Tropical convexity and its applications to zero-sum games

Minilecture, Part I

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Works with Akian, Allamigeon, Goubault, Guterman, Katz, Joswig, Meunier, Sergeev, Walsh; highlight: PhD of Benchimol and Qu.
In an exotic country, children are taught that:

\[ a + b = \max(a, b) \quad a \times b = a + b \]

So

\[ 2 + 3 = \]

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Tropical convexity and zero-sum games, I
Max-plus or tropical algebra

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- “2 × 3” =
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So

- \[ 2 + 3 = 3 \]
- \[ 2 \times 3 = 5 \]
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- "5/2" =
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- \[ "5/2" = 3 \]
- \[ "2^3" = \]
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\begin{align*}
\text{"a + b"} &= \max(a, b) \\
\text{"a \times b"} &= a + b
\end{align*}
\]

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- "2 \times 3" = 5
- "5/2" = 3
- "2^3" = "2 \times 2 \times 2" = 6
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- "\sqrt{-1}" =
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- “2 × 3” = 5
- “5/2” = 3
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- “\sqrt{-1}” = -0.5
The notation \( a \oplus b := \max(a, b) \), \( a \odot b := a + b \), \( 0 := -\infty \), \( 1 := 0 \) is also used in the tropical/max-plus literature.

Max-plus semiring: \( \mathbb{R}_{\text{max}} = (\mathbb{R} \cup \{-\infty\}, \max, +) \).
The sister algebra: min-plus

"a + b" = \text{min}(a, b) \quad "a \times b" = a + b

- "2 + 3" = 2
- "2 \times 3" = 5

Min-plus semiring: $\mathbb{R}_{\text{min}}$. 
Structures called idempotent

\[ a + a = a \]

or of characteristic one. Compare with

\[ (p + 1)a = a . \]
Find the roots of the max-plus polynomial 
\[1^{-1}X^3 + X^2 + 2X + 1^1\].
Nota bene: \[1^a = 1 \times a = a\] is unambiguous, compare \[1 = 0\] with \[1^1 = 1\].
Exercises (cont)

- Find the roots of the max-plus polynomial 
  \[1^{\text{st}} \cdot X^3 + X^2 + 2X + 1^{\text{st}}\].
  
  Nota bene: \(1^a = 1 \times a = a\) is unambiguous, compare \(1 = 0\) with \(1^1 = 1\).

- Answer: 
  \[
  \max(-1 + 3X, 2X, 2 + X, 1)
  = -1 + \max(X, -1) + 2 \max(X, 1.5)
  \]
Exercises (cont)

- Find the roots of the max-plus polynomial 
  \[ 1^{-1}X^3 + X^2 + 2X + 1^1 \].
  Nota bene: \( 1^a = 1 \times a = a \) is unambiguous, compare \( 1 = 0 \) with \( 1^1 = 1 \).

- Answer:
  \[ 1^{-1}X^3 + X^2 + 2X + 1^1 = 1^{-1}(X + 1^{-1})(X + 1^{3/2})^2 \]
Theorem (Cuninghame-Green & Meijer, 80)

A max-plus polynomial function \( p = "a_n X^n + \cdots + a_0" \) can be factored uniquely as

\[
p = "a_n(X + \alpha_1) \cdots (X + \alpha_n)"
\]

\[
= a_n + \max(X, \alpha_1) + \cdots + \max(X, \alpha_n)
\]

The \( \alpha_i \) are the (tropical) roots.

How to compute (tropical) roots?
Legendre-Fenchel = tropical Fourier transform

The map which sends coeffs: \((i \mapsto a_i)\) to the numerical function

\[ X \mapsto p(X) = \max_{0 \leq i \leq n} a_i + i \times X \]

is a special case of Legendre-Fenchel transform

\[ f : \mathbb{R}^n \to \mathbb{R}, \quad f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \]

\[ f^*(p) = \sup_{x \in \mathbb{R}^n} \langle p, x \rangle - f(x). \]

\[ f(i) = -a_i \text{ if } i \in \mathbb{N}, \quad f(i) = +\infty \text{ otherwise} \]

\[ f^* = g^* \text{ iff } \text{lscvex}(f) = \text{lscvex}(g) \]
Newton polygon

\[ \Delta(p) = \text{lower concave hull}\{(i, a_i) \mid 0 \leq i \leq n\}. \]

**Proposition**

Let \( p \in \mathbb{R}_{\max}[X] \) be a max-plus polynomial. Roots of \( p \) = minus slopes of \( \Delta(p) \). Multiplicity of root \( \alpha = \text{length of the interval of } \Delta(p) \text{ of slope } -\alpha. \)

\[ "p = 1^{-1}X^3 + 1^0X^2 + 1^2X + 1^1". \] The tropical roots are \(-1\) (multiplicity 1) and 1.5 (multiplicity 2).
Approximate essentially without computation the (usual) roots of

\[ p = 2^{-2} + 2^2 X - 2^5 X^4 + 2X^6 \in \mathbb{C}[X] \]
Exercises (cont.)

- Approximate essentially without computation the (usual) roots of

\[ p = 2^{-2} + 2^2 X - 2^5 X^4 + 2X^6 \in \mathbb{C}[X] \]

- Answer: \(-2^{-4}, 2^{-1}\{1, j, j^2\}, 2^2\{1, -1\}, -0.0625, -0.25-0.433i, -0.25+0.433i, 0.5, 4, -4\).

Check in Scilab:

\(-0.0624, -0.226-0.434i, -0.226+0.434i, 0.522, 4.00, -4.00\)
Approximate essentially without computation the (usual) roots of

\[ p = 2^{-2} + 2^2X - 2^5X^4 + 2X^6 \in \mathbb{C}[X] \]

Answer: \(-2^{-4}, 2^{-1}\{1, j, j^2\}, 2^2\{1, -1\}, \)

-0.0625 \(-0.25-0.433i\) \(-0.25+0.433i\) 0.5 4. - 4.

Check in Scilab:

-0.0624 \(-0.226-0.434i\) \(-0.226+0.434i\) 0.522 4.00 -4.00
Solution: Associate to $\sum_k a_k X^k \in \mathbb{C}[X]$ the max-plus polynomial

$$X \mapsto \max_k \log_2 |a_k| + kX.$$ 

$p = 2^{-2} + 2^2 X - 2^5 X^4 + 2^1 X^6$, tropical roots are $-4$ (mult. 1), $-1$ (mult. 3), $2$ (mult. 2)
Theorem (Hadamard, Ostrowski, Polyá)

Let \( p = \sum_k a_k X^k \) with roots \( \zeta_i \in \mathbb{C}, |\zeta_1| \geq \ldots \geq |\zeta_n|, \)
\( \alpha_1 \geq \ldots \geq \alpha_n \) tropical roots of \( \max_k \log |a_k| + kX. \)

\[
\frac{1}{C_n^k} \exp(\alpha_1 + \cdots + \alpha_k)
\leq |\zeta_1 \cdots \zeta_k| \leq \text{cst}_k \exp(\alpha_1 + \cdots + \alpha_k)
\]

Corollary

\[
cst''_{n,k} \exp(\alpha_k) \leq |\zeta_k| \leq \text{cst}'_{n,k} \exp(\alpha_k)
\]
Hadamard: $cst_k \leq k + 1$ (1891, memoir on Zeta function)

Ostrowski: lower bound, $cst_k \leq 2k + 1$ (1940, Graeffe method)

Polyá $cst_k < e\sqrt{k + 1}$ (reproduced by Ostrowski).

Proof = variation on Jensen formula

$$\left| a_0 \right| R^k \left| \zeta_n \cdots \zeta_{n-k+1} \right| \leq \exp\left( \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta \right), \quad \forall R > 0$$

Akian, SG, Sharify arXiv:1304.2967; more bounds, matrix extension; decomposition results in Bini, Noferini, Sharify arXiv:1206.3632
One more exercise

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix},$$

Eigenvalues ? $\varepsilon \to 0$
One more exercise

\[ A_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} , \]

Eigenvalues? $\varepsilon \to 0$

\[ L_\varepsilon^1 \sim \varepsilon^{-1/3}, \quad L_\varepsilon^2 \sim j\varepsilon^{-1/3}, \quad L_\varepsilon^3 \sim j^2\varepsilon^{-1/3}. \]
One more exercise

\[ A_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix}, \]

Eigenvalues? \( \varepsilon \to 0 \)

\[ L_1^\varepsilon \sim \varepsilon^{-1/3}, \ L_2^\varepsilon \sim j\varepsilon^{-1/3}, \ L_3^\varepsilon \sim j^2\varepsilon^{-1/3}. \]

Answer without computation using tropical algebra, solution later in this lecture.
A partial history of max-plus / tropical algebra
In the late 80’s in France, the term “algèbres exotiques” was used
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The term “exotic” appeared also in the User’s guide of viscosity solutions of Crandall, Ishii, Lions (Bull. AMS, 92)
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of its properties are given there. See also [104]. Its “magical properties” can be seen as related to the Lax formula for the solution of

$$\frac{\partial w}{\partial t} - \frac{1}{2} |\nabla w|^2 = 0 \quad \text{for } x \in \mathbb{R}^N, \ t \geq 0, \ w|_{t=0} = v \text{ on } \mathbb{R}^N,$$

which is

$$w(x, t) = \sup_y \left\{ v(y) - \frac{1}{2t} |x - y|^2 \right\}.$$ 

Indeed, the coincidence of this solution formula and solutions produced by the method of characteristics leads to the properties used. Of course, this is a heuristic connection, since characteristic methods require too much regularity to be rigorous here.

The inf convolution can also be seen as a nonlinear analogue of the standard mollification when replacing the “linear structure of $L^2$ and its duality” by the “nonlinear structure of $L^\infty$ or $C$.” One can also interpret this analogy in terms of the so-called exotic algebra ($\mathbb{R}$, max, +).
The term “tropical” is in the honor of Imre Simon, 1943 - 2009

who lived in Sao Paulo (south tropic).
These algebras were invented by various schools in the world
• Cuninghame-Green 1960- OR (scheduling, optimization)
• Vorobyev ∼65 . . . Zimmerman, Butkovic; Optimization
• Maslov ∼ 80’- . . . Kolokoltsov, Litvinov, Samborskii, Shpiz . . . Quasi-classic analysis, variations calculus
• Simon ∼ 78- . . . Hashiguchi, Leung, Pin, Krob, . . . Automata theory
• Gondran, Minoux ∼ 77 Operations research
• Cohen, Quadrat, Viot ∼ 83- . . . Olsder, Baccelli, S.G., Akian discrete event systems, optimal control, idempotent probabilities, linear algebra
• Nussbaum 86- Nonlinear analysis, dynamical systems, also related work in linear algebra, Friedland 88, Bapat ∼94
• Kim, Roush 84 Incline algebras
• Fleming, McEneaney ∼00- max-plus approximation of HJB
• Del Moral ∼95 Puhalskii ∼99, idempotent probabilities.
Since 2000’ in pure maths, tropical geometry: Viro, Mikhalkin, Passare, Sturmfels . . . , recent work by Connes, Consani
Menu: applied tropical geometry, connections between...

- tropical convexity
- dynamic programming / zero-sum games
- Perron-Frobenius theory
- metric geometry
Some elementary tropical geometry

A **tropical line** in the plane is the set of \((x, y)\) such that the \(\max\) in

\[ ax + by + c \]

is attained at least twice.

\[ \max(x, y, 0) \]
Some elementary tropical geometry

A tropical line in the plane is the set of \((x, y)\) such that the max in

\[
\max(a + x, b + y, c)
\]
is attained at least twice.
Two generic tropical lines meet at a unique point
By two generic points passes a unique tropical line.
non generic case
non generic case resolved by perturbation
Tropical segments:

\[[f, g] := \left\{ \lambda f + \mu g \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\}, \ \lambda + \mu = 1 \right\} \].

(The condition “\(\lambda, \mu \geq 0\)” is automatic.)
Tropical segments:

$$[f, g] := \{ \sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\}, \max(\lambda, \mu) = 0 \}.$$ 

(The condition $\lambda, \mu \geq -\infty$ is automatic.)
Tropical convex set: \( f, g \in C \implies [f, g] \in C \)
Tropical convex set: \( f, g \in C \implies [f, g] \in C \)

Tropical convex cone: omit "\( \lambda + \mu = 1 \)" , i.e., replace \([f, g]\) by \( \{ \sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\} \} \)
Homogeneization

A convex set $C$ in $\mathbb{R}^n_{\text{max}}$ is a cross section of a convex cone $\hat{C}$ in $\mathbb{R}^{n+1}_{\text{max}},$

$$\hat{C} := \{ (\lambda + u, \lambda) \mid u \in C, \lambda \in \mathbb{R}_{\text{max}} \}$$
A tropical polytope with four vertices

Structure of the polyhedral complex: Develin, Sturmfels
The previous drawing was generated by POLYMAKE of Gawrilow and Joswig, in which an extension allows one to handle easily tropical polyhedra. They were drawn with JAVAVIEW.

