$, $d \geqslant 1$. Sierksma conjectured that for every $(d+1)(q-1)+1$ points in $\mathbb{R}^d$ the number of unordered Tverberg partitions is at least $((q-1)!)^d$.

SG and Meunier, DCG09: True in the tropical setting! (development of a "bipartite analogue" of Tverberg's theorem due to Lindström, 1970 and Tverberg, 71)

Unfortunately, it is not clear whether this can be transferred to the classical case. In other words, there may be no "Mikhalkin's correspondence theorem" in the case of inequalities (?)

Menu of the next lectures

- Tropical linear programming, classical linear programming, and mean payoff games
- Non-linear Perron-Frobenius theory
- Infinite dimensional tropical convex sets,
- Metric geometry / boundaries
- max-plus approximation, curse of dim reduction in optimal control

Tropical convexity and its applications to zero-sum games

Minilecture, Part II

Stephane.Gaubert@inria.fr

INRIA and CMAP, École Polytechnique

JGA, Marseille
December 16-20, 2013

Works with Akian, Allamigeon, Goubault, Guterman, Katz, Joswig, Meunier, Sergeev, Walsh; highlight: PhD of Benchimol and Qu.

Today

Equivalence between tropical linear programming and mean payoff games

Tropical half-spaces

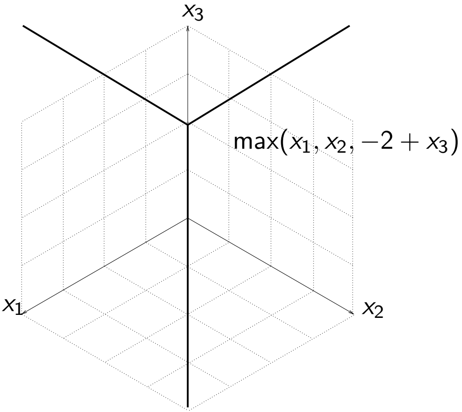
Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \text{"}ax \leqslant bx\text{"}\}$$

Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} a_i + x_i \leqslant \max_{1 \leqslant i \leqslant n} b_i + x_i\}$$



Tropical half-spaces

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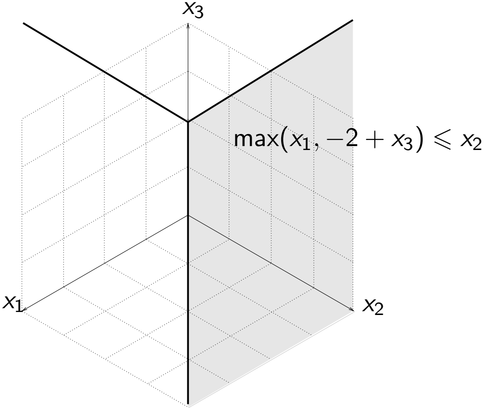

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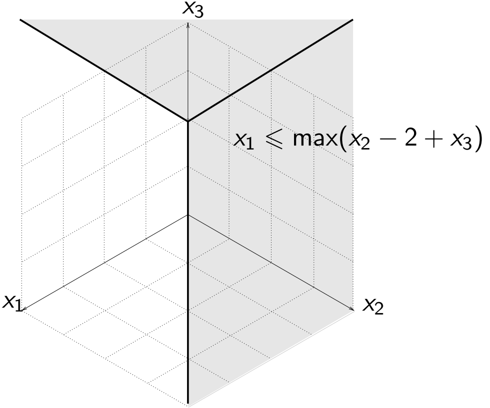

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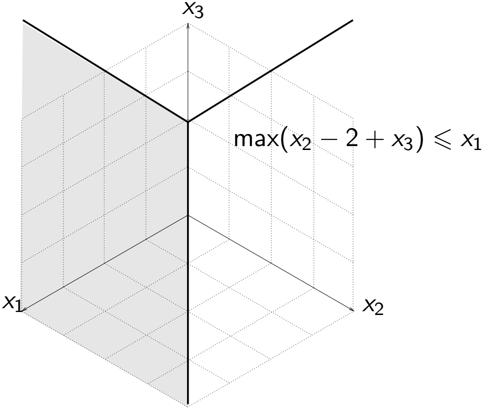


A halfspace can always be written as:

$$\max_{i \in I^-} a_i + x_i \leqslant \max_{j \in I^+} b_j + x_j, \qquad I^- \cap I^+ = \varnothing \quad .$$

Apex: $v_i := -\max(a_i, b_i)$.

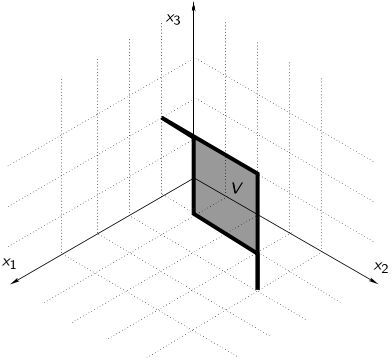
If $v \in \mathbb{R}^n$, H is the union of sectors of the tropical hyperplane with apex v:

$$\max_{1 \leqslant i \leqslant n} x_i - v_i \qquad \text{attained twice}$$

Halfspaces appeared in: Joswig 04; Cohen, Quadrat SG 00; Zimmermann 77, ...

Tropical polyhedral cones

can be defined as intersections of finitely many half-spaces



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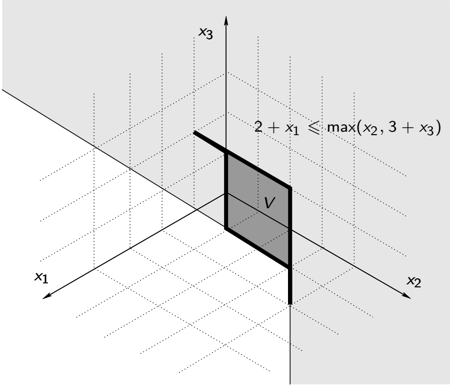

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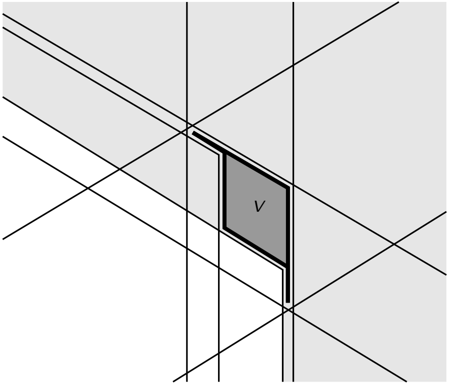


Feasibility in tropical LP

A **tropical polyhedral cone** is defined as

$$C = \{x \in \mathbb{R}^n_{\max} \mid \text{“}Ax \leqslant Bx\text{”}\}, \qquad A, B \in \mathbb{R}^{m\times n}_{\max}$$

$$\max_{j\in[n]} A_{ij} + x_j \leqslant \max_{j\in[n]} B_{ij} + x_j, \qquad \forall i \in [m]$$

A **tropical polyhedron** is defined as

$$P = \{x \in \mathbb{R}^n_{\max} \mid \text{“}Ax + c \leqslant Bx + d\text{”}\}$$

where $A, B \in \mathbb{R}^{m\times n}_{\max}$, $c, d \in \mathbb{R}^m_{\max}$.

Questions:

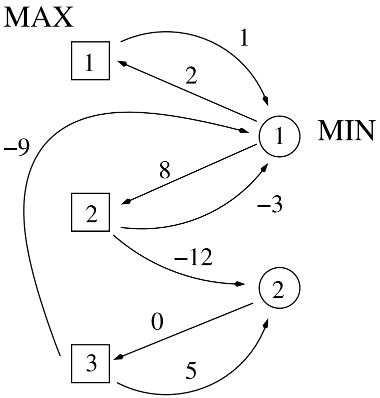
- is C reduced to “0” ? $\qquad$ “0” $= (-\infty, \dots, -\infty)^\top$
- does C contain a finite vector ?
- is P non-empty?

Example: mean payoff (deterministic) games

$G = (V, E)$ bipartite graph. r_{ij} price of the arc $(i, j) \in E$. "Max" and "Min" move a token. The player receives the amount of the arc.

v_i^k value of MAX, initial state (i, MIN).

$$v_1^k = \min(-2+1+v_1^{k-1}, -8+\max(-3+v_1^{k-1}, -12+v_2^{k-1}))$$
$$v_2^k = 0+\max(-9+v_1^{k-1}, 5+v_2^{k-1})$$



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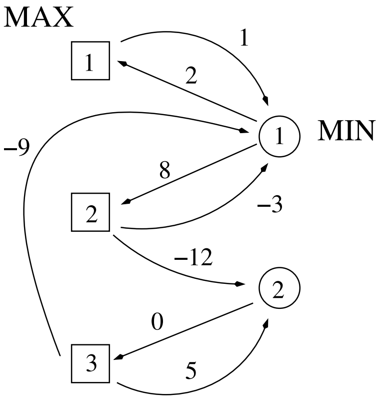
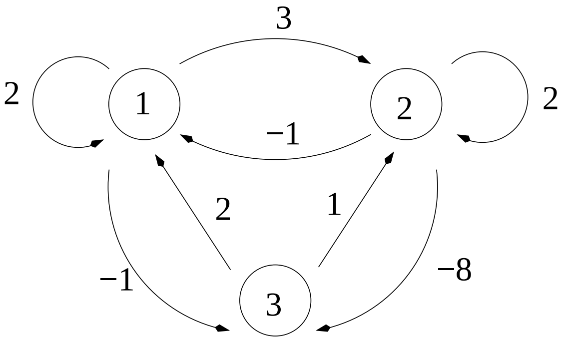


$\lim_k v^k/k = (-1, 5)$

Max and Min flip a coin to decide who makes the move. Min always pays.



Solving the game

- $v_i^k :=$ value of the k-horizon game starting from node i.