See Joswig arXiv:0809.4694 for more information.

Tropical polyhedra handled by ocaml TPLib, Allamigeon
Motivation ?
The tropical point of view arises with log glasses.
Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety \( V \subset (\mathbb{C}^*)^n \) to be the “log-log plot”

\[
A(V) := \{ (\log |z_1|, \ldots, \log |z_n|) \mid (z_1, \ldots, z_n) \in V \}.
\]

\[y + x + 1 = 0\]

\[\max(\log |x|, \log |y|, 0)\]

attained twice
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\]

\[ y + x + 1 = 0 \]
\[ \max(\log |x|, \log |y|, 0) \]
\[ \text{attained twice} \]
\[ |y| \leq |x| + 1, \quad |x| \leq |y| + 1, \quad 1 \leq |x| + |y| \]
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$$A(V) := \{(\log |z_1|, \ldots, \log |z_n|) \mid (z_1, \ldots, z_n) \in V\} .$$

$y + x + 1 = 0$

$$\max(\log |x|, \log |y|, 0)$$

attained twice

$X := \log |x|, \ Y := \log |y|$

$Y \leq \log(e^X + 1), \ X \leq \log(e^Y + 1), \ 0 \leq \log(e^X + e^Y)$
Viro’s log-glasses, related to Maslov’s dequantization

\[ a + h b := h \log \left( e^{a/h} + e^{b/h} \right), \quad h \to 0^+ \]

With \( h \)-log glasses, the amoeba of the line retracts to the tropical line as \( h \to 0^+ \)

\[ y + x + 1 = 0 \quad \text{max}(\log |x|, \log |y|, 0) \]

\[ \max(a, b) \leq a + h b \leq h \log 2 + \max(a, b) \]
Tropical convex sets are deformations of classical convex sets

\[ [a, b] := \{ \lambda a +_p \mu b, \lambda, \mu \geq 0, \lambda +_p \mu = 1 \} \]

\[ a +_p b = (a^p + b^p)^{1/p} \]
See Passare & Rullgard, Duke Math. 04

Introduction to amoebas: lecture notes by Alain Yger.

Nonarchimedean valuation point of view

Alternatively [Sturmfels’ point of view], the tropical line “max($X, Y, 0$) attained twice” can be seen as the image by the valuation of the line $x + y + 1$ over the field of complex Puiseux series, $\mathbb{C}\{\{t\}\}$, equipped with the valuation $\text{val} s = \text{smallest exponent of } s$.

E.g., $\text{val}(t^{-1/2} - t + 7t^{3/2} + \ldots) = 1/2$

$\text{val}(z_1 + z_2) \leq \max(\text{val}(z_1), \text{val}(z_2))$, with equality when $\text{val}(z_1) \neq \text{val}(z_2)$.

$\text{val}(z_1z_2) = \text{val}(z_1) + \text{val}(z_2)$
tropical hyperplanes (complex version)

Given $a \in \mathbb{R}^n_{\text{max}}$, $a \neq -\infty$,

$$H := \{ x \in \mathbb{R}^n_{\text{max}} \mid \text{“}ax = 0\text{”} \}$$
tropical hyperplanes (complex version)

Given $a \in \mathbb{R}^n_{\text{max}}, a \neq -\infty$, 

$$H := \{ x \in \mathbb{R}^n_{\text{max}} \mid \max_{1 \leq i \leq n} a_i + x_i \text{ attained twice} \}$$
The rational points of tropical hyperplanes are images of hyperplanes of $(\mathbb{C}\{\{t\}\})^n$ by the valuation, see:

**Theorem (Kapranov)**

Given $p = \sum_{\alpha} p_{\alpha} z^\alpha \in \mathbb{C}\{\{t\}\}[z_1, \ldots, z_n]$, and $Z \in \mathbb{Q}^n$,

$$\exists z \in (\mathbb{C}\{\{t\}\})^n, \quad p(z) = 0, \quad Z = \text{val } z$$

iff

$$\max_{\alpha} \text{val } p_{\alpha} + \langle \alpha, Z \rangle \text{ attained twice}$$

Restriction to $\mathbb{Q}$ can be avoided by working with Puiseux series with real exponents (Markwig) or Hahn series (well ordered support), cvg issues: van den Dries $\mathbb{R}_{an^*}$ o-minimal model.
real tropical hyperplanes

Given \( a, b \in \mathbb{R}^n_{\text{max}} \), \( a, b \not\equiv -\infty \), \( a_i = -\infty \) or \( b_i = -\infty \), \( \forall i \),

\[
H := \{ x \in \mathbb{R}^n_{\text{max}} \mid "ax = bx" \}
\]
real tropical hyperplanes

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$, $a_i = -\infty$ or $b_i = -\infty$, \forall i,

$$H := \{ x \in \mathbb{R}^n_{\max} \mid \max_{1 \leq i \leq n} a_i + x_i = \max_{1 \leq i \leq n} b_i + x_i \}$$
real tropical hyperplanes

Given \( a, b \in \mathbb{R}_\text{max}^n \), \( a, b \neq -\infty \), \( a_i = -\infty \) or \( b_i = -\infty \), \( \forall i \),

\[
H := \{ x \in \mathbb{R}_\text{max}^n \mid \max_{1 \leq i \leq n} a_i + x_i = \max_{1 \leq i \leq n} b_i + x_i \}
\]
real tropical hyperplanes

Given \( a, b \in \mathbb{R}_\text{max}^n, a, b \not\equiv -\infty, a_i = -\infty \) or \( b_i = -\infty \), \( \forall i \),

\[
H := \{ x \in \mathbb{R}_\text{max}^n \mid \max_{1 \leq i \leq n} a_i + x_i = \max_{1 \leq i \leq n} b_i + x_i \}
\]

\[-2 + x_3 = \max(x_1, x_2)\]
Real tropical hyperplanes are images of hyperplanes of \( \mathbb{R}\{t\} \) by the valuation.
Given any \( n \) points in \( \mathbb{R}^n_{\max} \) in general position, there is a unique (complex) affine tropical hyperplane passing through them. Richter-Gebert, Sturmfels, Theobalt, 05,

Given any \( n \) points in \( \mathbb{R}^n_{\max} \) in general position, there is a unique real affine tropical hyperplane passing through them. Max Plus, 90, see also Akian, SG, Guterman 09

Let these points be given by the columns of a \((n + 1) \times n\) matrix \( M \), in projective coordinates. The vector \( a \) such that \( H = \{ x \mid "a \cdot x = 0" \} \) contains the points is solution of "\( aM = 0 \)". Hence, \( a_i = "(-1)^i D_i" \), where \( D_i \) is the \( i \)th Cramer determinant (delete row \( i \) of \( M \)).
How are Cramer det defined?

For the complex (RGST 05) version, ignore sign

\[ \det A = \sum_{\sigma} \text{sgn} \prod_i A_{i\sigma(i)} = \max_{\sigma} \sum_i A_{i\sigma(i)} \]

This is an optimal assignment problem

For the real (Max Plus 90) version, the signs of the maximising permutations tells on which side of the equality “\( ax = bx \)” the coefficients should be put.

**general position:** only one opt assignment.
Proofs

- Extensions of tropical semiring, Maxplus 90, Akian, SG, Guterman, 09, 13, Izhakian, Rowen 09, related formalism: Krasner hyperfield 57 (different axioms but comparable expressivity)

- Coherent matching fields Richter-Gebert, Sturmfels, Theobalt, 05, building on Sturmfels and Zelevinsky, The Newton polytope of the product of maximal minors of a \((n + 1) \times n\) matrix is a transportation polytope.
The symmetrized tropical semiring

Recall that $\mathbb{Z} = \mathbb{N}^2/\triangledown$, where $(a', a'') \triangledown (b', b'')$ if $a' + b'' = a'' + b'$, $-(a', a'') = (a'', a')$. Replace $\mathbb{N}$ by $\mathbb{R}_{\max}$, $S_{\max} = \mathbb{R}_{\max}^2/\sim$

$a \sim b$ if $a = b$ or ($a \triangledown b$ and $a, b \not\triangledown 0$).

$S_{\max} = \mathbb{R}_{\max} \cup \ominus \mathbb{R}_{\max} \cup \mathbb{R}_{\max}^\bullet$; $u = (u, 0)$, $\ominus u = (0, u)$, $u^\bullet := u \ominus u = (u, u)$ with $u \in \mathbb{R}_{\max}$.

$S_{\max}^\lor := \mathbb{R}_{\max} \cup \ominus \mathbb{R}_{\max}$: signed elements.

E.g., $2 \oplus (\ominus 3) = \ominus 3$, but $3 \ominus 3 = 3^\bullet$. Think of $u$ as $\Theta(t^u)$, $u^\bullet = O(t^u)$, $\Theta(t^2) - \Theta(t^3) = -\Theta(t^3)$ but $\Theta(t^3) - \Theta(t^3) = O(t^3)$.
\[ x_1 = \max(2 + x_2, 7 + x_3) \iff x_1 \ominus 2x_2 \ominus 7x_3 \nabla 0 \]

Need to solve \( aM \nabla 0 \), where \( M \) is of \((n + 1) \times n\), \( a \in S_{\text{max}}^{n+1} \).

**Theorem (Transfer theorem)**

*Any polynomial identity valid in rings is valid in the extensions of semirings.*

Akian, SG, Guterman 09, using an idea of Reutenauer and Straubing 86.

E.g. \( P_A(A) = 0 \) becomes \( P^+_A(A) = P^-_A(A) \), or \( P_A(A) \nabla 0 \), where \( P_A \) characteristic polynomial.
Lemma (Elimination)

\[ x \triangleright b, \ cx \triangleright d, \ x \in S_{\max}^\triangleright \ \text{implies} \ cb \triangleright d. \]

Cramer theorem is proved by Gaussian elimination.

Izhakian introduced the bi-valued tropical semiring, \( 2 \oplus 2 = 2^\bullet \), to remind that \( \text{max} \) is attained twice. The “complex” tropical Cramer theorem is proved along the same lines.
Tropical half-spaces

Given \( a, b \in \mathbb{R}_{\text{max}}^n \), \( a, b \not\equiv -\infty \),

\[
H := \{ x \in \mathbb{R}_{\text{max}}^n \mid "ax \leq bx" \}
\]
Tropical half-spaces

Given \( a, b \in \mathbb{R}^n_{\text{max}}, a, b \not\equiv -\infty \),

\[
H := \{ x \in \mathbb{R}^n_{\text{max}} \mid \max_{1 \leq i \leq n} a_i + x_i \leq \max_{1 \leq i \leq n} b_i + x_i \}
\]
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Given $a, b \in \mathbb{R}_\text{max}^n$, $a, b \not\equiv -\infty$, 

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\]
Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}, a, b \not\equiv -\infty$, \[ H := \{ x \in \mathbb{R}^n_{\max} \mid \max_{1 \leq i \leq n} a_i + x_i \leq \max_{1 \leq i \leq n} b_i + x_i \} \]
Tropical sesquilinear form and Hilbert’s metric

\[
\begin{align*}
\frac{x}{v} &:= \max\{\lambda \mid \lambda v \leq x\} \\
&= \min_i (x_i - v_i) \quad \text{if } x, v \in \mathbb{R}^n.
\end{align*}
\]

\[
\delta(x, y) = \left(\frac{x}{y}\right) \left(\frac{y}{x}\right) = \min_i (x_i - y_i) + \min_j (y_j - x_i)
\]

\(d = -\delta\) is the (additive) Hilbert’s projective metric

\[
d(x, y) = \|x - y\|_H, \quad \|z\|_H := \max_{1 \leq i \leq d} z_i - \min_{1 \leq i \leq d} z_i.
\]
Hilbert’s metric on an open convex set

\[ d_H(a, b) = \log \frac{|b - \bar{a}||a - \bar{b}|}{|a - \bar{a}||b - \bar{b}|}. \]

**disc**: Klein model of the hyperbolic space;

**simplex**: \( d_H \) conjugate to the additive (tropical) Hilbert metric (take \( \exp: \mathbb{R}^n \to \mathbb{R}_+^n \) and a cross section of \( \mathbb{R}_+^n \)).
A ball in Hilbert’s metric is classically and tropically convex.
Projection on a tropical cone

If the tropical convex cone \( C \subset \mathbb{R}_{\text{max}}^n \) generated by \( U \) is stable by arbitrary sups (closed in Scott topology -non-Haussdorf-):

\[
P_C(x) = \max\{ v \in C \mid v \leq x \} \\
= \max_{u \in U} (x/u) + u .
\]

Similar to

\[
P_C(x) = \sum_{u \in U} \langle x, u \rangle u
\]

\( C = \text{Col}(A) \), \( [P_C(x)]_i = \max_{k \in [p]} \min_{j \in [n]} (A_{ik} - A_{jk} + x_j), \) \( i \in [n] \)

Cuninghame-Green; Gondran, Minoux; Cohen, SG, Quadrat; Ardila; Joswig; Sturmfels, Yu
Prop. (Cohen, SG, Quadrat, in Bensoussan Festschrift 01)

\[ d(x, P_V(x)) = \min_{y \in V} d(x, y) . \]
Separation

Goes back to Zimmermann 77, simple geometric construction in Cohen, SG, Quadrat in Ben01, LAA04. $C$ closed linear cone of $\mathbb{R}^d_{\text{max}}$, or complete semimodule.