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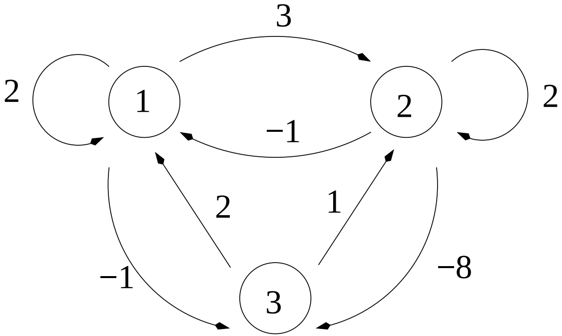
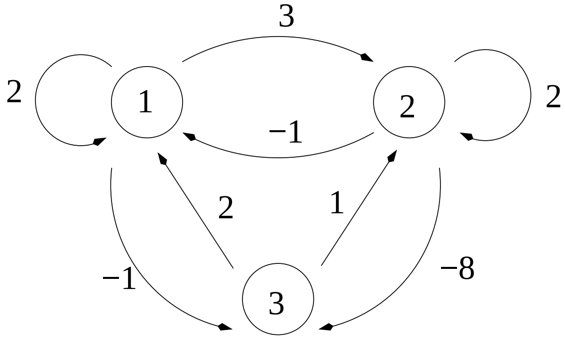
- $v^k = (v_i^k) \in \mathbb{R}^n$
- $v^0 = 0$
- $v^{k+1} = T(v^k)$

where $T : \mathbb{R}^n \to \mathbb{R}^n$ is the Shapley operator


3
2
1
2
2
−1
2
1
−1
3
−8



$$v_i^{k+1} = \frac{1}{2}\big(\max_{j:\, i\to j}(c_{ij} + v_j^k) + \min_{j:\, i\to j}(c_{ij} + v_j^k)\big) \quad .$$

Shapley operators

$X = \mathscr{C}(K)$, even $X = \mathbb{R}^n$, $K = [n]$; Shapley operator T,

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \leqslant j \leqslant n} P_{ij}^{ab} x_j \Big), \qquad i \in [n]$$

- $[n] := \{1, \dots, n\}$ set of states
- a action of Player I, b action of Player II
- r_i^{ab} payment of Player II to Player I
- P_{ij}^{ab} transition probability $i \to j$
- Nested max/min/mean can be reduced to the above.

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T is order preserving, additively homogeneous $\Rightarrow$ sup-norm nonexpansive:

$$x \leqslant y \implies T(x) \leqslant T(y)$$

$$T(\alpha + x) = \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}$$

$$\| T(x) - T(y) \| \leqslant \| x - y \|$$

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Conversely, any order preserving additively homogeneous operator is a Shapley operator (Kolokoltsov), even with degenerate transition probabilities (deterministic)

Gunawardena, Sparrow; Singer, Rubinov,

$$T_i(x) = \sup_{y \in \mathbb{R}} \Big(T_i(y) + \min_{1 \leqslant i \leqslant n} (x_i - y_i) \Big)$$

Repeated games

The value of the game in horizon k starting from state i is $(T^k(0))_i$.

We are interested in the long term payment per time unit

$$\chi(T) := \lim_{k\to\infty} T^k(0)/k$$

The mean payoff vector

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for $\|T^k(x) - T^k(0)\| \leqslant \|x - 0\| = \|x\|$

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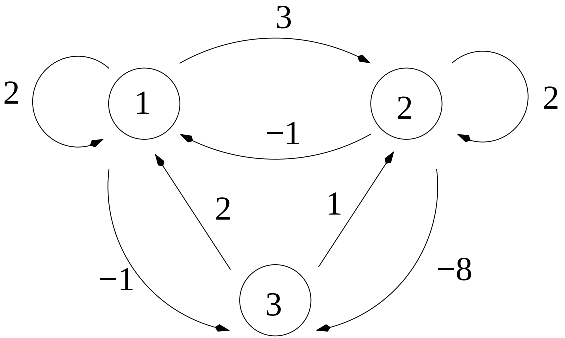
$$\text{for } \|T^k(x) - T^k(0)\| \leqslant \|x - 0\| = \|x\|$$

Think of x_i has a terminal bounty paid by Min to Max if the game ends in state i.



$$\begin{bmatrix}5\\0\\4\end{bmatrix}\quad \begin{aligned} v_1 &= \tfrac{1}{2}(\max(\overline{2+v_1}, 3+v_2, -1+v_3) + \min(2+v_1, \overline{3+v_2}, \overline{-1+v_3}) \\ v_2 &= \tfrac{1}{2}(\max(\overline{-1+v_1}, 2+v_2, -8+v_3) + \min(-1+v_1, 2+v_2, \overline{-8+v_3}) \\ v_3 &= \tfrac{1}{2}(\max(\overline{2+v_1}, 1+v_2) + \min(2+v_1, \overline{1+v_2}) \end{aligned}$$

this game is fair

Optimality certificates

More generally, for $u \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

$$T(u) \geqslant u \implies \chi(T) \geqslant 0 \qquad \text{superfair}$$

$$T(u) \leqslant u \implies \chi(T) \leqslant 0 \qquad \text{subfair}$$

$$T(u) = \lambda + u \implies \chi(T) = (\lambda, \dots, \lambda) \ .$$

Does $\chi(T) = \lim_k T^k(0)/k$ exist? Do such certificates exist?

$\chi_i = \chi_j$ is related to ergodicity. **May not hold for deterministic games, what are the certificates then?**

$\chi(T) = \lim_k T^k(0)/k$ may not exist if the action spaces are infinite (Kohlberg, Neyman). Counter example in dimension 3.

However. Let v_α denote the discounted value

$$v_\alpha = T(\alpha v_\alpha), \qquad 0 < \alpha < 1 \ .$$

Theorem (Neyman 04 - book edited with Sorin)

If $\alpha \mapsto (1-\alpha)v_\alpha$ has bounded variation as $\alpha \to 1$, then

$$\lim_k T^k(0)/k = \lim_{\alpha \to 1^-} (1-\alpha)v_\alpha \qquad \textit{does exist}$$

Corollary (Neyman 04, Bewley and Kohlberg 76)

If the graph of T is semi-algebraic, then $\chi(T)$ does exists.

Then, v_α is a semi-algebraic function of α, it has a Puiseux series expansion, and so $(1-\alpha)v_\alpha$ has BV. This is the case in particular if the action spaces are finite.

More generally, T definable in an o-minimal model (Bolte, SG, Vigeral 13).

By subadditivity, the following limits (indep of $x \in \mathbb{R}^n$) do exist

$$\lim_{k\to\infty} \frac{\|T^k(x)-x\|_\infty}{k} = \inf_{k\geqslant 1} \frac{\|T^k(x)-x\|_\infty}{k}$$

$$\overline{\chi}(T) := \lim_{k\to\infty} \frac{\mathrm{t}(T^k(x)-x)}{k} = \inf_{k\geqslant 1} \frac{\mathrm{t}(T^k(x)-x)}{k}$$

$$\underline{\chi}(T) := \lim_{k\to\infty} \frac{\mathrm{b}(T^k(x)-x)}{k} = \sup_{k\geqslant 1} \frac{\mathrm{b}(T^k(x)-x)}{k}$$

$$\mathrm{t}(z) := \max_{1\leqslant i\leqslant n} z_i, \qquad \mathrm{b}(z) := \min_{1\leqslant i\leqslant n} z_i \quad .$$

Think of T as a Perron-Frobenius operator in log-glasses:

$$F = \exp \circ T \circ \log, \qquad \mathbb{R}_+^n \to \mathbb{R}_+^n$$

F extends continuously from $\operatorname{int} \mathbb{R}_+^n$ to $\mathbb{R}_+^n$ Burbanks, Nussbaum, Sparrow.

Theorem (non-linear Collatz-Wielandt, Nussbaum, LAA 86)

$$\begin{aligned}
\rho(F) &= \lim_{k\to\infty} \|F^k(x)\|^{1/k}, \qquad x \in \operatorname{Int} \mathbb{R}_+^n \\
&= \max\{\mu \in \mathbb{R}_+ \mid F(v) = \mu v, v \in \mathbb{R}_+^n, v \neq 0\} \\
&= \max\{\mu \in \mathbb{R}_+ \mid F(v) \geqslant \mu v, v \in \mathbb{R}_+^n, v \neq 0\}
\end{aligned}$$

Stochastic games $\to$ tropical convex feasibility

Corollary

Let T be a Shapley operator. Then, $\lim_k \max_i [T^k(0)]_i/k \geqslant 0$ *iff there is* $u \in (\mathbb{R} \cup \{-\infty\})^n$, $u \neq (-\infty, \dots, -\infty)^\top$, $T(u) \geqslant u$.

Proposition

If T is a Shapley operator, $C = \{u \in \mathbb{R}^n_{\max} \mid T(u) \geqslant u\}$ *is a closed tropical convex cone.*

Proof.

$T(\sup(u, v)) \geqslant \sup(T(u), T(v)) \geqslant \sup(u, v)$,
$T(\alpha + u) = \alpha + T(u)$, $\alpha \in \mathbb{R}$. $\square$

Conversely, any closed tropical convex cone can be written as

$$C = \bigcap_{i \in I} H_i$$

where $(H_i)_{i \in I}$ is a family of tropical half-spaces.

$$H_i : \text{“}A_i x \leqslant B_i x\text{”}$$

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$$[T(x)]_j = \inf_{i \in I} -a_{ij} + \max_{1 \leqslant k \leqslant n} b_{ik} + x_k \quad .$$

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$$[T(x)]_j = \inf_{i \in I} -a_{ij} + \max_{1 \leqslant k \leqslant n} b_{ik} + x_k \quad .$$

$$x \leqslant T(x) \iff \max_{1 \leqslant j \leqslant n} a_{ij} + x_j \leqslant \max_{1 \leqslant k \leqslant n} b_{ik} + x_k , \ \forall i \in I \quad .$$

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$$[T(x)]_j = \inf_{i\in I} -a_{ij} + \max_{1\leqslant k\leqslant n} b_{ik} + x_k \ .$$

Interpretation of the game

- State of MIN: variable x_j, $j \in \{1, \dots, n\}$
- State of MAX: half-space H_i, $i \in I$
- In state x_j, Player MIN chooses a tropical half-space H_i with x_j in the LHS
- In state H_i, player MAX chooses a variable x_k at the RHS of H_i
- Payment $-a_{ij} + b_{ik}$.