If $y \not\in C$, then, the tropical half-space

$$\mathcal{H} := \{v \mid y/v \leq P_C(y)/v\}$$

contains $C$ and not $y$.

Compare with the optimality condition for the projection on a convex cone $C$: $\langle y - P_C(y), v \rangle \leq 0, \forall v \in C$
Corollary (Zimmermann; Samborski, Shpiz; Cohen, SG, Quadrat, Singer; Develin, Sturmfels; Joswig...)

A tropical convex cone closed (in the Euclidean topology) is the intersection of tropical half-spaces.

\( \mathbb{R}_{\text{max}} \) is equipped with the topology of the metric 
\( (x, y) \mapsto \max_i |e^{x_i} - e^{y_i}| \) inherited from the Euclidean topology by log-glasses.

The apex \(-P_C(y)\) of the algebraic separating half-space \( \mathcal{H} \) above may have some \(+\infty\) coordinates, and therefore may not be closed in the Euclidean topology (always Scott closed). The proof needs a perturbation argument, this is where the assumption that \( C \) is closed (and not only stable by arbitrary sups = Scott closed) is needed.
Separation of several convex sets / cyclic projections SG, Sergeev, Fund. i priklad. mat. 07

If $V_1 \cap \cdots \cap V_k = \{ "0" \}$, we can find half-spaces $H_i$ such that $H_i \supset V_i$ and $H_1 \cap \cdots \cap H_k = \{ "0" \}$. The apices of these half-spaces are obtained from an eigenvector $u$ of the cyclic projector

"$P_{V_1} \cdots P_{V_k}(u) = \lambda u$"
The existence of such an eigenvector is obtained by a technique from nonlinear analysis, case of \(-\infty\) entries dealt with by perturbation (Collatz-Wielandt theorem), more on this next lecture.
Theorem (tropical Helly theorem, Briec and Horvath, 04)

If a finite collection of cones of $\mathbb{R}^d_{\max}$ has a “zero” intersection, then a subfamily of at most $d$ of them also has a “zero” intersection.

- Proved by “dequantization” from classical Helly (passing to the limit).
- Alt proof by SG and Sergeev 07 by cyclic projection (at each projection one coordinate decreases)
Tropical Radon

In SG and Meunier, DCG09, Helly deduced from the tropical Radon’s theorem: a subset of $d + 1$ vectors in dimension $d$ can be partitioned in two subsets generating cones with a “non-zero” intersection. Radon theorem follows from tropical Cramer theory (signs provide partition).
More advanced results of tropical convex geometry, SG and Meunier, DCG09

Barany’s Colorful Caratheodory Theorem
More advanced results of tropical convex geometry, SG and Meunier, DCG09

Barany’s Colorful Caratheodory Theorem . . . Tropical
Tropical Tverberg’s theorem, SG and Meunier, DCG09.
Let $X$ be a set of $(d + 1)(q - 1) + 1$ points in $\mathbb{R}^d_{\max}$.
Then there are $q$ pairwise disjoint subsets $X_1, X_2, \ldots, X_q$ of $X$ whose tropical convex hulls have a common point.

Deduced from the classical one by a limit argument, no direct combinatorial proof known.
Tropical Tverberg
Dutch cheese conjecture

Let $q \geq 2$, $d \geq 1$. Sierksma conjectured that for every $(d + 1)(q - 1) + 1$ points in $\mathbb{R}^d$ the number of unordered Tverberg partitions is at least $((q - 1)!)^d$.

SG and Meunier, DCG09: True in the tropical setting!
(development of a “bipartite analogue” of Tverberg’s theorem due to Lindström, 1970 and Tverberg, 71)

Unfortunately, it is not clear whether this can be transferred to the classical case. In other words, there may be no “Mikhalkin’s correspondence theorem” in the case of inequalities (?)
Menu of the next lectures

- Tropical linear programming, classical linear programming, and mean payoff games
- Non-linear Perron-Frobenius theory
- Infinite dimensional tropical convex sets,
- Metric geometry / boundaries
- max-plus approximation, curse of dim reduction in optimal control
Tropical convexity and its applications to zero-sum games

Minilecture, Part II

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INRIA and CMAP, École Polytechnique

JGA, Marseille
December 16-20, 2013

Works with Akian, Allamigeon, Goubault, Guterman, Katz, Joswig, Meunier, Sergeev, Walsh; highlight: PhD of Benchimol and Qu.
Equivalence between tropical linear programming and mean payoff games
Tropical half-spaces

Given $a, b \in \mathbb{R}_\text{max}^n$, $a, b \not\equiv -\infty$,

$$H := \{ x \in \mathbb{R}_\text{max}^n \mid \text{“}ax \leq bx\text{”} \}$$
Tropical half-spaces

Given \( a, b \in \mathbb{R}^n_{\text{max}}, a, b \not\equiv -\infty \),

\[
H := \{ x \in \mathbb{R}^n_{\text{max}} \mid \max_{1 \leq i \leq n} a_i + x_i \leq \max_{1 \leq i \leq n} b_i + x_i \}
\]
Tropical half-spaces

Given $a, b \in \mathbb{R}_\text{max}^n$, $a, b \not\equiv -\infty$, 

$$H := \left\{ x \in \mathbb{R}_\text{max}^n \mid \max_{1 \leq i \leq n} a_i + x_i \leq \max_{1 \leq i \leq n} b_i + x_i \right\}$$
Tropical half-spaces

Given \( a, b \in \mathbb{R}^{n}_{\text{max}} \), \( a, b \not\equiv -\infty \),

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H := \left\{ x \in \mathbb{R}^{n}_{\text{max}} \mid \max_{1 \leq i \leq n} a_i + x_i \leq \max_{1 \leq i \leq n} b_i + x_i \right\}
\]
Tropical half-spaces

Given $a, b \in \mathbb{R}_\text{max}^n$, $a, b \not\equiv -\infty$,

$$H := \{ x \in \mathbb{R}_\text{max}^n \mid \max_{1 \leq i \leq n} a_i + x_i \leq \max_{1 \leq i \leq n} b_i + x_i \}$$
A halfspace can always be written as:

$$\max_{i \in I^-} a_i + x_i \leq \max_{j \in I^+} b_j + x_j, \quad I^- \cap I^+ = \emptyset.$$ 

Apex: $v_i := -\max(a_i, b_i)$.

If $\nu \in \mathbb{R}^n$, $H$ is the union of sectors of the tropical hyperplane with apex $\nu$:

$$\max_{1 \leq i \leq n} x_i - v_i \quad \text{attained twice}$$

Halfspaces appeared in: Joswig 04; Cohen, Quadrat SG 00; Zimmermann 77, . . .
Tropical polyhedral cones can be defined as intersections of finitely many half-spaces.
Tropical polyhedral cones can be defined as intersections of finitely many half-spaces.
Tropical polyhedral cones can be defined as intersections of finitely many half-spaces.
Feasibility in tropical LP

A tropical polyhedral cone is defined as

\[ C = \{ x \in \mathbb{R}^{n}_{\max} \mid "Ax \leq Bx" \} , \quad A, B \in \mathbb{R}^{m \times n}_{\max} \]

\[ \max A_{ij} + x_j \leq \max B_{ij} + x_j, \quad \forall i \in [m] \]

A tropical polyhedron is defined as

\[ P = \{ x \in \mathbb{R}^{n}_{\max} \mid "Ax + c \leq Bx + d" \} \]

where \( A, B \in \mathbb{R}^{m \times n}_{\max}, c, d \in \mathbb{R}^{m}_{\max} \).

Questions:

- is \( C \) reduced to “0”? \( "0" = (-\infty, \ldots, -\infty)^{\top} \)
- does \( C \) contain a finite vector?
- is \( P \) non-empty?
Example: mean payoff (deterministic) games

\[ G = (V, E) \text{ bipartite graph. } r_{ij} \text{ price of the arc } (i, j) \in E. \]

“Max” and “Min” move a token. The player receives the amount of the arc.
\( v^k_i \) value of MAX, initial state \((i, MIN)\).

\[
\begin{align*}
 v^k_1 &= \min(-2 + 1 + v^{k-1}_1, -8 + \max(-3 + v^{k-1}_1, -12 + v^{k-1}_2)) \\
 v^k_2 &= 0 + \max(-9 + v^{k-1}_1, 5 + v^{k-1}_2)
\end{align*}
\]
$\nu_i^k$ value of MAX, initial state $(i, MIN)$.

$$\nu_1^k = \min(-2 + 1 + \nu_1^{k-1}, -8 + \max(-3 + \nu_1^{k-1}, -12 + \nu_2^{k-1}))$$

$$\nu_2^k = 0 + \max(-9 + \nu_1^{k-1}, 5 + \nu_2^{k-1})$$

$$\lim_k \nu^k / k = (-1, 5)$$
Max and Min flip a coin to decide who makes the move. Min always pays.
Solving the game

\[ v_i^k := \text{value of the } k\text{-horizon game starting from node } i. \]
Solving the game

- \( v^k_i \) := value of the \( k \)-horizon game starting from node \( i \).
- Value is defined as the mean reward of Max, assuming both players play optimally.
Solving the game

- \( v^k_i \) := value of the \( k \)-horizon game starting from node \( i \).
- value is defined as the mean reward of Max, assuming both players play optimally
- \( v^k = (v^k_i) \in \mathbb{R}^n \)
Solving the game

- \( v^k_i := \text{value} \) of the \( k \)-horizon game starting from node \( i \).
- Value is defined as the mean reward of Max, assuming both players play optimally.
- \( v^k = (v^k_i) \in \mathbb{R}^n \)
- \( v^0 = 0 \)
Solving the game

- $v_i^k := \text{value of the } k\text{-horizon game starting from node } i$.
- value is defined as the mean reward of Max, assuming both players play optimally
- $v^k = (v_i^k) \in \mathbb{R}^n$
- $v^0 = 0$
- $v^{k+1} = T(v^k)$

where $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the Shapley operator.
Stephane Gaubert (INRIA and CMAP)

Tropical convexity and zero-sum games, II

CIRM 11 / 40
\[ v_i^{k+1} = \frac{1}{2} \left( \max_{j: i \to j} (c_{ij} + v_j^k) + \min_{j: i \to j} (c_{ij} + v_j^k) \right). \]
Shapley operators

\[ X = \mathcal{C}(K), \text{ even } X = \mathbb{R}^n, \ K = [n]; \text{ Shapley operator } T, \]

\[ T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \left( r_{i}^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \right), \quad i \in [n] \]

- \([n] := \{1, \ldots, n\}\) set of states
- \(a\) action of Player I, \(b\) action of Player II
- \(r_{i}^{ab}\) payment of Player II to Player I
- \(P_{ij}^{ab}\) transition probability \(i \rightarrow j\)
- Nested max/min/mean can be reduced to the above.
Shapley operators

$X = \mathcal{C}(K)$, even $X = \mathbb{R}^n$, $K = [n]$; Shapley operator $T$,

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \left( r_{i}^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \right), \quad i \in [n]$$

$T$ is order preserving, additively homogeneous $\Rightarrow$ sup-norm nonexpansive:

$$x \preceq y \implies T(x) \preceq T(y)$$

$$T(\alpha + x) = \alpha + T(x), \quad \forall \alpha \in \mathbb{R}$$

$$\|T(x) - T(y)\| \leq \|x - y\|$$
Shapley operators

\[ X = \mathcal{C}(K), \text{ even } X = \mathbb{R}^n, K = [n]; \text{ Shapley operator } T, \]

\[ T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \left( r_i^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \right), \quad i \in [n] \]

Conversely, any order preserving additively homogeneous operator is a Shapley operator (Kolokoltsov), even with degenerate transition probabilities (deterministic)

Gunawardena, Sparrow; Singer, Rubinov,

\[ T_i(x) = \sup_{y \in \mathbb{R}} \left( T_i(y) + \min_{1 \leq i \leq n} (x_i - y_i) \right) \]
Repeated games

The value of the game in horizon $k$ starting from state $i$ is $(T^k(0))_i$.