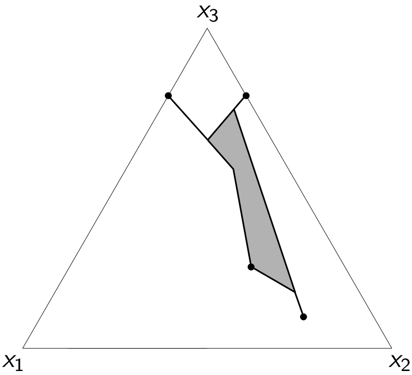
Correspondence between tropical convexity and zero-sum games

Theorem (Akian, SG, Guterman, IJAC 2012)

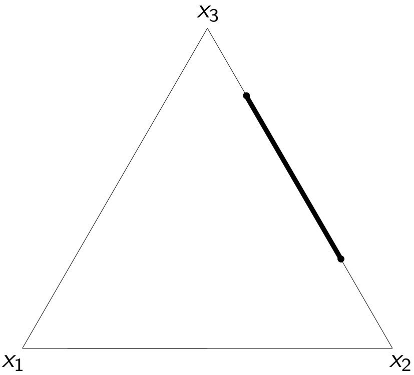
TFAE:

- *C closed tropical convex cone*
- *$C = \{u \in (\mathbb{R} \cup \{-\infty\})^n \mid u \leqslant T(u)\}$ for some Shapley operator T*

and MAX has at least one winning state ($\exists i, \chi_i(T) \geqslant 0$) if and only if $C \neq \{(-\infty, \dots, -\infty)\}$. Moreover, tropical polyhedra correspond to deterministic games with finite action spaces. Then, state i is winning iff $u_i \neq -\infty$ for some $u \in C$.



states 1,2,3 winning



states 2,3 winning

Polyhedral part relies on Kohlberg's theorem 1980.

A nonexpansive piecewise affine map $T:\mathbb{R}^n \to \mathbb{R}^n$ admits an invariant half-line

$$\exists v \in \mathbb{R}^n,\ \vec{\eta} \in \mathbb{R}^n, \quad T(v + t\vec{\eta}) = v + (t+1)\vec{\eta} \ .$$

The vector u such that $T(u) \geqslant u$ is obtained from v, η (hint: $u_i = -\infty$ if $\vec{\eta}_i < 0$).

- Strategy of MAX $\sigma : \{H_1, \ldots, H_m\} \to \{x_1, \ldots, x_n\}$, in state H_i choose coordinate $x_{\sigma(i)}$

Duality theorem (coro of Kohlberg)

$$\chi(T) = \max_{\sigma} \chi(T^{\sigma}) = \min_{\pi} \chi(T_{\pi}) \quad .$$

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- One player Shapley operators

$$[T^\sigma(x)]_j = \inf_{1 \leqslant i \leqslant m} -a_{ij} + b_{i\sigma(i)} + x_{\sigma(i)} \quad .$$

$$[T_\pi(x)]_j = -a_{\pi(j)j} + \max_{1 \leqslant k \leqslant n} b_{\pi(j)k} + x_k \quad .$$

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Duality theorem (coro of Kohlberg)

$$\chi(T) = \max_\sigma \chi(T^\sigma) = \min_\pi \chi(T_\pi) \enspace .$$

Every $\chi(T^\sigma)$ and $\chi(T_\pi)$ can be computed in polynomial time.

$$\chi(T) = \max_{\sigma} \chi(T^{\sigma}) = \min_{\pi} \chi(T_{\pi}) \enspace .$$

- "$Ax \leqslant Bx$" unfeasible iff $\exists \pi$, $\overline{\chi}(T_\pi) < 0$.

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- "$Ax \leqslant Bx$" unfeasible iff $\exists \pi$, $\overline{\chi}(T_{\pi}) < 0$.
- "$Ax \leqslant Bx$" feasible iff $\exists \sigma$, $\overline{\chi}(T^{\sigma}) \geqslant 0$.

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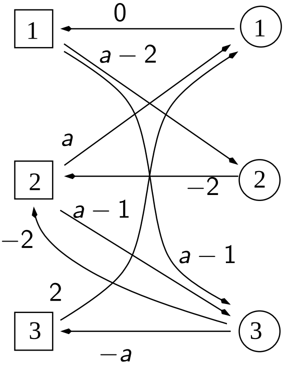
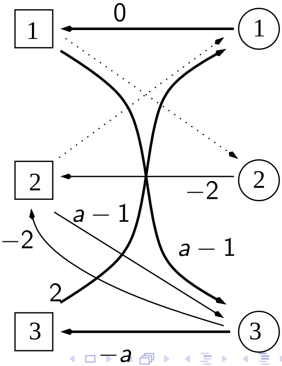
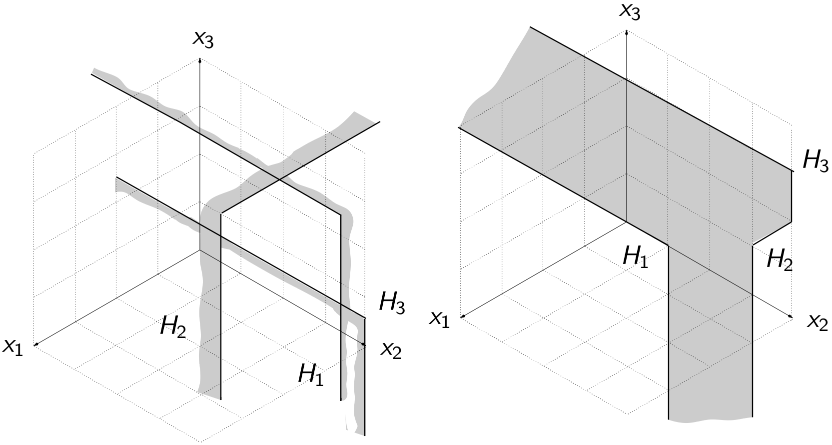
Strategies are Lagrange multipliers!

$$x_1 \leqslant a + \max(x_2 - 2, x_3 - 1) \qquad (H_1)$$
$$-2 + x_2 \leqslant a + \max(x_1, x_3 - 1) \qquad (H_2)$$
$$\max(x_2 - 2, x_3 - a) \leqslant x_1 + 2 \qquad (H_3)$$

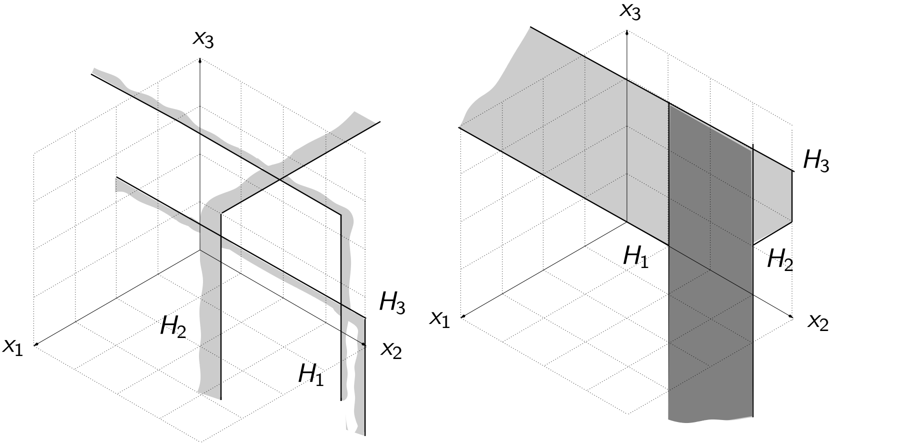
value $\chi(T)_j = (2a+1)/2, \forall j.$







$a = -3/2$, victorious strategy of Min: certificate of emptyness involving $\leqslant n$ inequalities (Helly)



$a = 1$, victorious strategy of Max: tropical polytrope $\neq \emptyset$ included in the convex set

Tropical Farkas (Allamigeon, SG, Katz, LAA11)

- Check "$Ax \leqslant Bx \implies cx \leqslant dx$"?

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Implication holds in Farkas iff $\overline{\chi}(T) < 0$, where T is the Shapley operator associated to the system "$Ax \leqslant Bx, dx \leqslant \alpha cx$".

Compare with classical Farkas

- Given $a_1, \ldots, a_m, c \in \mathbb{Q}^n$, is it true that

$$x \in \mathbb{R}^n, \qquad a_i \cdot x \geqslant 0 \qquad \forall 1 \leqslant i \leqslant m \implies c \cdot x \geqslant 0 \ ?$$

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- λ can be required to be sparse: $\lambda_i = 0$ except for n values of $1 \leqslant i \leqslant m$. (Carathéodory / Helly by duality). ANALOGOUS
- λ can actually be found in polynomial time (Linear programming: Khachyan 79, Karmarkar 84,...). DONT KNOW!

The Mean Payoff Game problem

Compute $\chi(T)$ where T Shapley operator of deterministic game with finite action spaces?

Existence of polynomial time algorithm open since Gurvich, Karzanov, Khachyan 86. One of the few natural pbs in NP $\cap$ coNP not known to be in P (with factoring!).

Pseudo polynomial algorithm (value iteration) Zwick-Paterson 96, experimentally efficient policy iteration algorithms but worst case exponential Friedmann 10.

The following problems are all equivalent to mean payoff games

- Feasibility: $\exists x \not\equiv -\infty?$, "$Ax \leqslant Bx$"

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- Farkas: "$Ax \leqslant Bx \implies cx \leqslant dx$"
- Separation: given finite sets $X, Y \subset \mathbb{R}^n_{\max}$, $\operatorname{cone} X \cap \operatorname{cone} Y = \{\text{"}0\text{"}\}$?

Tropical simplex

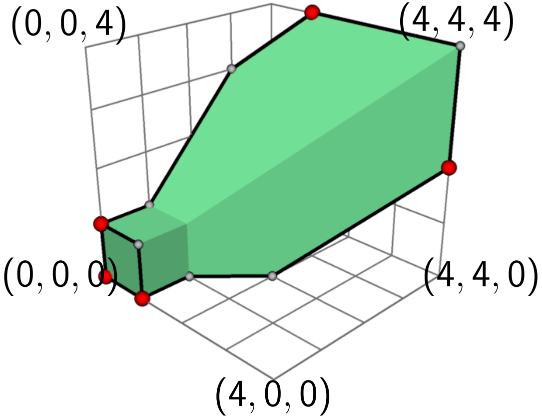
A tropical LP

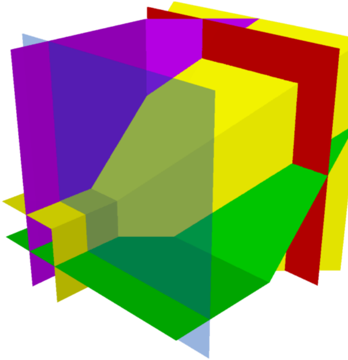
$$\min \text{“}f \cdot x\text{”}; \quad \text{“}Ax + c \leqslant Bx + d\text{”}$$

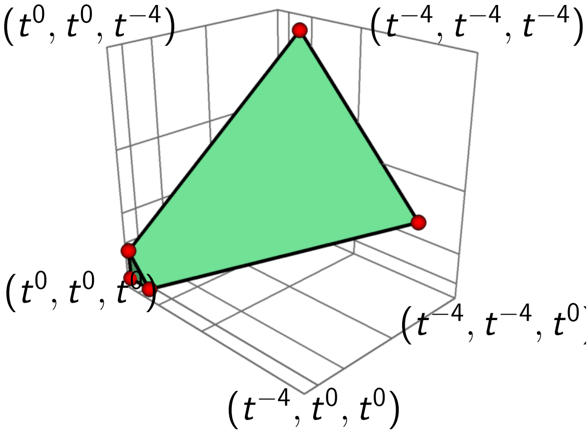
$A, B \in \mathbb{R}_{\max}^{m\times n}$, $b, c \in \mathbb{R}_{\max}^{m}$, $f \in \mathbb{R}_{\max}^{n}$, the inqualities "$x \geqslant 0$" being included in "$Ax + c \leqslant Bx + d$", can be lifted to a classical LP over Puiseux series

$$\min \mathbf{f} \cdot \mathbf{x}; \quad \mathbf{A}\mathbf{x} + \mathbf{c} \leqslant \mathbf{B}\mathbf{x} + \mathbf{d}$$

$\mathbf{A}, \mathbf{B} \in \mathbb{K}^{m\times n}$, $\mathbf{b}, \mathbf{c} \in \mathbb{K}^{m}, \mathbf{f} \in \mathbb{K}^{n}$,
meaning that $\operatorname{val} \mathbf{A} = A$, $\operatorname{val} \mathbf{B} = B$, etc. Recall that $\operatorname{val} 7t^{-1/2} - 1 + t^{1/2} + 7t + \cdots = 1/2$.


(0, 0, 4)
(4, 4, 4)
(0, 0, 0)
(4, 4, 0)
(4, 0, 0)




(t^0, t^0, t^{-4})
(t^{-4}, t^{-4}, t^{-4})
(t^0, t^0, t^0)
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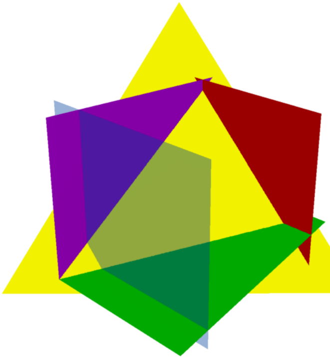

Assume that the data are in general position. This can be defined in terms of tropical Cramer subdeterminants of "$(A + B, c + d)$".

A **tropical basic point** is obtained by saturating n inequalities.

Theorem (Allamigeon, Benchimol, SG, Joswig arXiv:1308.0454)

The valuation of the path of the simplex algorithm over Puiseux series can be computed tropically (with a compatible pivoting rule). One iteration takes $O(n(m+n))$ *time.*

Tropical Cramer determinants = opt. assignment used to compute reduce costs.

Example of compatible pivoting rule. A rule is **combinatorial** if any entering/leaving inequalities are functions of the history (sequence of bases) and of the signs of the minors of the matrix

$$M = \begin{pmatrix} \text{“}A - B\text{”} & \text{“}c - d\text{”} \\ f^\top & \text{“}0\text{”} \end{pmatrix} .$$

(eg signs of reduced costs).

Corollary (Allamigeon, Benchimol, SG, Joswig arXiv:1309.5925)

If any combinatorial rule in classical linear programming would run in polynomial time, then, mean payoff games could be solved in strongly polynomial time.

Concluding remarks

- Complexity of tropical LP = deterministic mean payoff games is open
- Stochastic mean payoff games: a fortiori (a pseudo polynomial algorithm is not known).
- Games with fixed discount rate: strongly polynomial, Ye; Hansen, Miltersen, Zwick.
- Current work (Allamigeon, Benchimol, SG, Joswig): tropicalization of central path.

Tropical convexity and its applications to zero-sum games

Minilecture, Part III

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INRIA and CMAP, École Polytechnique

JGA, Marseille
December 16-20, 2013

Works with Akian, Allamigeon, Goubault, Guterman, Katz, Joswig, Meunier, Sergeev, Walsh; highlight: PhD of Benchimol and Qu.

Tropical Minkowski-Weyl

Theorem (SG, Katz, RelMiCS 06, JACO 2011)
A tropical polyhedral convex set can be written as

$$\mathcal{K} = \text{“conv}(X) + \text{cone}(Y)\text{”}$$

with X, Y finite, and vice versa.

- Inequalities to vertices: finiteness can be proved by elimination Butkovic and Hegedus, 84; tropical double description Allamigeon, SG, Goubault DCG 2012.

- Vertices to inequalities: the set of valid inequalities is itself a polyhedron (tropical polar; SG, Katz).

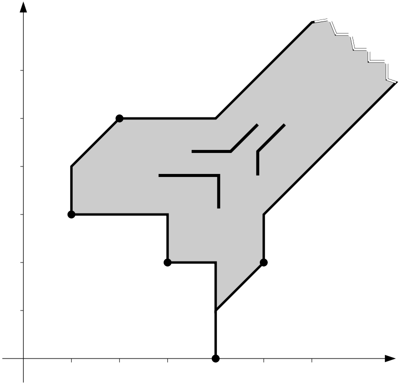
Extreme points and rays

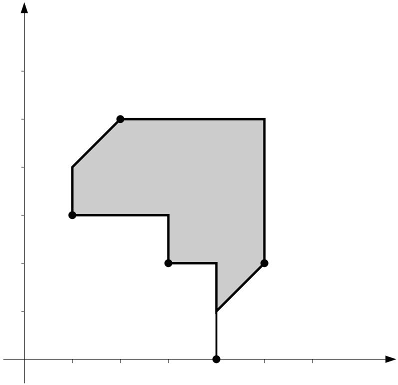
Definition
$u \in \mathcal{V}$ is an extreme generator if $u =$ "$v + w$" with $v, w \in \mathcal{V}$ implies $u = v$ or $u = w$. (ie u join irreducible)

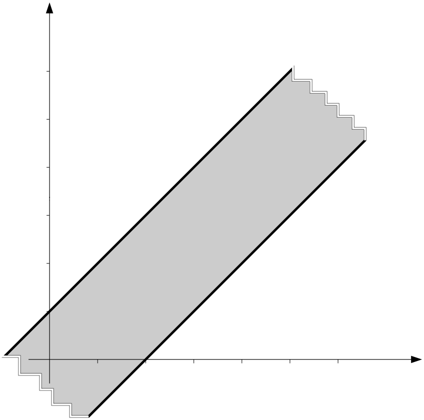
Theorem (Tropical Minkowski SG, Katz RELMICS 06, LAA07; Butkovič, Sergeev, Schneider LAA07**)**
Every element of a closed tropical cone of $\mathbb{R}^d_{\max}$ is a sum of at most d extreme generators.

Affine version. If $\mathcal{K}$ is a closed convex set, then,

$$\mathcal{K} = \text{"}\mathrm{conv}(\mathrm{ext}(K)) + \mathrm{rec}(K)\text{"}$$



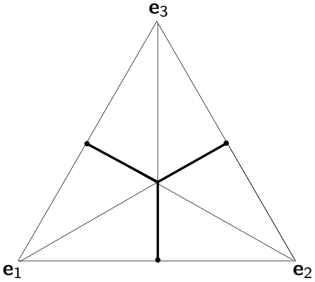




It is one of the places where $-\infty$ is needed; generators do have $-\infty$ coordinates.

E.g., the tropical line, $\max(x_1, x_2, x_3)$ attained twice, is generated by

$$(0, 0, -\infty), \qquad (0, -\infty, 0), \qquad (-\infty, 0, 0)$$



Number of extreme rays

A classical question . . .

- What is the maximal number of facets of a polytope of dimension d with p vertices?

or (equivalent by duality)

- What is the maximal number of vertices of a polytope of dimension d with p facets ?

Theorem (McMullen upper bound 1970)

Among the polytopes of dimension d with p vertices, the cyclic polytope maximizes the number of faces of each dimension.

The cyclic polytope $C(p, d)$ is the convex hull of p points of the moment curve $t \mapsto z(t) := (t, t^2, \ldots, t^d)$.

Definition

A 0/1 sequence satisfies Gale's evenness condition if the number of 1 between any two 0 is even.

Eg., 0110111100011001111110000111

Let $C(p, d) := \mathrm{co}(z(t_1), \dots, z(t_p))$ with $t_1 < t_2 < \cdots < t_p$.

Fact

The points $z(t_{i_1}), \dots, z(t_{i_{d+1}})$ define a facet iff the associated word satisfies Gale's evenness condition.