We are interested in the long term payment per time unit

$$\chi(T) := \lim_{k \to \infty} \frac{T^k(0)}{k}$$
The mean payoff vector

\[ \chi(T) := \lim_{k \to \infty} \frac{T^k(0)}{k} \]
The mean payoff vector

\[ \chi(T) := \lim_{k \to \infty} T^k(0)/k \]

\[ \chi_i(T) = \text{mean payoff per turn if initial state is } i \]
The mean payoff vector

\[ \chi(T) := \lim_{k \to \infty} T^k(0)/k \]

\[ \chi_i(T) = \text{mean payoff per turn if initial state is } i \]

\[ \chi(T) = \lim_{k \to \infty} T^k(x)/k, \quad \forall x \in \mathbb{R}^n \]

\[ \text{for } \| T^k(x) - T^k(0) \| \leq \| x - 0 \| = \| x \| \]
The mean payoff vector

\[ \chi(T) := \lim_{k \to \infty} \frac{T^k(0)}{k} \]

\[ \chi_i(T) = \text{mean payoff per turn if initial state is } i \]

\[ \chi(T) = \lim_{k \to \infty} \frac{T^k(x)}{k}, \quad \forall x \in \mathbb{R}^n \]

for \( \| T^k(x) - T^k(0) \| \leq \| x - 0 \| = \| x \| \)

Think of \( x_i \) has a terminal bounty paid by Min to Max if the game ends in state \( i \).
\[
\begin{bmatrix}
5 \\
0 \\
4
\end{bmatrix}
\]
\[
\begin{align*}
v_1 &= \frac{1}{2} (\max(2 + v_1, 3 + v_2, -1 + v_3) + \min(2 + v_1, 3 + v_2, -1 + v_3)) \\
v_2 &= \frac{1}{2} (\max(-1 + v_1, 2 + v_2, -8 + v_3) + \min(-1 + v_1, 2 + v_2, -8 + v_3)) \\
v_3 &= \frac{1}{2} (\max(2 + v_1, 1 + v_2) + \min(2 + v_1, 1 + v_2))
\end{align*}
\]

this game is fair
Optimality certificates

More generally, for $u \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

\[
T(u) \geq u \implies \chi(T) \geq 0 \quad \text{superfair}
\]

\[
T(u) \leq u \implies \chi(T) \leq 0 \quad \text{subfair}
\]

\[
T(u) = \lambda + u \implies \chi(T) = (\lambda, \ldots, \lambda)
\]

Does $\chi(T) = \lim_k T^k(0)/k$ exist? Do such certificates exist?

$\chi_i = \chi_j$ is related to ergodicity. May not hold for deterministic games, what are the certificates then?
\( \chi(T) = \lim_{k} T^{k}(0)/k \) may not exist if the action spaces are infinite (Kohlberg, Neyman). Counter example in dimension 3.

However. Let \( v_{\alpha} \) denote the discounted value

\[
v_{\alpha} = T(\alpha v_{\alpha}), \quad 0 < \alpha < 1.
\]

**Theorem** (Neyman 04 - book edited with Sorin)

If \( \alpha \mapsto (1 - \alpha)v_{\alpha} \) has bounded variation as \( \alpha \to 1 \), then

\[
\lim_{k} T^{k}(0)/k = \lim_{\alpha \to 1^{-}} (1 - \alpha)v_{\alpha} \quad \text{does exist}
\]
Corollary (Neyman 04, Bewley and Kohlberg 76)

If the graph of $T$ is semi-algebraic, then $\chi(T)$ does exists.

Then, $v_\alpha$ is a semi-algebraic function of $\alpha$, it has a Puiseux series expansion, and so $(1 - \alpha)v_\alpha$ has BV. This is the case in particular if the action spaces are finite.

More generally, $T$ definable in an o-minimal model (Bolte, SG, Vigeral 13).
By subadditivity, the following limits (indep of $x \in \mathbb{R}^n$) do exist:

$$\lim_{k \to \infty} \frac{\| T^k(x) - x \|_\infty}{k} = \inf_{k \geq 1} \frac{\| T^k(x) - x \|_\infty}{k}$$

$$\chi(T) := \lim_{k \to \infty} \frac{t(T^k(x) - x)}{k} = \inf_{k \geq 1} \frac{t(T^k(x) - x)}{k}$$

$$\underline{\chi}(T) := \lim_{k \to \infty} \frac{b(T^k(x) - x)}{k} = \sup_{k \geq 1} \frac{b(T^k(x) - x)}{k}$$

$$t(z) := \max_{1 \leq i \leq n} z_i, \quad b(z) := \min_{1 \leq i \leq n} z_i .$$
Think of $T$ as a Perron-Frobenius operator in log-glasses:

$$F = \exp \circ T \circ \log, \quad \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+$$

$F$ extends continuously from $\text{int} \mathbb{R}^n_+$ to $\mathbb{R}^n_+$ Burbanks, Nussbaum, Sparrow.

Theorem (non-linear Collatz-Wielandt, Nussbaum, LAA 86)

$$\rho(F) = \lim_{k \to \infty} \|F^k(x)\|^{1/k}, \quad x \in \text{Int} \mathbb{R}^n_+$$

$$= \max\{\mu \in \mathbb{R}_+ \mid F(\nu) = \mu \nu, \nu \in \mathbb{R}^n_+, \nu \neq 0\}$$

$$= \max\{\mu \in \mathbb{R}_+ \mid F(\nu) \succeq \mu \nu, \nu \in \mathbb{R}^n_+, \nu \neq 0\}$$
Corollary

Let $T$ be a Shapley operator. Then,
\[\lim_k \max_i [T^k(0)]_i/k \geq 0 \text{ iff there is } u \in (\mathbb{R} \cup \{-\infty\})^n,\]
where $u \neq (-\infty, \ldots, -\infty)^\top$, $T(u) \succeq u$.

Proposition

If $T$ is a Shapley operator, $C = \{ u \in \mathbb{R}^n_{\max} \mid T(u) \succeq u \}$ is a closed tropical convex cone.

Proof.

$T(\sup(u, v)) \succeq \sup(T(u), T(v)) \succeq \sup(u, v),$
\[T(\alpha + u) = \alpha + T(u), \alpha \in \mathbb{R}.\]
Conversely, any closed tropical convex cone can be written as
\[ C = \bigcap_{i \in I} H_i \]
where \((H_i)_{i \in I}\) is a family of tropical half-spaces.

\[ H_i : "A_i x \leq B_i x" \]
Conversely, any closed tropical convex cone can be written as

\[ C = \bigcap_{i \in I} H_i \]

where \((H_i)_{i \in I}\) is a family of tropical half-spaces.

\[ H_i : \max_{1 \leq j \leq n} a_{ij} + x_j \leq \max_{1 \leq k \leq n} b_{ik} + x_k, \quad a_{ij}, b_{ik} \in \mathbb{R} \cup \{-\infty\} \]

\[
[T(x)]_j = \inf_{i \in I} -a_{ij} + \max_{1 \leq k \leq n} b_{ik} + x_k .
\]
Conversely, any closed tropical convex cone can be written as

\[ C = \bigcap_{i \in I} H_i \]

where \((H_i)_{i \in I}\) is a family of tropical half-spaces.

\[ H_i : \max_{1 \leq j \leq n} a_{ij} + x_j \leq \max_{1 \leq k \leq n} b_{ik} + x_k, \quad a_{ij}, b_{ik} \in \mathbb{R} \cup \{-\infty\} \]

\[ [T(x)]_j = \inf_{i \in I} -a_{ij} + \max_{1 \leq k \leq n} b_{ik} + x_k. \]

\[ x \leq T(x) \iff \max_{1 \leq j \leq n} a_{ij} + x_j \leq \max_{1 \leq k \leq n} b_{ik} + x_k, \quad \forall i \in I. \]
\[ H_i : \max_{1 \leq j \leq n} a_{ij} + x_j \leq \max_{1 \leq k \leq n} b_{ik} + x_k \]

\[ [T(x)]_j = \inf_{i \in I} -a_{ij} + \max_{1 \leq k \leq n} b_{ik} + x_k. \]

Interpretation of the game

- State of MIN: variable \( x_j, j \in \{1, \ldots, n\} \)
- State of MAX: half-space \( H_i, i \in I \)
- In state \( x_j \), Player MIN chooses a tropical half-space \( H_i \) with \( x_j \) in the LHS
- In state \( H_i \), player MAX chooses a variable \( x_k \) at the RHS of \( H_i \)
- Payment \(-a_{ij} + b_{ik}\).
Correspondence between tropical convexity and zero-sum games

Theorem (Akian, SG, Guterman, IJAC 2012)

TFAE:

- $C$ closed tropical convex cone
- $C = \{ u \in (\mathbb{R} \cup \{-\infty\})^n \mid u \leq T(u) \}$ for some Shapley operator $T$

and $\text{MAX}$ has at least one winning state ($\exists i, \chi_i(T) \geq 0$) if and only if $C \neq \{(-\infty, \ldots, -\infty)\}$. Moreover, tropical polyhedra correspond to deterministic games with finite action spaces. Then, state $i$ is winning iff $u_i \neq -\infty$ for some $u \in C$. 

Stephane Gaubert (INRIA and CMAP) Tropical convexity and zero-sum games, II
states 1, 2, 3 winning

states 2, 3 winning
Polyhedral part relies on Kohlberg’s theorem 1980.

A nonexpansive piecewise affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ admits an invariant half-line

$$\exists v \in \mathbb{R}^n, \eta \in \mathbb{R}^n, \quad T(v + t\eta) = v + (t + 1)\eta.$$  

The vector $u$ such that $T(u) \succeq u$ is obtained from $v, \eta$ (hint: $u_i = -\infty$ if $\eta_i < 0$).
Strategy of MAX $\sigma : \{ H_1, \ldots, H_m \} \rightarrow \{ x_1, \ldots, x_n \}$, in state $H_i$ choose coordinate $x_{\sigma(i)}$

Duality theorem (coro of Kohlberg)

$$\chi(T) = \max_{\sigma} \chi(T^\sigma) = \min_{\pi} \chi(T^\pi)$$
Strategy of MAX $\sigma : \{H_1, \ldots, H_m\} \rightarrow \{x_1, \ldots, x_n\}$, in state $H_i$ choose coordinate $x_{\sigma(i)}$

Strategy of MIN $\pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, m\}$, in state $x_j$ choose hyperplane $H_{\pi(j)}$

Duality theorem (coro of Kohlberg)

$$\chi(T) = \max_\sigma \chi(T^\sigma) = \min_\pi \chi(T_\pi)$$
Strategy of MAX $\sigma : \{H_1, \ldots, H_m\} \to \{x_1, \ldots, x_n\}$, in state $H_i$ choose coordinate $x_{\sigma(i)}$

Strategy of MIN $\pi : \{1, \ldots, n\} \to \{1, \ldots, m\}$, in state $x_j$ choose hyperplane $H_{\pi(j)}$

One player Shapley operators

$$[T^\sigma(x)]_j = \inf_{1 \leq i \leq m} -a_{ij} + b_{i\sigma(i)} + x_{\sigma(i)} .$$

$$[T^\pi(x)]_j = -a_{\pi(j)j} + \max_{1 \leq k \leq n} b_{\pi(j)k} + x_k .$$

Duality theorem (coro of Kohlberg)

$$\chi(T) = \max_{\sigma} \chi(T^\sigma) = \min_{\pi} \chi(T^\pi) .$$
• Strategy of MAX $\sigma : \{H_1, \ldots, H_m\} \rightarrow \{x_1, \ldots, x_n\}$, in state $H_i$ choose coordinate $x_{\sigma(i)}$

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• One player Shapley operators

$$\left[ T^\sigma(x) \right]_j = \inf_{1 \leq i \leq m} -a_{ij} + b_{i\sigma(i)} + x_{\sigma(i)} .$$

$$\left[ T^\pi(x) \right]_j = -a_{\pi(j)j} + \max_{1 \leq k \leq n} b_{\pi(j)k} + x_k .$$

Duality theorem (coro of Kohlberg)

$$\chi(T) = \max_{\sigma} \chi(T^\sigma) = \min_{\pi} \chi(T^\pi) .$$

Every $\chi(T^\sigma)$ and $\chi(T^\pi)$ can be computed in polynomial time.
\[
\chi(T) = \max_{\sigma} \chi(T^\sigma) = \min_{\pi} \chi(T^\pi).
\]

“Ax \leq Bx” unfeasible iff \( \exists \pi, \chi(T^\pi) < 0 \).
\( \chi(T) = \max_{\sigma} \chi(T^\sigma) = \min_{\pi} \chi(T^{\pi}) \).