Eg., $i_1 = 2, i_2 = 3,\ i_3 = 5,\ p = 5, d = 2 \rightarrow 01101$

Corollary (classical)

The number of facets of a polytope of dimension d with p vertices is at most

$$U(p,d) := \binom{p-d/2}{d/2} + \binom{p-d/2-1}{d/2-1} \qquad \textit{for } d \textit{ even}$$

$$U(p,d) := 2\binom{p-(d+1)/2}{(d-1)/2} \qquad \textit{for } d \textit{ odd.}$$

This is $\Theta(p^{d/2})$ has $p \to \infty$, keeping d fixed, so much smaller than the naive bound $\binom{p}{d} = \Theta(p^d)$.

The same questions can be raised for max-plus or tropical convex sets/ cones

Theorem (Allamigeon, SG, Katz, JCTA 11)

The number of extreme rays of a tropical cone $\mathcal{V}$ defined by p inequalities in dimension d cannot exceed $U(p+d, d-1)$.

$$\mathcal{V} := \{x \in \mathbb{R}^d_{\max} \mid \max_{j\in[d]} a_{ij} + x_j \leqslant \max_{j\in[d]} b_{ij} + x_j, \ i \in [p]\} \ .$$

The bound is $\Theta(p^{\lfloor (d-1)/2 \rfloor})$ for d fixed and $p \to \infty$.

Proof (by dequantization)

For $\beta > 0$, consider the classical convex cone $\mathcal{V}(\beta)$ defined by the $p + d$ inequalities

$$
\begin{aligned}
y_j \geqslant 0 \ , \qquad & j \in [d] \ , \\
\frac{1}{d} \sum_{j \in [d]} \exp(\beta a_{ij}) y_j \leqslant \sum_{j \in [d]} \exp(\beta b_{ij}) y_j \ , & i \in [p] \ .
\end{aligned}
$$

By the McMullen upper bound theorem, $\mathcal{V}(\beta)$ has a generating family $(u_k(\beta))_{k \in [K]}$ with $K \leqslant U(p + d, d - 1)$.

If $x \in \mathcal{V}$, then $E_\beta(x) := (\exp(\beta x_j)) \in \mathcal{V}(\beta)$. WLOG, normalize $u_k(\beta)$ (entries sum to one). Let $v_k(\beta) := E_\beta^{-1}(u_k)$.

$$\max_{j\in[d]} v_k(\beta)_j \leqslant 0 \leqslant \beta^{-1}\log d + \max_{j\in[d]} v_k(\beta)_j \enspace ,$$

$$-\beta^{-1}\log d + \max_{j\in[d]} a_{ij} + v_k(\beta)_j \leqslant \beta^{-1}\log d + \max_{j\in[d]} b_{ij} + v_k(\beta)_j \enspace .$$

Then, it can be checked that any accumulation point of the family $(v_k(\beta))_{k\in[K]}$ yields a generating family of $\mathcal{V}$ (use $\mathcal{V}(\beta) \supset \mathcal{V}$ thanks to the $1/d$ trick). $\square$

Is the tropical upper bound attained?

The usual bound of $\sharp$ vertices for a dim d polytope with p facets is attained by the polar of the cyclic polytope

$$C(p,d)^\circ := \{y \mid z(t_i)\cdot(y-w) \leqslant 1, i \in [p]\},\ w \in \mathrm{int}(C(p,d)) \enspace .$$

In the tropical case

$$z(t) := \text{“}(1, t, \ldots, t^{d-1})\text{”} = (1, t, \ldots, (d-1)t) \in \mathbb{R}^d_{\max} \enspace ,$$

Homogeneizing naively $C(p,d)^\circ$ yields

$$\{y \mid \text{“}z(t_i)\cdot y \leqslant 0\text{”}, i \in [p]\}$$

which is trivial. To make it less trivial, we may add signs.

- Introduce a **sign pattern** $\epsilon_{ij} \in \{\pm 1\}$
- Set formally "$z(t_i)_j = \epsilon_{ij} t_i^{j-1}$" in the symmetrized maxplus semiring $\mathbb{S}_{\max}$, so $z(t_i) =$ "$z^+(t_i) - z^-(t_i)$" where

$$z^\pm(t_i) \in \mathbb{R}^d_{\max}, \qquad z^\pm(t_i)_j = \begin{cases} \text{“}t_l^{j-1}\text{”} & \text{if } \epsilon_{ij} = \pm 1 \\ \text{“}0\text{”} & \text{otherwise} \end{cases}$$

Definition

The signed cyclic polyhedral cone $C(p, d; \epsilon)$, is generated by p pairs of vectors $(z^-(t_i), z^+(t_i)) \in (\mathbb{R}^d_{\max})^2$, $i \in [p]$. Its polar $\mathcal{K}(p, d; \epsilon)$ is the set of vectors $x \in \mathbb{R}^d_{\max}$ such that

$$\text{“}z^-(t_i) \cdot x \leqslant z^-(t_i) \cdot x\text{”}, \quad i \in [p], \text{ i.e.}$$

$$\max_{j \in [d], \epsilon_{ij} = -1} (j-1)t_i + x_j \leqslant \max_{j \in [d], \epsilon_{ij} = +1} (j-1)t_i + x_j, \qquad i \in [p] \ .$$

The analogy with the classical case . . .

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may suggest that there should be some choice of sign ϵ such that the polar $\mathcal{K}(p, d; \epsilon)$ of the signed cyclic polyhedral cone has exactly $U(p + d, d - 1)$ extreme rays. . .

The analogy with the classical case ...

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we shall see that this is not true.

Some lattice paths

southward / eastward paths in the sign pattern ϵ_{ij}

$$\begin{array}{cc}
 & \begin{matrix} & j_1 & & & j_2 & j_3 & & & & j_4 & & j_5 & j_6 & & \end{matrix} \\
\begin{matrix} \\ \\ i_1 \\ \\ \\ i_2 \\ \\ i_3 \\ \\ \\ \\ i_4 \\ \\ \\ \\ \\ i_5 \\ \\ \\ \\ \end{matrix} &
\begin{pmatrix}
\cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & + & \star & \star & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & - & \star & \star & \star & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & \star & + & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot
\end{pmatrix}
\end{array}$$

A lattice path for the sign pattern ϵ_{ij} is tropically allowed if

(i) every sign occurring on the initial vertical segment, except possibly the sign at the bottom of the segment, is positive;

(ii) every sign occurring on the final vertical segment, except possibly the sign at the top of the segment, is positive;

(iii) every sign occurring in some other vertical segment, except possibly the signs at the top and bottom of this segment, is positive;

(iv) for every horizontal segment, the pair of signs consisting of the signs of the leftmost and rightmost positions of the segment is of the form $(+,-)$ or $(-,+)$;

(v) as soon as a pair $(-,+)$ occurs as the extreme signs of an horizontal segment, the pairs of the next horizontals segments must also be equal to $(-,+)$.

If only (i)–(iv) hold, we say that the path is classically allowed.

Theorem (Allamigeon, SG, Katz, JCTA 11)

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When deforming a polytope into a tropical polytope, some extreme points vanish.

This apparently mysterious result relies on a characterization of the extreme points of a tropical polyhedron in terms of the inequalities which define it:
Allamigeon, SG, Goubault, DCG 2012

Recall first.

Fact (See Butkovič, Sergeev, Schneider; SG, Katz, both LAA 07)

A vector g of a tropical cone $\mathcal{C} \in \mathbb{R}^d_{\max}$ is extreme iff $\exists t \in [d]$ such that g is a minimal element of the set $\{ x \in \mathcal{C} \mid x_t = g_t \}$. In that case, g is said to be extreme of type t.

Definition

The tangent cone of $\mathcal{C} := \{x \mid \text{“}Ax \leqslant Bx\text{”}\}$ at g is defined as the tropical cone $\mathcal{T}(g, \mathcal{C})$ of $\mathbb{R}^d_{\max}$ given by the system of inequalities

$$\max_{i \in \arg\max(A_k g)} x_i \leqslant \max_{j \in \arg\max(B_k g)} x_j$$

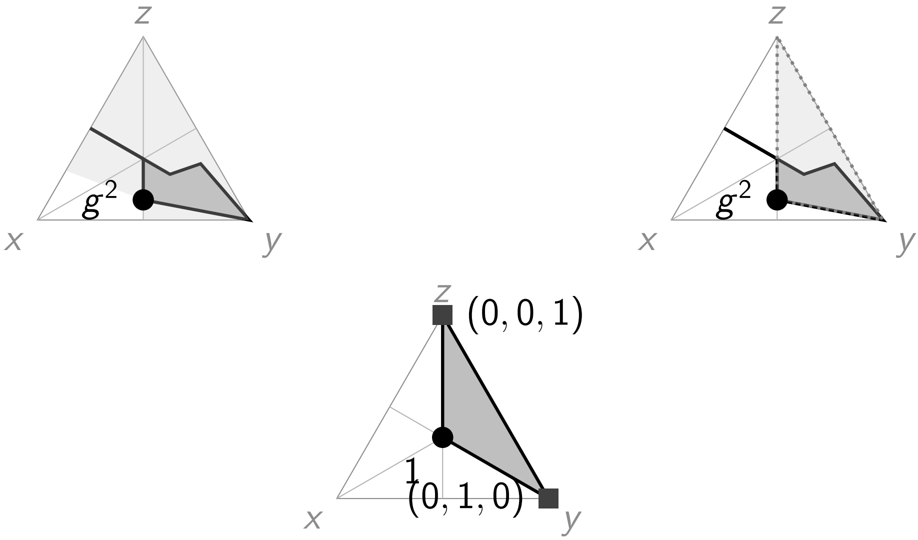
for all $k \in [p]$ such that $A_k g = B_k g$.

Fact (Allamigeon, SG, Goubault)

There exists a neighborhood N of g such that for all $x \in N$, x belongs to $\mathcal{C}$ if and only if it is an element of $g + \mathcal{T}(g, \mathcal{C})$.

Fact (ibid.)

The element g is extreme in $\mathcal{C}$ if and only if the vector 1 is extreme in $\mathcal{T}(g, \mathcal{C})$.


z
x
y
g^2
z
x
y
g^2
z
(0, 0, 1)
1
(0, 1, 0)
x
y

Theorem (Allamigeon,SG, Goubault, ibid.)

A vector $y \in \mathbb{R}^d_{\max}$ belongs to an extreme ray of a tropical polyhedral cone $\mathcal{C}$ if, and only if, there exists $s \in \{1, \dots, d\}$ such that

$$(x \in \mathcal{T}(\mathcal{C}, y) \cap \{1, 0\}^d \text{ and } x_s = 1) \Rightarrow (x_r = 1 \text{ or } y_r = 0)$$

for all $r \in \{1, \dots, d\}$.

Corollary

If t entries of y are zero, then y must saturate at least $d - t - 1$ inequalities among $A_r x \leqslant B_r x$, $r \in [p]$.

Recall our characterization: a vector $y \in \mathbb{R}^d_{\max}$ belongs to an extreme ray of a tropical polyhedral cone $\mathcal{C}$ if, and only if, there exists $s \in \{1, \dots, d\}$ such that

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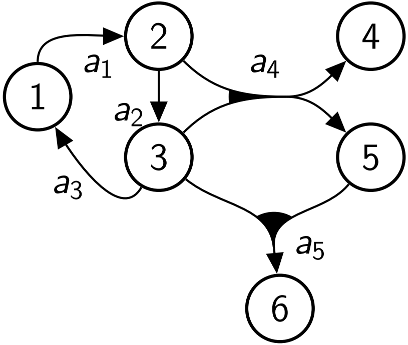
for all $r \in \{1, \dots, d\}$.

Eg, when y is finite, does there exists s such that, for $x \in \{0, 1\}^d$,

$$x_s = 1 \text{ and } \max_{i \in \arg\max(A_k y)} x_i \leqslant \max_{j \in \arg\max(B_k y)} x_j \implies x \equiv 1?$$

This is expressed as an hypergraph reachability problem. Given a node set N, an (oriented) hyperedge is a pair (T, H) (tail, head) with $T, H \subset N$. We say that v is reachable from u if $u = v$, or there exists $e \in E$ such that $v \in H(e)$ and all the elements of $T(e)$ are reachable from u. Here, $T = \arg\max(A_k y)$ and $H = \arg\max(B_k y)$.



Proposition (ibid.)

TFAE

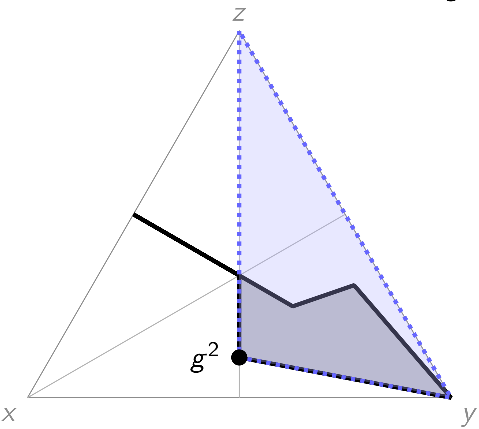
(a) *i is reachable from j in the hypergraph arising from the tangent cone at point v;*

(b) *for all $x \in \mathcal{T}(v, \mathcal{C}) \cap \{0,1\}^d$, $x_i \leqslant x_j$,*

Theorem (ibid.)

A vector y belongs to an extreme ray iff the hypergraph arising from its tangent cone has only one terminal strongly connected component.

$g^2 = (2,2,0),\ \mathcal{T}(g^2, \mathcal{C})$



$x_3 \leqslant x_1 + 2$

$\underline{x_1} \leqslant \max(\underline{x_2}, x_3)$

$\underline{x_1} \leqslant \underline{x_3 + 2}$

$x_3 \leqslant \max(x_1, x_2 - 1)$

$x_1 \leqslant x_2$

$x_1 \leqslant x_3$

- The first theorem (that the extreme rays of the tropical signed cyclic polyhedral cone correspond bijectively to the tropically allowed lattice paths) is obtained as a corollary

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- the proof uses also the tropical Cramer theorem in the signed tropical semiring (M. Plus (1990), Akian, SG, Guterman 09).

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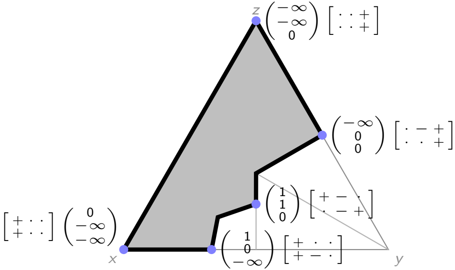
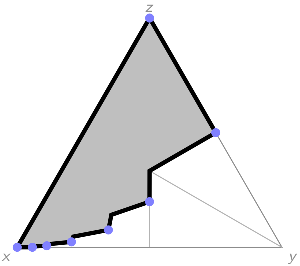
- the proof uses also the tropical Cramer theorem in the signed tropical semiring (M. Plus (1990), Akian, SG, Guterman 09).
- the tangent cones turn out to be described by "line" directed graphs, which must have a unique terminal node. This explains the mysterious condition (v)





$$\begin{pmatrix} 0 & -\infty & 0 \\ 0 & -\infty & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geqslant \begin{pmatrix} -\infty & 0 & -\infty \\ -\infty & 1 & -\infty \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} .$$

Usually, a point y in $\{x \mid Ax \leqslant b\}$ is extreme iff the family of rows A_k arising from active constraint is of full rank. The same is not true in the tropical case.

- $N^{\text{tpath}}(\epsilon)$ (resp. $N^{\text{path}}(\epsilon)$) $:= \sharp$ tropically (resp. non-tropically) allowed lattice paths for the sign pattern ϵ.
- $N^{\text{trop}}(p,d) :=$ maximal $\sharp$ extreme rays of a tropical cone in dimension d defined as the intersection of p half-spaces.

$$\max_{\epsilon\in\{\pm1\}^{p\times d}} N^{\text{tpath}}(\epsilon) \leqslant N^{\text{trop}}(p,d) \leqslant U(p+d,d-1) = \max_{\epsilon\in\{\pm1\}^{p\times d}} N^{\text{path}}(\epsilon) \enspace .$$

We initially thought that the maximum of the $\sharp$ of extreme points is attained among the polars of signed cyclic polyhedra:

$$\max_{\epsilon\in\{\pm1\}^{p\times d}} N^{\text{tpath}}(\epsilon) = N^{\text{trop}}(p,d)?$$

Not true! Finding the maximizing model for N^{trop} is an open problem.

However. . .

Fact

For $d \geqslant 2p+1$, we have

$$N^{tpath}(p,d) \geqslant U(d, d-p-1) \ . \qquad (2)$$

It follows that the tropical upper bound is asymptotically tight for a fixed number of constraints p, as the dimension tends to infinity

$$N^{\mathrm{trop}}(p,d) \sim U(p+d, d-1) \qquad \text{as } d \to \infty \ .$$

Lower and upper bounds for $N^{\text{trop}}(p,d)$, the maximal number of extreme rays of a tropical polyhedral cone defined by p inequalities in dimension d.

$d \setminus p$	1	2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10	11	12
4	6	8	10	12	14	16	18	20	22
5	9	14	20	[26, 27]	[32, 35]	[38, 44]	[44, 54]	[50, 65]	[56, 77]
6	12	20	30	42	[55, 56]	[68, 72]	[82, 90]	[96, 110]	[110, 132]
7	16	30	50	[71, 77]	[96, 112]	[124, 156]	[152, 210]	[180, 275]	[208, 352]
8	20	40	70	112	[159, 168]	[216, 240]	[280, 330]	[340, 440]	[401, 572]
9	25	55	105	[172, 182]	[250, 294]	[321, 450]	[436, 660]	[613, 935]	[751, 1287]
10	30	70	140	252	[370, 420]	[538, 660]	[668, 990]	[898, 1430]	[1320, 2002]
11	36	91	196	[363, 378]	[584, 672]	[805, 1122]	[1122, 1782]	[1357, 2717]	[1799, 4004]

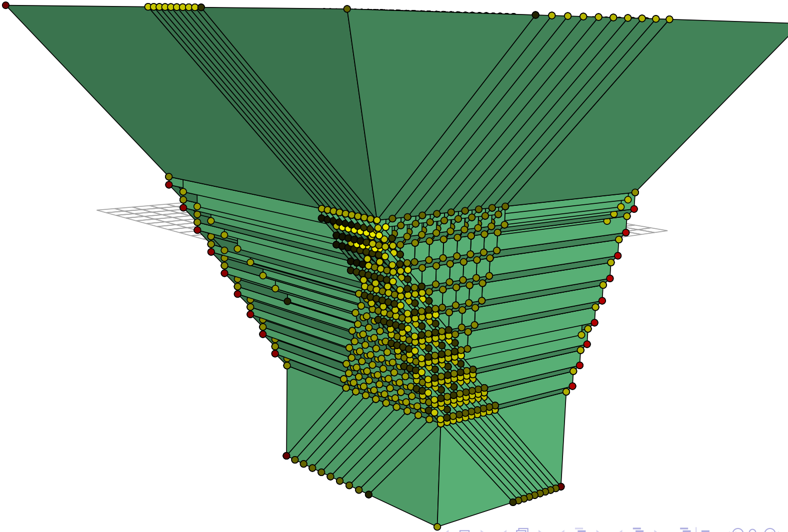
We do not know whether $N^{\text{trop}}(4,5) = 26$ or 27 (!)

When computing, use vertices, not pseudo vertices!

The $\natural$ pattern yields $(p-2d+7)(2^{d-2}-2)$ extreme rays.

$$\begin{pmatrix}
+ & + & - & - & - & - & + \\
+ & + & + & - & - & - & + \\
+ & + & + & + & - & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & - & + & + & + & + \\
+ & - & - & - & + & + & + \\
+ & - & - & - & - & + & +
\end{pmatrix}$$

For $d = 4$ and $p = 10$, 24 vertices, 1215 pseudo-vertices



The previous lattice path theorem has the following suprising corollary.