- “\( Ax \leq Bx \)” unfeasible iff \( \exists \pi, \chi(T^{\pi}) < 0 \).
- “\( Ax \leq Bx \)” feasible iff \( \exists \sigma, \chi(T^\sigma) \geq 0 \).
\[ \chi(T) = \max_{\sigma} \chi(T^{\sigma}) = \min_{\pi} \chi(T_{\pi}). \]

- "Ax ≤ Bx" unfeasible iff \( \exists \pi, \chi(T_{\pi}) < 0. \)
- "Ax ≤ Bx" feasible iff \( \exists \sigma, \chi(T^{\sigma}) \geq 0. \)
- \( \exists x \in \mathbb{R}^n_{\max}, Ax \leq Bx? \) is in \( \text{NP} \cap \text{co-NP} \) (Edmonds’ good characterization)
\[ \chi(T) = \max_{\sigma} \chi(T^{\sigma}) = \min_{\pi} \chi(T^{\pi}) . \]

- "Ax \leq Bx" unfeasible iff \( \exists \pi \), \( \chi(T^{\pi}) < 0 \).
- "Ax \leq Bx" feasible iff \( \exists \sigma \), \( \chi(T^{\sigma}) \geq 0 \).
- \( \exists x \in \mathbb{R}^n_{\text{max}}, Ax \leq Bx \) is in NP \( \cap \) co-NP (Edmonds’ good characterization)

Strategies are Lagrange multipliers!
\[ x_1 \leq a + \max(x_2 - 2, x_3 - 1) \quad (H_1) \]
\[ -2 + x_2 \leq a + \max(x_1, x_3 - 1) \quad (H_2) \]
\[ \max(x_2 - 2, x_3 - a) \leq x_1 + 2 \quad (H_3) \]

value \( \chi(T)_j = (2a + 1)/2, \ \forall j. \)
\[ a = -\frac{3}{2}, \text{ victorious strategy of Min: certificate of emptiness involving } \leq n \text{ inequalities (Helly)} \]
\( a = 1 \), victorious strategy of Max: tropical polytrope \( \neq \emptyset \) included in the convex set
Tropical Farkas (Allamigeon, SG, Katz, LAA11)

Check "\( Ax \preceq Bx \implies cx \preceq dx \)"?
Tropical Farkas (Allamigeon, SG, Katz, LAA11)

- Check "\(Ax \leq Bx \implies cx \leq dx\)"?
- A counter example is a vector \(x \not\equiv -\infty\),

\[
\text{"Ax} \leq \text{Bx" "dx} \leq \alpha cx\), \quad \alpha < 0
\]
Tropical Farkas (Allamigeon, SG, Katz, LAA11)

Check “$Ax \leq Bx \implies cx \leq dx$”?

A counter example is a vector $x \not\equiv -\infty$, 

“$Ax \leq Bx$” “$dx \leq \alpha cx$”, $\alpha < 0$

Assume $A, B, c, d$ is prepared (technical condition about supports, may occur that $cx = dx = -\infty$!)
Tropical Farkas \textbf{(Allamigeon, SG, Katz, LAA11)}

- Check \textit{“Ax ⩽ Bx ⇒ cx ⩽ dx”}\?
- A \textbf{counter example} is a vector $x \not\equiv -\infty$,

\[ Ax ⩽ Bx \Rightarrow dx ⩽ \alpha cx, \quad \alpha < 0 \]

- Assume $A, B, c, d$ is \textbf{prepared} (technical condition about supports, may occur that $cx = dx = -\infty$!)
- Suffices to take $\alpha = -1$ if $A, B, c, d$ have entries in $\mathbb{Z} \cup \{-\infty\}$. 
Tropical Farkas (Allamigeon, SG, Katz, LAA11)

- Check “\(Ax \leq Bx \Rightarrow cx \leq dx\)”?
- A counter example is a vector \(x \not\equiv -\infty\),

\[
\text{“}Ax \leq Bx, \text{ “}dx \leq \alpha cx\text{”, } \alpha < 0
\]

- Assume \(A, B, c, d\) is prepared (technical condition about supports, may occur that \(cx = dx = -\infty\)!)
- Suffices to take \(\alpha = -1\) if \(A, B, c, d\) have entries in \(\mathbb{Z} \cup \{-\infty\}\).

Implication holds in Farkas iff \(\overline{\chi}(T) < 0\), where \(T\) is the Shapley operator associated to the system

“\(Ax \leq Bx, dx \leq \alpha cx\)”. 
Compare with classical Farkas

- Given \( a_1, \ldots, a_m, c \in \mathbb{Q}^n \), is it true that
  
  \[
  x \in \mathbb{R}^n, \quad a_i \cdot x \geq 0 \quad \forall 1 \leq i \leq m \implies c \cdot x \geq 0?
  \]
Compare with classical Farkas

- Given $a_1, \ldots, a_m, c \in \mathbb{Q}^n$, is it true that
  
  $$x \in \mathbb{R}^n, \quad a_i \cdot x \geq 0 \quad \forall 1 \leq i \leq m \implies c \cdot x \geq 0$$

- Yes iff $\exists \lambda \in \mathbb{Q}_+^m$, $c = \lambda_1 a_1 + \cdots + \lambda_m a_m$. 

\[ \lambda \text{ can be required to be concise. ANALOGOUS.} \]
\[ \lambda \text{ can be required to be sparse: } \lambda_i = 0 \text{ except for } n \text{ values of } 1 \leq i \leq m. \text{ (Carathéodory / Helly by duality). ANALOGOUS} \]
\[ \lambda \text{ can actually be found in polynomial time (Linear programming: Khachyan 79, Karmarkar 84, \ldots).} \text{ DONT KNOW!} \]
Compare with classical Farkas

- Given $a_1, \ldots, a_m, c \in \mathbb{Q}^n$, is it true that
  
  $x \in \mathbb{R}^n, \quad a_i \cdot x \geq 0 \quad \forall 1 \leq i \leq m \implies c \cdot x \geq 0$ ?

- Yes iff $\exists \lambda \in \mathbb{Q}_+^m, \quad c = \lambda_1 a_1 + \cdots + \lambda_m a_m$. 

$\lambda$ can be required to be sparse: $\lambda_i = 0$ except for $n$ values of $1 \leq i \leq m$. (Carathéodory / Helly by duality). Analogue.
Compare with classical Farkas

- Given \( a_1, \ldots, a_m, c \in \mathbb{Q}^n \), is it true that
  \[
  x \in \mathbb{R}^n, \quad a_i \cdot x \geq 0 \quad \forall 1 \leq i \leq m \implies c \cdot x \geq 0
  \]
  Yes iff \( \exists \lambda \in \mathbb{Q}_+^m \),
  \[
  c = \lambda_1 a_1 + \cdots + \lambda_m a_m.
  \]
  \( \lambda \) can be required to be concise. ANALOGOUS.

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Tropical convexity and zero-sum games, II

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Compare with classical Farkas

- Given $a_1, \ldots, a_m, c \in \mathbb{Q}^n$, is it true that
  
  $$x \in \mathbb{R}^n, \quad a_i \cdot x \geq 0 \quad \forall 1 \leq i \leq m \implies c \cdot x \geq 0$$

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- Given $a_1, \ldots, a_m, c \in \mathbb{Q}^n$, is it true that
  \[
  x \in \mathbb{R}^n, \quad a_i \cdot x \geq 0 \quad \forall 1 \leq i \leq m \implies c \cdot x \geq 0
  \]?

- Yes iff $\exists \lambda \in \mathbb{Q}^m_+, \quad c = \lambda_1 a_1 + \cdots + \lambda_m a_m$.
- $\lambda$ can be required to be concise. ANALOGOUS.
- $\lambda$ can be required to be sparse: $\lambda_i = 0$ except for $n$ values of $1 \leq i \leq m$. (Carathéodory / Helly by duality). ANALOGOUS
- $\lambda$ can actually be found in polynomial time (Linear programming: Khachyan 79, Karmarkar 84, \ldots).

DONT KNOW!
The Mean Payoff Game problem

Compute $\chi(T)$ where $T$ Shapley operator of deterministic game with finite action spaces?

Existence of polynomial time algorithm open since Gurvich, Karzanov, Khachyan 86. One of the few natural pbs in $\text{NP} \cap \text{coNP}$ not known to be in $\text{P}$ (with factoring!).

Pseudo polynomial algorithm (value iteration) Zwick-Paterson 96, experimentally efficient policy iteration algorithms but worst case exponential Friedmann 10.
The following problems are all equivalent to mean payoff games

- **Feasibility**: \( \exists x \neq -\infty \), “\( Ax \leq Bx \)”
The following problems are all equivalent to mean payoff games

- **Feasibility:** $\exists x \neq -\infty, \ "Ax \leq Bx"$
- **Affine feasibility:** $\exists x, \ "Ax + b \leq Cx + d"$

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The following problems are all equivalent to mean payoff games

- **Feasibility**: \( \exists x \not\equiv -\infty \), “\( Ax \leq Bx \)”
- **Affine feasibility**: \( \exists x \), “\( Ax + b \leq Cx + d \)”
- **Farkas**: “\( Ax \leq Bx \implies cx \leq dx \)”
The following problems are all equivalent to mean payoff games

- **Feasibility**: \( \exists x \neq -\infty, \quad "Ax \leq Bx" \)
- **Affine feasibility**: \( \exists x, \quad "Ax + b \leq Cx + d" \)
- **Farkas**: \( "Ax \leq Bx \implies cx \leq dx" \)
- **Separation**: given finite sets \( X, Y \subset \mathbb{R}^n_{\max} \),
  \( \text{cone } X \cap \text{cone } Y = \{ "0" \}? \)
Tropical simplex

A tropical LP

\[
\min \ 'f \cdot x'; \quad 'Ax + c \leq Bx + d'
\]

\(A, B \in \mathbb{R}^{m \times n}, \ b, c \in \mathbb{R}^m, \ f \in \mathbb{R}^n\), the inequalities \(x \geq 0\) being included in \(Ax + c \leq Bx + d\), can be lifted to a classical LP over Puiseux series

\[
\min f \cdot x; \quad Ax + c \leq Bx + d
\]

\(A, B \in \mathbb{K}^{m \times n}, \ b, c \in \mathbb{K}^m, f \in \mathbb{K}^n\), meaning that val\(A = A\), val\(B = B\), etc. Recall that val\(7t^{-1/2} - 1 + t^{1/2} + 7t + \cdots = 1/2\).
\[(t^0, t^0, t^{-4})\]

\[(t^{-4}, t^{-4}, t^{-4})\]

\[(t^{-4}, t^0, t^0)\]

\[(t^{-4}, t^{-4}, t^0)\]
Assume that the data are in general position. This can be defined in terms of tropical Cramer subdeterminants of “$(A + B, c + d)$”.

A **tropical basic point** is obtained by saturating $n$ inequalities.

**Theorem (Allamigeon, Benchimol, SG, Joswig arXiv:1308.0454)**

*The valuation of the path of the simplex algorithm over Puiseux series can be computed tropically (with a compatible pivoting rule). One iteration takes $O(n(m + n))$ time.*

Tropical Cramer determinants $=$ opt. assignment used to compute reduce costs.
Example of compatible pivoting rule. A rule is **combinatorial** if any entering/leaving inequalities are functions of the history (sequence of bases) and of the signs of the minors of the matrix

\[
M = \begin{pmatrix}
"A - B" & "c - d" \\
\text{f}^\top & "0"
\end{pmatrix}.
\]

(eg signs of reduced costs).

**Corollary (Allamigeon, Benchimol, SG, Joswig arXiv:1309.5925)**

*If any combinatorial rule in classical linear programming would run in polynomial time, then, mean payoff games could be solved in strongly polynomial time.*
Concluding remarks

- Complexity of tropical LP = deterministic mean payoff games is open
- Stochastic mean payoff games: a fortiori (a pseudo polynomial algorithm is not known).
- Games with fixed discount rate: strongly polynomial, Ye; Hansen, Miltersen, Zwick.
- Current work (Allamigeon, Benchimol, SG, Joswig): tropicalization of central path.
Tropical convexity and its applications to zero-sum games

Minilecture, Part III

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JGA, Marseille
December 16-20, 2013

Works with Akian, Allamigeon, Goubault, Guterman, Katz, Joswig, Meunier, Sergeev, Walsh; highlight: PhD of Benchimol and Qu.
Tropical Minkowski-Weyl

Theorem (SG, Katz, Relmics 06, JACO 2011)

A tropical polyhedral convex set can be written as

$$K = \text{"conv}(X) + \text{cone}(Y)"$$

with $X$, $Y$ finite, and vice versa.

- Inequalities to vertices: finiteness can be proved by elimination Butkovic and Hegedus, 84; tropical double description Allamigeon, SG, Goubault DCG 2012.