Corollary (Allamigeon, SG, Goubault, Katz LAA 11)

The tropical (unsigned) cylic polyhedral cone $C(p,d)$*, i.e., the row space of the matrix* $(t_i^{j-1})_{1\leqslant i\leqslant p, 1\leqslant j\leqslant d}$*, can be defined by a family of* $O(pd^3)$ *inequalities.,*

Compare with the classical analogue $O(p^{\lfloor (d-1)/2\rfloor})$.

Dual problem: minimal defining systems of inequalities for a polyhedron.

There is a minimal representation (unique modulo certain exchanges), Alllamigeon, Katz JCTA 13

An application of tropical convexity in infinite dimension

tropical approximation in optimal control, attenuation of the curse of dimensionality

Lagrange problem / Lax-Oleinik semigroup

$$v(t,x) = \sup_{\mathbf{x}(0)=x,\ \mathbf{x}(\cdot)} \int_0^t L(\mathbf{x}(s),\dot{\mathbf{x}}(s))ds + \phi(\mathbf{x}(t))$$

Lax-Oleinik semigroup: $(S_t)_{t\geqslant 0}$, $S_t\phi := v(t,\cdot)$.

Superposition principle: $\forall\lambda\in\mathbb{R}$, $\forall\phi,\psi$,

$$\begin{aligned} S_t(\sup(\phi,\psi)) &= \sup(S_t\phi, S_t\psi) \\ S_t(\lambda+\phi) &= \lambda + S_t\phi \end{aligned}$$

So S_t is max-plus linear.

Lagrange problem / Lax-Oleinik semigroup

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$$\begin{aligned} S_t(\text{“}\phi+\psi\text{''}) &= \text{“}S_t\phi + S_t\psi\text{''} \\ S_t(\text{“}\lambda\phi\text{''}) &= \text{“}\lambda S_t\phi\text{''} \end{aligned}$$

So S_t is max-plus linear.

The function v is solution of the Hamilton-Jacobi equation

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}) \qquad v(0, \cdot) = \phi$$

Max-plus linearity $\Leftrightarrow$ Hamiltonian convex in p

$$H(x, p) = \sup_u (L(x, u) + p \cdot u)$$

Hopf formula, when $L = L(u)$ concave:

$$v(t, x) = \sup_{y \in \mathbb{R}^n} tL(\frac{x-y}{t}) + \phi(y) \ .$$

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Hopf formula, when $L = L(u)$ concave:

$$v(t, x) = \text{“} \int G(x - y)\phi(y) dy \text{”} \quad .$$

Max-plus basis / finite-element method

Fleming, McEneaney 00-; Akian, Lakhoua, SG 04-

Approximate the value function by a "linear comb." of "basis" functions with coeffs. $\lambda_i(t) \in \mathbb{R}$:

$$v(t,\cdot) \simeq \text{"}\sum_{i\in[p]} \lambda_i(t) w_i\text{"}$$

The w_i are given finite elements, to be chosen depending on the regularity of $v(t,\cdot)$

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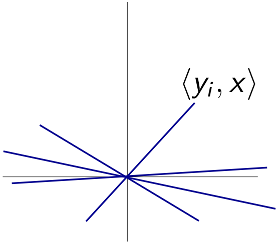
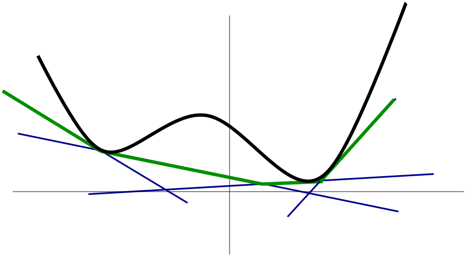
Best max-plus approximation

$$P(f) := \max\{g \leqslant f \mid g \text{ “linear comb.” of } w_i\}$$

linear forms $w_i : x \mapsto \langle y_i, x\rangle$





adapted if v is convex

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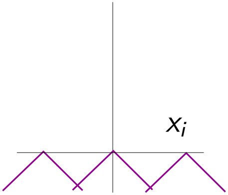
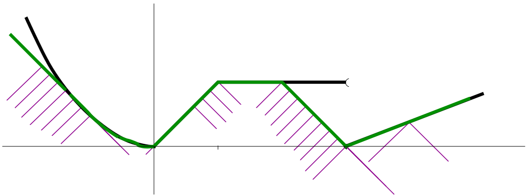
cone like functions $w_i : x \mapsto -C\|x - x_i\|$





adapted if v is C-Lip

Max-plus basis propagation

Max-plus linearity is essential in max-plus basis method:

$$V_t \simeq \tilde{V}_t = \sup_i \lambda_i^t + \mathbf{w}_i$$

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&= \sup_i \lambda_i^t + S_\tau[\mathbf{w}_i] && \text{maxplus linearity} \\
&\simeq \sup_i \lambda_i^t + \tilde{S}_\tau[\mathbf{w}_i] && \text{semigroup approximation step}
\end{aligned}$$

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&= \sup_i \lambda_i^t + S_\tau[\mathbf{w}_i] && \text{maxplus linearity}\\
&\simeq \sup_i \lambda_i^t + \tilde{S}_\tau[\mathbf{w}_i] && \text{semigroup approximation step}\\
&\simeq \sup_i \lambda_i^{t+\tau} + \mathbf{w}_i && \text{maxplus projection step}
\end{aligned}$$

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&\simeq \sup_i \lambda_i^{t+\tau} + \mathbf{w}_i && \text{maxplus projection step}
\end{aligned}$$

In summary:

$$\boxed{V_T(x) = \{S_\tau\}^N[V_0] \simeq \{\mathcal{P} \circ \tilde{S}_\tau\}^N[V_0] \ .}$$

Max-plus basis methods

Several max-plus basis methods have been proposed:

- [Fleming,McEneaney 00]:
  A first development of max-plus basis method
- [Akian,Gaubert,Lakhoua 06]:
  A finite element max-plus basis method
- [McEneaney 07]:
  A curse of dimensionality free method
- [McEneaney,Deshpande,Gaubert 08],
  [Sridharan,James,McEneaney 10], [Dower,McEneaney 11], ...

Switched optimal control problem

- Infinite horizon switched optimal control problem [McEneaney 07]:

$$V(x) = \sup_{\mu} \sup_{\mathbf{u}} \int_0^\infty \frac{1}{2}\mathbf{x}(t)' D^{\mu(t)} \mathbf{x}(t) - \frac{\gamma^2}{2} |\mathbf{u}(t)|^2 dt,$$

where

$$\begin{aligned} \mathcal{D}_\infty &\doteq \{\mu : [0,\infty) \to \{1,\ldots,M\} : \text{measurable}\} \ , \\ W &\doteq L_2^{\text{loc}}([0,\infty); \mathbb{R}^k) \ , \end{aligned}$$

and $\mathbf{x}(\cdot)$ satisfies:

$$\dot{\mathbf{x}}(t) = A^{\mu(t)}\mathbf{x}(t) + \sigma^{\mu(t)}\mathbf{u}(t), \ \ \mathbf{x}(0) = x \in \mathbb{R}^d \ ,$$

arising from H_∞ robust control, nonconvex ($D^1, \ldots, D^M \succcurlyeq 0$).

McEneaney's curse of dimensionality free method

- Semigroup approximation:

$$S_\tau \simeq \tilde{S}_\tau = \sup_m S_\tau^m$$

- S_t^m is the semigroup associated to the control problem by letting the switching control μ equal to $m \in \{1, \ldots, M\}$:

$$S_t^m[\phi](x) = \sup_{\mathbf{u}} \int_0^t \frac{1}{2}\mathbf{x}(t)' D^m \mathbf{x}(t) - \frac{\gamma^2}{2}|\mathbf{u}(t)|^2 dt + \phi(\mathbf{x}(t)).$$

$$\dot{\mathbf{x}}(s) = A^m \mathbf{x}(s) + \sigma^m \mathbf{u}(s);\ \ \mathbf{x}(0) = x \in \mathbb{R}^d \quad .$$

- $S_t^m[\phi]$ is a quadratic function if ϕ is. (Riccati)

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] =$$
$$\sup_{i_N,\ldots,i_1} S_\tau^{i_N} \circ \ldots S_\tau^{i_1}[V_0] \ .$$

Arborescent propagation

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\dots,i_1} S_\tau^{i_N} \circ \dots S_\tau^{i_1}[V_0] \enspace .$$

$$\boxed{V_0}$$

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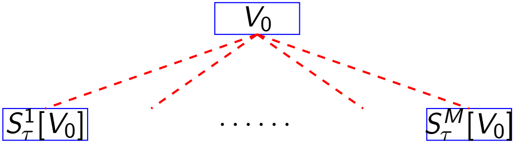

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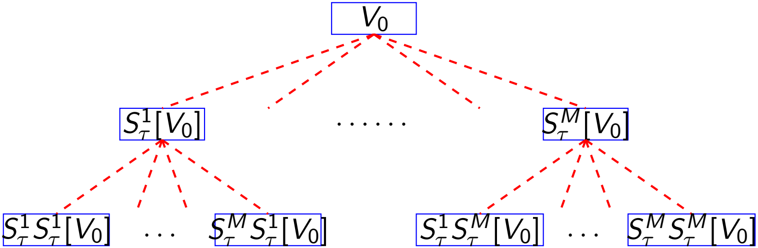

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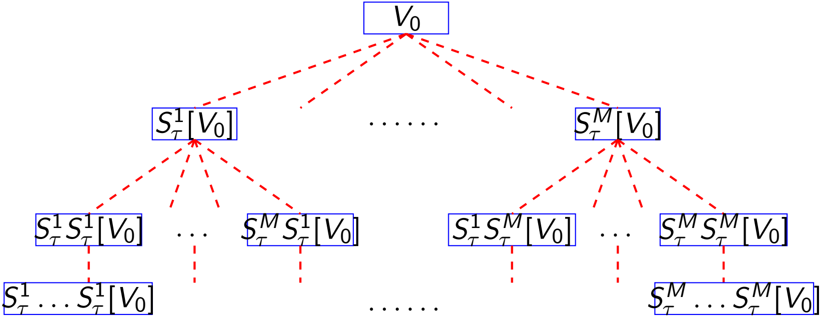

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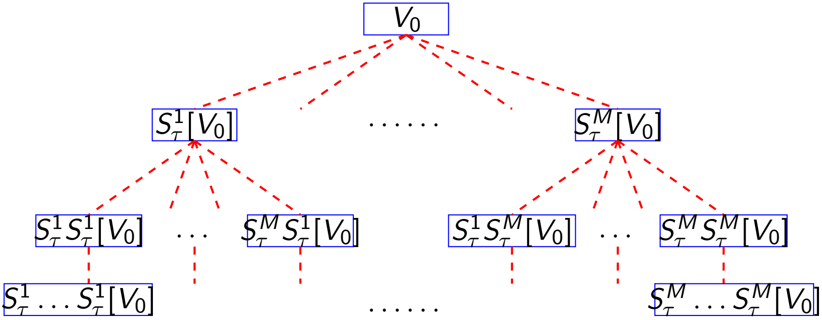


Computational complexity: $O(M^N d^3) \Rightarrow$curse of dimensionality

Tree approximation + pruning

Computational complexity: $O(M^N d^3)$

The method has been applied to solve approximately problems of

- dimension $d = 4$, number of switches $M = 3$, in [McEneaney 07]
- dimension $d = 6$, number of switches $M = 6$, in [McEneaney,Deshpande,Gaubert 08] (with a semidefinite programming pruning technique)
- dimension $d = 15$, number of switches $M = 6$, in [Sridharan,James,McEneaney 10] (quantum optimal gate synthesis, $SU(4)$)

Switched infinite horizon optimal control problem

Static HJ equation:

$$H(x, \nabla V) = 0, \ \ \forall x \in \mathbb{R}^d; V(0) = 0 \ \ .$$

where $H(x,p) = \sup_{m \in \{1,\dots,M\}} \frac{1}{2} x' D^m x + \frac{1}{2} p' \Sigma^m p + (A^m x)' p.$

Assumption (existence)

$$0 \prec D^m \preccurlyeq c_D I_d, \ \ 0 \prec \Sigma^m \preccurlyeq c_\Sigma I_d, \ \forall m$$
$$x' A^m x \leqslant -c_A |x|^2, \ \ \forall x \in \mathbb{R}^d, \ \forall m. \ \ c_A^2 > c_D c_\Sigma.$$

Assumption Σ:

$$\Sigma^m = \Sigma, \ \ m = 1, \dots, M \ \ .$$

Assumption contraction:

$$D^m \geqslant m_D I_d, \ \ m_D c_\Sigma > (c_A - \sqrt{c_A^2 - c_D c_\Sigma})^2.$$

Error bound

Theorem (Zheng Qu, PhD 2013)

Under Assumption existence and Assumption contraction, the computational complexity to reach an error of order ϵ is

$$O(M^{-\log(\epsilon)/\epsilon} d^3) \ .$$

Compare with $O(1/\epsilon^{d/r})$ for a grid scheme with an error of order $(\Delta x)^r$.

Invariant metrics on the cone of positive matrices

- Thompson's part metric:

$$d_T(A,B) = \|\operatorname{spec}(\log B^{-\frac{1}{2}} A B^{-\frac{1}{2}})\|_\infty, \quad A, B \succ 0$$

Thompson's part metric is an invariant Finsler metric:

$$d_T(UAU', UBU') = d_T(A,B) \ \ , U \in GL(n)$$

$$d_T(A,B) = \inf_\gamma \int_0^1 \|\dot\gamma(t)\gamma(t)^{-1}\|_\infty \, dt.$$

- Riemannian metric:

$$d_2(A,B) = \inf_\gamma \int_0^1 \|\dot\gamma(t)\gamma(t)^{-1}\|_2 dt$$

- Standard Riccati operator (flow) is a strict contraction mapping in Riemannian metric ([Bougerol 93]), in Thompson's part metric ([Liverani and Wojtkowski.94, Lawson and Lim 07]) and in all invariant Finsler metric ([Lee and Lim 07]). (symplectic

Main ingredient: contraction property of Riccati flow

For all $m \in \{1, \ldots, M\}$, the semigroup $\{S_t^m\}_t$ corresponds to the flow of an indefinite Riccati equation:

$$\dot{P} = (A^m)'P + PA^m + D^m + P\Sigma^m P \ . \tag{3}$$

Theorem (Indefinite Riccati flow is a strict local contraction)

Under Assumption existence and Assumption contraction, there is $P_0 \succ 0$ and $\alpha > 0$ such that for all solutions $P_1(\cdot), P_2(\cdot) : [0, T] \to (0, P_0)$ of the indefinite Riccati flow (3) we have:

$$d_T(P_1(t), P_2(t)) \leqslant e^{-\alpha t} d_T(P_1(0), P_2(0)), \ \ \forall t \in [0, T] \ .$$

Curse of dimensionality is unavoidable

Qu's error bound $O(M^{-\log(\epsilon)/\epsilon}d^3)$ shows that for fixed ϵ, execution time is polynomial in d.

However, we recover a curse of dimensionality, when $\epsilon \to 0$.

Theorem (coro of Grüber, polyhedral approximation of convex bodies)

The minimal number of affine minorant functions to approximate a C^2 convex function $f : \mathbb{R}^d \to \mathbb{R}$ is equivalent to:

$$\frac{C}{\epsilon^{d/2}} \quad , \qquad \text{as} \quad \epsilon \to 0 \quad ,$$

Current bottleneck: pruning representation. Given

$$f = \sup_{i \in [p]} \phi_i , \qquad \phi_i \text{ quadratic } \mathbb{R}^d \to \mathbb{R}$$

and $k \ll p$, find $I \subset [p]$, $|I| = k$, with a best approximation of f by

$$\sup_{i \in I} \phi_i \quad .$$

Heuristics, SDP relaxations, reduction to a discrete facility location problem (curse of dim dependent).

Coming back to the first exercise

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} \quad ,$$

Eigenvalues ? $\epsilon \to 0$

Coming back to the first exercise

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} ,$$

Eigenvalues ? $\epsilon \to 0$

$$\mathcal{L}_\varepsilon^1 \sim \varepsilon^{-1/3},\ \mathcal{L}_\varepsilon^2 \sim j\varepsilon^{-1/3},\ \mathcal{L}_\varepsilon^3 \sim j^2\varepsilon^{-1/3}.$$

Coming back to the first exercise

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} \quad ,$$

Eigenvalues ? $\epsilon \to 0$

$$\mathcal{L}^1_\varepsilon \sim \varepsilon^{-1/3},\ \mathcal{L}^2_\varepsilon \sim j\varepsilon^{-1/3},\ \mathcal{L}^3_\varepsilon \sim j^2\varepsilon^{-1/3}.$$

Answer without computation using tropical algebra.

Give $A \in \mathbb{R}_{\max}^{n\times n}$, λ is a *geometric eigenvalue* if

$$Au = \lambda u, \qquad u \in \mathbb{R}_{\max}^{n} \setminus \{\text{“}0\text{”}\}$$

λ is an *algebraic eigenvalue* if

$$\text{“}\det(A - \lambda I) = 0\text{”}$$

meaning that λ is a nondifferentiability point of value of parametric optimal assignment problem

$$t \mapsto \max_{\sigma} \sum_i M_{i,\sigma}(t),$$

M(t)= “A+ t I”, $\mathrm{M}_{ij} = A_{ij}$, $M_{ii} = \max(A_{ii}, t)$.

Theorem (Max-plus spectral theorem, Cuninghame-Green, 61, Gondran & Minoux 77, Cohen et al. 83)

Assume $G(A)$ is strongly connected. Then

- *the eigenvalue is unique:*

$$\rho_{\max}(A) := \max_{i_1,\ldots,i_k} \frac{A_{i_1 i_2} + \cdots + A_{i_k i_1}}{k}$$

- *Assume WLOG $\rho_{\max}(A) = 0$, then, $\exists \alpha_j \in \mathbb{R} \cup \{-\infty\}$,*

$$u = \max_{j \in \text{maximizing circuits}} \alpha_j + A^*_{\cdot,j}$$

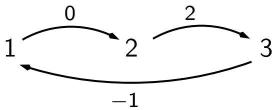
$$A^*_{ij} := \textit{max weight path arbitrary length } i \to j.$$

Arc $i \to j$ in $G(A)$ if $A_{ij} \neq -\infty$.

The first exercise solved

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & 0 & -4 \\ -\infty & -1 & 2 \\ -1 & -2 & -\infty \end{bmatrix}.$$

We have $\lambda = 1/3$, corresponding to the critical circuit:



Eigenvalues:

$$\mathcal{L}^1_\varepsilon \sim \varepsilon^{-1/3}, \mathcal{L}^2_\varepsilon \sim j\varepsilon^{-1/3}, \mathcal{L}^3_\varepsilon \sim j^2\varepsilon^{-1/3}.$$

Akian, Bapat, SG, CRAS 04, generalizes Lidski's theorem

One can show eigenspaces are isomorphic to spaces of Lipschitz functions (wrt to non symmetric metrics).

Relation with horoboundaries of metric spaces. The tropically extreme Lipschitz functions are Busemann points (limits of geodesics).

Concluding remarks

- All the convexity you like works: Helly, Carathéodory, Radon, Tverberg, Double Description, Hahn-Banach, Krein-Milman, Choquet, ...
- Some combinatorial aspects (counting extreme points and faces) are not understood.
- Complexity of trop LP $=$ complexity of mean payoff games (is it polynomial?)
- useful: metric estimates of amoebas, bounds for matrix eigenvalues, scaling in matrix analysis
- emerging max-plus curse of dimensionality attenuation for HJ equation, open question: extension to stochastic control.

Tropical problems are simpler (combinatorial)

. . . but not too simple.

Much remains to do . . .

Thank you !

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