- Vertices to inequalities: the set of valid inequalities is itself a polyhedron (tropical polar; SG, Katz).
Extreme points and rays

Definition

\( u \in \mathcal{V} \) is an extreme generator if \( u = "v + w" \) with \( v, w \in \mathcal{V} \) implies \( u = v \) or \( u = w \). (ie \( u \) join irreducible)

Theorem (Tropical Minkowski SG, Katz RELMICS 06, LAA07; Butkovič, Sergeev, Schneider LAA07)

*Every element of a closed tropical cone of \( \mathbb{R}_\text{max}^d \) is a sum of at most \( d \) extreme generators.*

Affine version. If \( \mathcal{K} \) is a closed convex set, then,

\[
\mathcal{K} = "\text{conv}(\text{ext}(K)) + \text{rec}(K)"
\]
It is one of the places where $-\infty$ is needed; generators do have $-\infty$ coordinates.

E.g., the tropical line, \( \max(x_1, x_2, x_3) \) attained twice, is generated by

\[
(0, 0, -\infty), \quad (0, -\infty, 0), \quad (-\infty, 0, 0)
\]
Number of extreme rays

A classical question . . .

- What is the maximal number of facets of a polytope of dimension $d$ with $p$ vertices?

or (equivalent by duality)

- What is the maximal number of vertices of a polytope of dimension $d$ with $p$ facets?
Theorem (McMullen upper bound 1970)

Among the polytopes of dimension $d$ with $p$ vertices, the cyclic polytope maximizes the number of faces of each dimension.

The cyclic polytope $C(p, d)$ is the convex hull of $p$ points of the moment curve $t \mapsto z(t) := (t, t^2, \ldots, t^d)$. 
Definition
A 0/1 sequence satisfies Gale’s evenness condition if the number of 1 between any two 0 is even.

Eg., 0110111100011001111110000111
Let $C(p, d) := \text{co}(z(t_1), \ldots, z(t_p))$ with $t_1 < t_2 < \cdots < t_p$.

Fact
The points $z(t_{i_1}), \ldots, z(t_{i_{d+1}})$ define a facet iff the associated word satisfies Gale’s evenness condition.

Eg., $i_1 = 2, i_2 = 3, i_3 = 5, p = 5, d = 2 \rightarrow 01101$
Corollary (classical)

The number of facets of a polytope of dimension $d$ with $p$ vertices is at most

$$U(p, d) := \binom{p - d/2}{d/2} + \binom{p - d/2 - 1}{d/2 - 1} \quad \text{for } d \text{ even}$$

$$U(p, d) := 2\binom{p - (d + 1)/2}{(d - 1)/2} \quad \text{for } d \text{ odd.}$$

This is $\Theta(p^{d/2})$ has $p \to \infty$, keeping $d$ fixed, so much smaller than the naive bound $\binom{p}{d} = \Theta(p^d)$. 
The same questions can be raised for max-plus or tropical convex sets/cones

Theorem (Allamigeon, SG, Katz, JCTA 11)

The number of extreme rays of a tropical cone $\mathcal{V}$ defined by $p$ inequalities in dimension $d$ cannot exceed $U(p + d, d - 1)$.

$$\mathcal{V} := \{ x \in \mathbb{R}^d_{\max} \mid \max_{j \in [d]} a_{ij} + x_j \leq \max_{j \in [d]} b_{ij} + x_j, \ i \in [p] \}.$$

The bound is $\Theta(p^{\lfloor (d-1)/2 \rfloor})$ for $d$ fixed and $p \to \infty$. 
Proof (by dequantization)

For $\beta > 0$, consider the classical convex cone $\mathcal{V}(\beta)$ defined by the $p + d$ inequalities

\[ y_j \geq 0 , \quad j \in [d] , \]
\[
\frac{1}{d} \sum_{j \in [d]} \exp(\beta a_{ij}) y_j \leq \sum_{j \in [d]} \exp(\beta b_{ij}) y_j , \quad i \in [p] .
\]

By the McMullen upper bound theorem, $\mathcal{V}(\beta)$ has a generating family $(u_k(\beta))_{k \in [K]}$ with $K \leq U(p + d, d - 1)$. 
If \( x \in \mathcal{V} \), then \( E_\beta(x) := (\exp(\beta x)) \in \mathcal{V}(\beta) \). WLOG, normalize \( u_k(\beta) \) (entries sum to one). Let \( v_k(\beta) := E_\beta^{-1}(u_k) \).

\[
\max_{j \in [d]} v_k(\beta)_j \leq 0 \leq \beta^{-1} \log d + \max_{j \in [d]} v_k(\beta)_j ,
\]

\[-\beta^{-1} \log d + \max_{j \in [d]} a_{ij} + v_k(\beta)_j \leq \beta^{-1} \log d + \max_{j \in [d]} b_{ij} + v_k(\beta)_j .
\]

Then, it can be checked that any accumulation point of the family \((v_k(\beta))_{k \in [K]}\) yields a generating family of \( \mathcal{V} \) (use \( \mathcal{V}(\beta) \supset \mathcal{V} \) thanks to the \( 1/d \) trick).
Is the tropical upper bound attained?

The usual bound of $\frac{p}{d}$ vertices for a dim $d$ polytope with $p$ facets is attained by the polar of the cyclic polytope

$$C(p, d)^\circ := \{y \mid z(t_i) \cdot (y - w) \leq 1, i \in [p]\}, \quad w \in \text{int}(C(p, d)).$$

In the tropical case

$$z(t) := "(1, t, \ldots, t^{d-1})" = (1, t, \ldots, (d - 1)t) \in \mathbb{R}_\text{max}^d,$$

Homogeneizing naively $C(p, d)^\circ$ yields

$$\{y \mid "z(t_i) \cdot y \leq 0", i \in [p]\}$$

which is trivial. To make it less trivial, we may add signs.
Introduce a sign pattern \( \epsilon_{ij} \in \{ \pm 1 \} \)

Set formally "\( z(t_i)_j = \epsilon_{ij} t_{i}^{j-1} \)" in the symmetrized maxplus semiring \( \mathcal{S}_{\text{max}} \), so \( z(t_i) = \) "\( z^+(t_i) - z^-(t_i) \)" where

\[
\begin{align*}
z^\pm(t_i) &\in \mathbb{R}_{\text{max}}^d, \\
z^\pm(t_i)_j &= \begin{cases} 
"t_{i}^{j-1}" & \text{if } \epsilon_{ij} = \pm 1 \\
"0" & \text{otherwise}
\end{cases}
\end{align*}
\]

Definition

The signed cyclic polyhedral cone \( C(p, d; \epsilon) \), is generated by \( p \) pairs of vectors \( (z^-(t_i), z^+(t_i)) \in (\mathbb{R}_{\text{max}}^d)^2, i \in [p] \). Its polar \( K(p, d; \epsilon) \) is the set of vectors \( x \in \mathbb{R}_{\text{max}}^d \) such that

\[
\begin{align*}
"z^-(t_i) \cdot x \leq z^-(t_i) \cdot x", & \quad i \in [p], \text{ i.e.} \\
\max_{j \in [d], \epsilon_{ij} = -1} (j - 1)t_i + x_j &\leq \max_{j \in [d], \epsilon_{ij} = +1} (j - 1)t_i + x_j, \quad i \in [p].
\end{align*}
\]
The analogy with the classical case . . .
The analogy with the classical case . . .

may suggest that there should be some choice of sign $\epsilon$ such that the polar $\mathcal{K}(p, d; \epsilon)$ of the signed cyclic polyhedral cone has exactly $U(p + d, d - 1)$ extreme rays. . .
The analogy with the classical case . . .

may suggest that there should be some choice of sign \( \epsilon \) such that the polar \( \mathcal{K}(p, d; \epsilon) \) of the signed cyclic polyhedral cone has exactly \( U(p + d, d - 1) \) extreme rays.

we shall see that this is not true.
Some lattice paths

southward / eastward paths in the sign pattern $\epsilon_{ij}$

$$
\begin{array}{ccccccc}
  & j_1 & j_2 & j_3 & j_4 & j_5 & j_6 \\
\hline
i_1 & + & . & . & . & & \\
     & + & . & . & . & & \\
     & . & * & * & - & . & . \\
     & . & * & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
i_2 & . & . & + & - & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
i_3 & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
i_4 & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
i_5 & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
     & . & . & . & . & . & . \\
\end{array}
$$

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A lattice path for the sign pattern $\varepsilon_{ij}$ is tropically allowed if

(i) every sign occurring on the initial vertical segment, except possibly the sign at the bottom of the segment, is positive;

(ii) every sign occurring on the final vertical segment, except possibly the sign at the top of the segment, is positive;

(iii) every sign occurring in some other vertical segment, except possibly the signs at the top and bottom of this segment, is positive;

(iv) for every horizontal segment, the pair of signs consisting of the signs of the leftmost and rightmost positions of the segment is of the form $(+,-)$ or $(-,+)$;

(v) as soon as a pair $(-,+)$ occurs as the extreme signs of an horizontal segment, the pairs of the next horizontals segments must also be equal to $(-,+)$.

If only (i)–(iv) hold, we say that the path is classically allowed.
Theorem (Allamigeon, SG, Katz, JCTA 11)

The extreme rays of the polar of the tropical signed cyclic polyhedral cone correspond bijectively to the tropically allowed lattice paths.

For $t_1 \ll t_2 \ll \cdots \ll t_p$, the extreme rays of the classical analogue of this polar correspond bijectively to the classically allowed lattice paths.

When deforming a polytope into a tropical polytope, some extreme points vanish.
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- For $t_1 \ll t_2 \ll \cdots \ll t_p$, the extreme rays of the classical analogue of this polar correspond bijectively to the classically allowed lattice paths.

When deforming a polytope into a tropical polytope, some extreme points vanish.
This apparently mysterious result relies on a characterization of the extreme points of a tropical polyhedron in terms of the inequalities which define it: Allamigeon, SG, Goubault, DCG 2012

Recall first.

Fact (See Butkovič, Sergeev, Schneider; SG, Katz, both LAA 07)

A vector \( g \) of a tropical cone \( C \in \mathbb{R}_\text{max}^d \) is extreme iff 
\[ \exists t \in [d] \text{ such that } g \text{ is a minimal element of the set } \{ x \in C \mid x_t = g_t \}. \] In that case, \( g \) is said to be extreme of type \( t \).
Definition

The tangent cone of $\mathcal{C} := \{x \mid "Ax \leq Bx"\}$ at $g$ is defined as the tropical cone $\mathcal{T}(g, \mathcal{C})$ of $\mathbb{R}^d_{\max}$ given by the system of inequalities

$$\max_{i \in \arg \max (A_k g)} x_i \leq \max_{j \in \arg \max (B_k g)} x_j$$

for all $k \in [p]$ such that $A_k g = B_k g$. 

Stephane Gaubert (INRIA and CMAP)
Fact (Allamigeon, SG, Goubault)

There exists a neighborhood \( N \) of \( g \) such that for all \( x \in N \), \( x \) belongs to \( C \) if and only if it is an element of \( g + T(g, C) \).

Fact (ibid.)

The element \( g \) is extreme in \( C \) if and only if the vector \( 1 \) is extreme in \( T(g, C) \).
Theorem (Allamigeon, SG, Goubault, ibid.)

A vector $y \in \mathbb{R}_{\max}^d$ belongs to an extreme ray of a tropical polyhedral cone $C$ if, and only if, there exists $s \in \{1, \ldots, d\}$ such that
\[
(x \in \mathcal{T}(C, y) \cap \{1, 0\}^d \text{ and } x_s = 1) \Rightarrow (x_r = 1 \text{ or } y_r = 0)
\]
for all $r \in \{1, \ldots, d\}$.

Corollary

If $t$ entries of $y$ are zero, then $y$ must saturate at least $d - t - 1$ inequalities among $A_r x \leq B_r x$, $r \in [p]$. 

Stephane Gaubert (INRIA and CMAP) Tropical convexity and zero-sum games, III
Recall our characterization: a vector \( y \in \mathbb{R}_{max}^d \) belongs to an extreme ray of a tropical polyhedral cone \( C \) if, and only if, there exists \( s \in \{1, \ldots, d\} \) such that

\[
(x \in T(C, y) \cap \{1, 0\}^d \text{ and } x_s = 1) \Rightarrow (x_r = 1 \text{ or } y_r = 0)
\]

for all \( r \in \{1, \ldots, d\} \).

Eg, when \( y \) is finite, does there exists \( s \) such that, for \( x \in \{0, 1\}^d \),

\[
x_s = 1 \text{ and } \max_{i \in \arg \max(A_ky)} x_i \leq \max_{j \in \arg \max(B_ky)} x_j \implies x \equiv 1?
\]
This is expressed as an **hypergraph reachability problem**. Given a node set \( N \), an (oriented) hyperedge is a pair \((T, H)\) (tail, head) with \( T, H \subset N \). We say that \( v \) is **reachable from** \( u \) if \( u = v \), or there exists \( e \in E \) such that \( v \in H(e) \) and all the elements of \( T(e) \) are reachable from \( u \). Here, \( T = \text{arg max}(A_k y) \) and \( H = \text{arg max}(B_k y) \).
Proposition (ibid.)

**TFAE**

(a) \( i \) is reachable from \( j \) in the hypergraph arising from the tangent cone at point \( v \);

(b) for all \( x \in \mathcal{T}(v, C) \cap \{0, 1\}^d \), \( x_i \leq x_j \).

Theorem (ibid.)

A vector \( y \) belongs to an extreme ray iff the hypergraph arising from its tangent cone has only one terminal strongly connected component.
\[ g^2 = (2, 2, 0), \ T(g^2, C) \]

\[ x_3 \leq x_1 + 2 \]
\[ x_1 \leq \max(x_2, x_3) \]
\[ x_1 \leq x_3 + 2 \]
\[ x_3 \leq \max(x_1, x_2 - 1) \]

Figure ?? illustrates that the cones \( C \) and \( g^2 + T(g^2, C) \) locally coincide in a neighborhood of \( g^2 \). The tangent directed hypergraph \( H(g^2, C) \) associated to the vector \( g^2 \) is formed by the two hyperarcs \( (\{2\}, \{1\}) \) and \( (\{3\}, \{1\}) \). The node 1 consequently forms the greatest strongly component of the hypergraph.
The first theorem (that the extreme rays of the tropical signed cyclic polyhedral cone correspond bijectively to the tropically allowed lattice paths) is obtained as a corollary.
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The proof uses also the tropical Cramer theorem in the signed tropical semiring (M. Plus (1990), Akian, SG, Guterman 09).
The first theorem (that the extreme rays of the tropical signed cyclic polyhedral cone correspond bijectively to the tropically allowed lattice paths) is obtained as a corollary.

The proof uses also the tropical Cramer theorem in the signed tropical semiring (M. Plus (1990), Akian, SG, Guterman 09).

The tangent cones turn out to be described by “line” directed graphs, which must have a unique terminal node. This explains the mysterious condition (v).
\[
\begin{pmatrix}
0 & -\infty & 0 \\
0 & -\infty & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\geq
\begin{pmatrix}
-\infty & 0 & -\infty \\
-\infty & 1 & -\infty
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}.
\]
Usually, a point $y$ in $\{x \mid Ax \leq b\}$ is extreme iff the family of rows $A_k$ arising from active constraint is of full rank. The same is not true in the tropical case.
\begin{itemize}
  \item $N^{\text{trop}}(\epsilon)$ (resp. $N^{\text{path}}(\epsilon)$) := \# tropically (resp. non-tropically) allowed lattice paths for the sign pattern $\epsilon$.
  \item $N^{\text{trop}}(p, d) :=$ maximal \# extreme rays of a tropical cone in dimension $d$ defined as the intersection of $p$ half-spaces.
\end{itemize}

\[
\max_{\epsilon \in \{-1, 1\}^{p \times d}} N^{\text{trop}}(\epsilon) \leq N^{\text{trop}}(p, d) \leq U(p+d, d-1) = \max_{\epsilon \in \{-1, 1\}^{p \times d}} N^{\text{path}}(\epsilon).
\]

We initially thought that the maximum of the \# of extreme points is attained among the polars of signed cyclic polyhedra:

\[
\max_{\epsilon \in \{-1, 1\}^{p \times d}} N^{\text{trop}}(\epsilon) = N^{\text{trop}}(p, d) ?
\]

Not true! Finding the maximizing model for $N^{\text{trop}}$ is an open problem.
However...

Fact

For $d \geq 2p + 1$, we have

$$N_{tpath}(p, d) \geq U(d, d - p - 1).$$

(2)

It follows that the tropical upper bound is asymptotically tight for a fixed number of constraints $p$, as the dimension tends to infinity

$$N_{trop}(p, d) \sim U(p + d, d - 1) \quad \text{as } d \to \infty.$$
Lower and upper bounds for $N^{\text{trop}}(p, d)$, the maximal number of extreme rays of a tropical polyhedral cone defined by $p$ inequalities in dimension $d$.

<table>
<thead>
<tr>
<th>$d \setminus p$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<td>[44, 54]</td>
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<td>[56, 77]</td>
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<tr>
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<td>30</td>
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<td>[55, 56]</td>
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<td>[159, 168]</td>
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<td>[250, 294]</td>
<td>[321, 450]</td>
<td>[436, 660]</td>
<td>[613, 935]</td>
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<td>[898, 1430]</td>
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<td>36</td>
<td>91</td>
<td>196</td>
<td>[363, 378]</td>
<td>[584, 672]</td>
<td>[805, 1122]</td>
<td>[1122, 1782]</td>
<td>[1357, 2717]</td>
<td>[1799, 4004]</td>
</tr>
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</table>

We do not know whether $N^{\text{trop}}(4, 5) = 26$ or 27 (!)
When computing, use vertices, not pseudo vertices!

The pattern yields \((p - 2d + 7)(2^{d-2} - 2)\) extreme rays.

\[
\begin{pmatrix}
+ & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + \\
\end{pmatrix}
\]
For $d = 4$ and $p = 10$, 24 vertices, 1215 pseudo-vertices
The previous lattice path theorem has the following surprising corollary.

**Corollary (Allamigeon, SG, Goubault, Katz LAA 11)**

The tropical (unsigned) cyclic polyhedral cone $C(p, d)$, i.e., the row space of the matrix $(t_i^{j-1})_{1 \leq i \leq p, 1 \leq j \leq d}$, can be defined by a family of $O(pd^3)$ inequalities.

Compare with the classical analogue $O(p^{\lfloor(d-1)/2\rfloor})$. 
Dual problem: minimal defining systems of inequalities for a polyhedron.

There is a minimal representation (unique modulo certain exchanges), Allamigeon, Katz JCTA 13
An application of tropical convexity in infinite dimension

tropical approximation in optimal control, attenuation of the curse of dimensionality
Lagrange problem / Lax-Oleinik semigroup

\[ v(t, x) = \sup_{\dot{x}(s) = \dot{x}, x(0) = x} \int_0^t L(x(s), \dot{x}(s)) \, ds + \phi(x(t)) \]

Lax-Oleinik semigroup: \((S_t)_{t \geq 0}, S_t \phi := v(t, \cdot).\)

Superposition principle: \(\forall \lambda \in \mathbb{R}, \forall \phi, \psi,\)

\[ S_t(\sup(\phi, \psi)) = \sup(S_t \phi, S_t \psi) \]

\[ S_t(\lambda + \phi) = \lambda + S_t \phi \]

So \(S_t\) is max-plus linear.
\[ v(t, x) = \sup_{x(0) = x, \ x(\cdot)} \int_{0}^{t} L(x(s), \dot{x}(s)) \, ds + \phi(x(t)) \]

Lax-Oleinik semigroup: \((S_t)_{t \geq 0}, \ S_t \phi := v(t, \cdot)\).

Superposition principle: \(\forall \lambda \in \mathbb{R}, \forall \phi, \psi\),

\[ S_t(\phi + \psi) = S_t \phi + S_t \psi \]
\[ S_t(\lambda \phi) = \lambda S_t \phi \]

So \(S_t\) is max-plus linear.
The function $v$ is solution of the Hamilton-Jacobi equation

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}) \quad v(0, \cdot) = \phi$$

Max-plus linearity $\iff$ Hamiltonian convex in $p$

$$H(x, p) = \sup_u (L(x, u) + p \cdot u)$$

Hopf formula, when $L = L(u)$ concave:

$$v(t, x) = \sup_{y \in \mathbb{R}^n} tL\left(\frac{x - y}{t}\right) + \phi(y).$$
The function $v$ is solution of the Hamilton-Jacobi equation

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}) \quad v(0, \cdot) = \phi$$

Max-plus linearity $\iff$ Hamiltonian convex in $p$

$$H(x, p) = \sup_u (L(x, u) + p \cdot u)$$

Hopf formula, when $L = L(u)$ concave:

$$v(t, x) = \left\langle \int G(x - y)\phi(y)dy \right\rangle.$$
Approximate the value function by a “linear comb.” of “basis” functions with coeffs. $\lambda_i(t) \in \mathbb{R}$:

$$v(t, \cdot) \simeq \sum_{i \in [p]} \lambda_i(t) w_i$$

The $w_i$ are given finite elements, to be chosen depending on the regularity of $v(t, \cdot)$.
Max-plus basis / finite-element method

Fleming, McEneaney 00-; Akian, Lakhoua, SG 04-

Approximate the value function by a “linear comb.” of “basis” functions with coeffs. \( \lambda_i(t) \in \mathbb{R} \):

\[
v(t, \cdot) \simeq \sup_{i \in [p]} \lambda_i(t) + w_i
\]

The \( w_i \) are given finite elements, to be chosen depending on the regularity of \( v(t, \cdot) \)
Best max-plus approximation

\[ P(f) := \max\{g \leq f \mid g \text{ “linear comb.” of } w_i\} \]

linear forms \( w_i : x \mapsto \langle y_i, x \rangle \)

adapted if \( v \) is convex
Best max-plus approximation

\[ P(f) := \max \{ g \leq f \mid g \quad \text{“linear comb.” of } w_i \} \]

cone like functions \( w_i : x \mapsto -C\|x - x_i\| \)

adapted if \( v \) is \( C\)-Lip
Max-plus basis propagation

Max-plus linearity is essential in max-plus basis method:

\[ V_t \simeq \tilde{V}_t = \sup_i \lambda_i^t + w_i \]
Max-plus basis propagation

Max-plus linearity is essential in max-plus basis method:

\[
V_t \simeq \tilde{V}_t = \sup_{i} \lambda^t_i + w_i
\]

\[
V_{t+\tau} \simeq S_\tau[\tilde{V}_t]
\]

dynamic programming principle
Max-plus basis propagation

Max-plus linearity is essential in max-plus basis method:

\[ V_t \simeq \tilde{V}_t = \sup \lambda^t_i + w_i \]

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dynamic programming principle
Max-plus basis propagation

Max-plus linearity is essential in max-plus basis method:

\[ V_t \simeq \tilde{V}_t = \sup_i \lambda_i^t + w_i \]

\[ V_{t+\tau} \simeq S_\tau[\tilde{V}_t] \]

\[ = \sup_i S_\tau[\lambda_i^t + w_i] \]

\[ = \sup_i \lambda_i^t + S_\tau[w_i] \]

dynamic programming principle

maxplus linearity

In summary:

\[ V_T(x) = \{S_\tau[V_0] \} \]

\[ \simeq \{P \circ \tilde{S}_\tau[V_0] \} \]
Max-plus basis propagation

Max-plus linearity is essential in max-plus basis method:

\[ V_t \simeq \tilde{V}_t = \sup_{i} \lambda_i^t + w_i \]

\[ V_{t+\tau} \simeq S_{\tau}[\tilde{V}_t] = \sup_{i} S_{\tau}[\lambda_i^t + w_i] = \sup_{i} \lambda_i^t + S_{\tau}[w_i] \]

dynamic programming principle

maxplus linearity

\[ \simeq \sup_{i} \lambda_i^t + \tilde{S}_{\tau}[w_i] \]

semigroup approximation step
Max-plus basis propagation

Max-plus linearity is essential in max-plus basis method:

\[ V_t \simeq \tilde{V}_t = \sup_i \lambda^t_i + w_i \]

\[ V_{t+\tau} \simeq S_\tau[\tilde{V}_t] \]

\[ = \sup \left( \lambda^t_i + w_i \right) \]

\[ = \sup \lambda^t_i + S_\tau[w_i] \]

\[ \simeq \sup \lambda^{t+\tau}_i + \tilde{S}_\tau[w_i] \]

\[ \simeq \sup \lambda^{t+\tau}_i + w_i \]

dynamic programming principle
maxplus linearity
semigroup approximation step
maxplus projection step
Max-plus basis propagation

Max-plus linearity is essential in max-plus basis method:

\[
V_t \simeq \tilde{V}_t = \sup_{i} \lambda_i^t + w_i \\
V_{t+\tau} \simeq S_{\tau}[\tilde{V}_t] \\
= \sup_{i} S_{\tau}[\lambda_i^t + w_i] \\
= \sup_{i} \lambda_i^t + S_{\tau}[w_i] \\
\simeq \sup_{i} \lambda_i^t + \tilde{S}_{\tau}[w_i] \\
\simeq \sup_{i} \lambda_i^{t+\tau} + w_i
\]

dynamic programming principle

maxplus linearity

semigroup approximation step

maxplus projection step

In summary:

\[
V_T(x) = \{S_{\tau}\}^N[V_0] \simeq \{P \circ \tilde{S}_{\tau}\}^N[V_0].
\]
Several max-plus basis methods have been proposed:

- [Fleming, McEneaney 00]:
  A first development of max-plus basis method

- [Akian, Gaubert, Lakhoua 06]:
  A finite element max-plus basis method

- [McEneaney 07]:
  A curse of dimensionality free method

- [McEneaney, Deshpande, Gaubert 08],
  [Sridharan, James, McEneaney 10], [Dower, McEneaney 11], ……
Switched optimal control problem

- Infinite horizon switched optimal control problem

[McEneaney 07]:

\[ V(x) = \sup_{\mu} \sup_{u} \int_{0}^{\infty} \frac{1}{2} x(t)' D^{\mu(t)} x(t) - \frac{\gamma^2}{2} |u(t)|^2 dt, \]

where

\[ D_{\infty} = \{ \mu : [0, \infty) \rightarrow \{1, \ldots, M\} : \text{measurable} \}, \]

\[ W = L_{2}^{\text{loc}}([0, \infty); \mathbb{R}^k), \]

and \( x(\cdot) \) satisfies:

\[ \dot{x}(t) = A^{\mu(t)} x(t) + \sigma^{\mu(t)} u(t), \quad x(0) = x \in \mathbb{R}^d, \]

arising from \( H_\infty \) robust control, nonconvex \( (D^1, \ldots, D^M \succcurlyeq 0) \).
McEneaney’s curse of dimensionality free method

- Semigroup approximation:
  \[ S_\tau \simeq \tilde{S}_\tau = \sup_{m} S^m_\tau \]

- \( S^m_\tau \) is the semigroup associated to the control problem by letting the switching control \( \mu \) equal to \( m \in \{1, \ldots, M\} \):
  \[ S^m_\tau[\phi](x) = \sup_u \int_0^t \frac{1}{2} x(t)' D^m x(t) - \frac{\gamma^2}{2} u(t)^2 dt + \phi(x(t)). \]

- \( \dot{x}(s) = A^m x(s) + \sigma^m u(s); \ x(0) = x \in \mathbb{R}^d \).

- \( S^m_\tau[\phi] \) is a quadratic function if \( \phi \) is. (Riccati)

\[
V \simeq V_T = \{ S_\tau \}^N[V_0] \simeq \{ \tilde{S}_\tau \}^N[V_0] = \\
\sup_{i_N, \ldots, i_1} S^{i_N}_\tau \circ \ldots \circ S^{i_1}_\tau [V_0].
\]

Stephane Gaubert (INRIA and CMAP)
Arborescent propagation

\[ V \simeq V_T = \{ S_\tau \}^N[V_0] \simeq \{ \tilde{S}_\tau \}^N[V_0] = \sup_{i_N, \ldots, i_1} S_{i_N}^{\circ} \cdots S_{i_1}^{\circ} [V_0]. \]
Arborescent propagation

\[ V \simeq V_T = \{ S_\tau \}^N [V_0] \simeq \{ \tilde{S}_\tau \}^N [V_0] = \sup_{i_N, \ldots, i_1} S_{i_N}^{i_1} \circ \ldots \circ S_{i_1}[V_0]. \]
Arborescent propagation

\[ V \simeq V_T = \{ S_\tau \}^N[V_0] \simeq \{ \tilde{S}_\tau \}^N[V_0] = \sup_{i_N, \ldots, i_1} S_{\tau}^{i_N} \circ \ldots \circ S_{\tau}^{i_1}[V_0]. \]
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Arborescent propagation

\[
V \simeq V_T = \{ S_\tau \}^N[V_0] \simeq \{ \tilde{S}_\tau \}^N[V_0] = \\
\sup_{i_N, \ldots, i_1} S_{i_N}^N \circ \ldots \circ S_{i_1}^1 [V_0].
\]

Computational complexity: \( O(M^N d^3) \) ⇒ curse of dimensionality
Problem

Tree approximation + pruning

Computational complexity: $O(M^N d^3)$

The method has been applied to solve approximately problems of

- dimension $d = 4$, number of switches $M = 3$, in [McEneaney 07]
- dimension $d = 6$, number of switches $M = 6$, in [McEneaney, Deshpande, Gaubert 08] (with a semidefinite programming pruning technique)
- dimension $d = 15$, number of switches $M = 6$, in [Sridharan, James, McEneaney 10] (quantum optimal gate synthesis, $SU(4)$)
Switched infinite horizon optimal control problem

Static HJ equation:

$$H(x, \nabla V) = 0, \quad \forall x \in \mathbb{R}^d; \quad V(0) = 0.$$  

where $H(x, p) = \sup_{m \in \{1, \ldots, M\}} \frac{1}{2} x' D^m x + \frac{1}{2} p' \Sigma^m p + (A^m x)' p.$

Assumption (existence)

$0 \prec D^m \preccurlyeq c_D l_d, \quad 0 \prec \Sigma^m \preccurlyeq c_\Sigma l_d, \quad \forall m$

$x' A^m x \leq -c_A |x|^2, \quad \forall x \in \mathbb{R}^d, \quad \forall m. \quad c_A^2 > c_D c_\Sigma.$

Assumption $\Sigma$:

$\Sigma^m = \Sigma, \quad m = 1, \ldots, M.$

Assumption contraction:

$D^m \succeq m_D l_d, \quad m_D c_\Sigma > (c_A - \sqrt{c_A^2 - c_D c_\Sigma})^2.$
Theorem (Zheng Qu, PhD 2013)

Under Assumption existence and Assumption contraction, the computational complexity to reach an error of order $\epsilon$ is

$$O(M^{-\log(\epsilon)/\epsilon} d^3)$$

Compare with $O(1/\epsilon^{d/r})$ for a grid scheme with an error of order $(\Delta x)^r$. 
Invariant metrics on the cone of positive matrices

- Thompson’s part metric:
  \[ d_T(A, B) = \| \text{spec}(\log B^{-\frac{1}{2}} AB^{-\frac{1}{2}}) \|_\infty, \ A, B \succ 0 \]

  Thompson’s part metric is an invariant Finsler metric:
  \[ d_T(UAU', UBU') = d_T(A, B), \ U \in GL(n) \]

  \[ d_T(A, B) = \inf_\gamma \int_0^1 \| \dot{\gamma}(t) \gamma(t)^{-1} \|_\infty \, dt. \]

- Riemannian metric:
  \[ d_2(A, B) = \inf_\gamma \int_0^1 \| \dot{\gamma}(t) \gamma(t)^{-1} \|_2 \, dt \]

- Standard Riccati operator (flow) is a strict contraction mapping in Riemannian metric ([Bougerol 93]), in Thompson’s part metric ([Liverani and Wojtkowski 94, Lawson and Lim 07]) and in all invariant Finsler metric ([Lee and Lim 07]). (symplectic)
Main ingredient: contraction property of Riccati flow

For all $m \in \{1, \ldots, M\}$, the semigroup $\{S_t^m\}_t$ corresponds to the flow of an indefinite Riccati equation:

$$
\dot{P} = (A^m)'P + PA^m + D^m + P\Sigma^m P.
$$

(3)

Theorem (Indefinite Riccati flow is a strict local contraction)

Under Assumption existence and Assumption contraction, there is $P_0 \succ 0$ and $\alpha > 0$ such that for all solutions $P_1(\cdot), P_2(\cdot) : [0, T] \to (0, P_0)$ of the indefinite Riccati flow (3) we have:

$$
d_T(P_1(t), P_2(t)) \leq e^{-\alpha t} d_T(P_1(0), P_2(0)), \ \forall t \in [0, T].
$$
Introduction

Curse of dimensionality is unavoidable

Qu’s error bound $O(M^{-\log(\epsilon)/\epsilon} d^3)$ shows that for fixed $\epsilon$, execution time is polynomial in $d$. However, we recover a curse of dimensionality, when $\epsilon \to 0$.

Theorem (coro of Grüber, polyhedral approximation of convex bodies)

The minimal number of affine minorant functions to approximate a $C^2$ convex function $f : \mathbb{R}^d \to \mathbb{R}$ is equivalent to:

$$\frac{C}{\epsilon^{d/2}} , \quad \text{as} \quad \epsilon \to 0 .$$
Current bottleneck: pruning representation. Given

\[ f = \sup_{i \in [p]} \phi_i, \quad \phi_i \text{ quadratic } \mathbb{R}^d \rightarrow \mathbb{R} \]

and \( k \ll p \), find \( I \subset [p], |I| = k \), with a best approximation of \( f \) by

\[ \sup_{i \in I} \phi_i. \]

Heuristics, SDP relaxations, reduction to a discrete facility location problem (curse of dim dependent).
Coming back to the first exercise

\[ A_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix}, \]

Eigenvalues? \( \varepsilon \to 0 \)
Coming back to the first exercise

\[ A_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix}, \]

Eigenvalues? \( \varepsilon \to 0 \)

\[ \mathcal{L}_\varepsilon^1 \sim \varepsilon^{-1/3}, \; \mathcal{L}_\varepsilon^2 \sim j\varepsilon^{-1/3}, \; \mathcal{L}_\varepsilon^3 \sim j^2\varepsilon^{-1/3}. \]
Coming back to the first exercise

\[ A_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix}, \]

Eigenvalues? \( \varepsilon \to 0 \)

\[ L_\varepsilon^1 \sim \varepsilon^{-1/3}, \quad L_\varepsilon^2 \sim j\varepsilon^{-1/3}, \quad L_\varepsilon^3 \sim j^2\varepsilon^{-1/3}. \]

Answer without computation using tropical algebra.
Give $A \in \mathbb{R}^{n \times n}_{\text{max}}$, $\lambda$ is a **geometric eigenvalue** if

$$Au = \lambda u, \quad u \in \mathbb{R}^n_{\text{max}} \setminus \{ "0" \}$$

$\lambda$ is an **algebraic eigenvalue** if

"$\det(A - \lambda I) = 0$"

meaning that $\lambda$ is a nondifferentiability point of value of parametric optimal assignment problem

$$t \mapsto \max_\sigma \sum_i M_{i,\sigma}(t),$$

$M(t) = "A + t I")$, $M_{ij} = A_{ij}$, $M_{ii} = \max(A_{ii}, t)$.
Theorem (Max-plus spectral theorem, Cuninghame-Green, 61, Gondran & Minoux 77, Cohen et al. 83)

Assume $G(A)$ is strongly connected. Then

- the eigenvalue is unique:

$$\rho_{\text{max}}(A) := \max_{i_1, \ldots, i_k} \frac{A_{i_1 i_2} + \cdots + A_{i_k i_1}}{k}$$

- Assume WLOG $\rho_{\text{max}}(A) = 0$, then, $\exists \alpha_j \in \mathbb{R} \cup \{-\infty\}$, $u = \max_{j \in \text{maximizing circuits}} \alpha_j + A^*_j$,

$$A^*_j := \max \text{ weight path arbitrary length } i \rightarrow j.$$
The first exercise solved

\[ A_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & -4 \\ -\infty & -1 & 2 \\ -1 & -2 & -\infty \end{bmatrix}. \]

We have \( \lambda = 1/3 \), corresponding to the critical circuit:

\begin{center}
\begin{tikzpicture}
    \node (1) at (0,0) {1};
    \node (2) at (2,0) {2};
    \node (3) at (4,0) {3};
    \draw (1) edge[->] node[above] {0} (2);
    \draw (2) edge[->] node[above] {2} (3);
    \draw (3) edge[->, bend left] node[below] {-1} (1);
\end{tikzpicture}
\end{center}

Eigenvalues:

\[ \mathcal{L}_\varepsilon^1 \sim \varepsilon^{-1/3}, \mathcal{L}_\varepsilon^2 \sim j\varepsilon^{-1/3}, \mathcal{L}_\varepsilon^3 \sim j^2\varepsilon^{-1/3}. \]

Akian, Bapat, SG, CRAS 04, generalizes Lidski’s theorem.
One can show eigenspaces are isomorphic to spaces of Lipschitz functions (wrt to non symmetric metrics).

Relation with horoboundaries of metric spaces. The tropically extreme Lipschitz functions are Busemann points (limits of geodesics).
Concluding remarks

- All the convexity you like works: Helly, Carathéodory, Radon, Tverberg, Double Description, Hahn-Banach, Krein-Milman, Choquet, . . .
- Some combinatorial aspects (counting extreme points and faces) are not understood.
- Complexity of trop LP = complexity of mean payoff games (is it polynomial?)
- useful: metric estimates of amoebas, bounds for matrix eigenvalues, scaling in matrix analysis
- emerging max-plus curse of dimensionality attenuation for HJ equation, open question: extension to stochastic control.
Introduction

Tropical problems are simpler (combinatorial)

...but not too simple.

Much remains to do ...

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