Normal and curvature estimation using the k-Voronoi covariance measure

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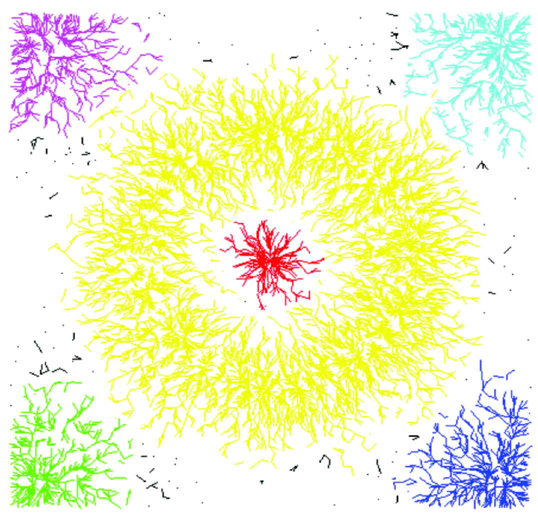
Geometric inference

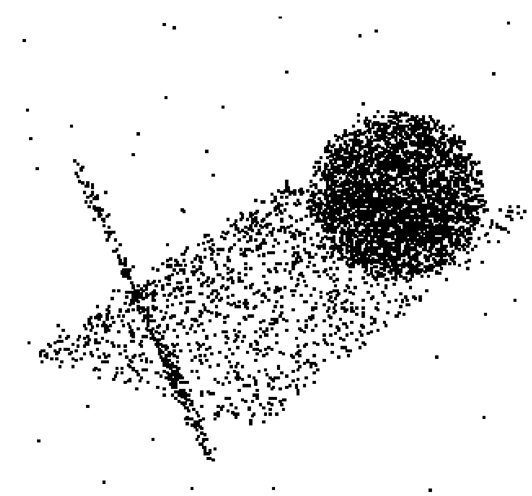
Given : — An unknown object K (compact set) of $\mathbb{R}^d$

— A finite set of points $P \subseteq \mathbb{R}^d$ approximating K.

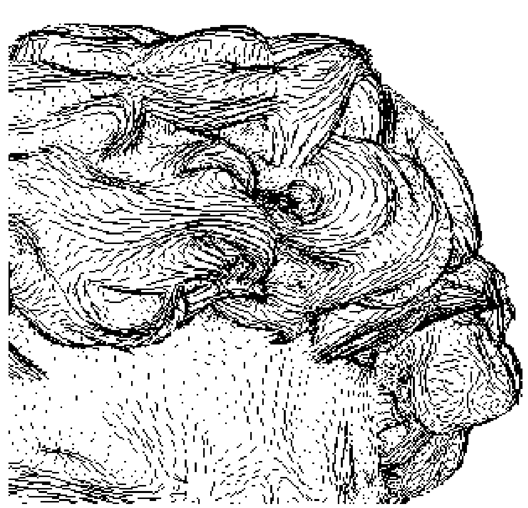
What can we say about the geometry and the topology of K?



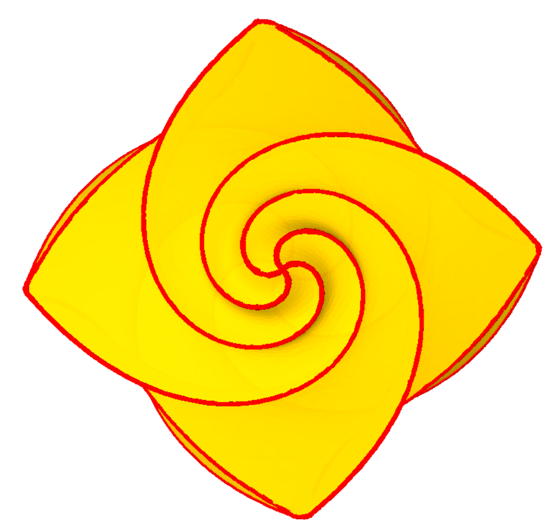
#connected components



intrinsec dimension



curvature



sharp feature

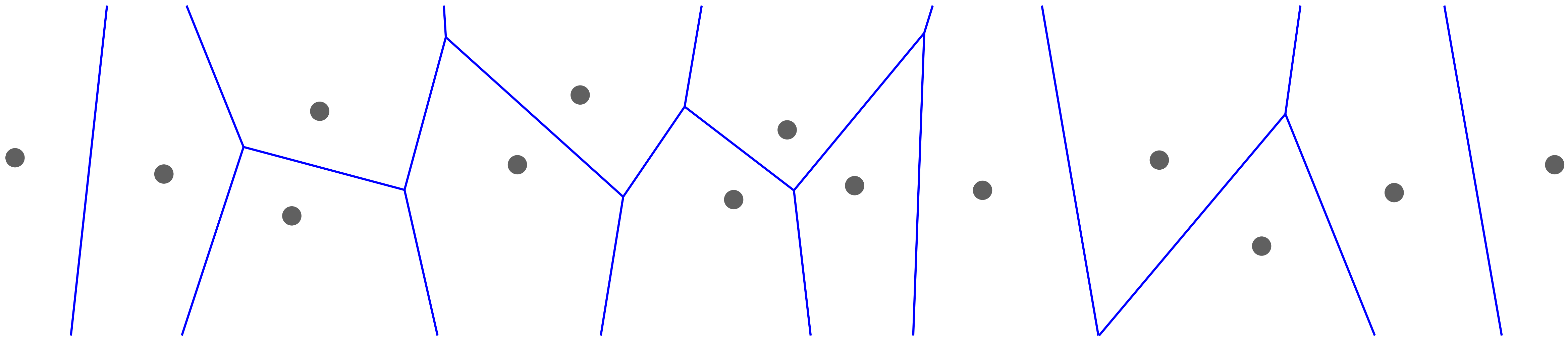
Previous work

1. Voronoi-based algorithms :

– Amenta et Bern, 1999, Surface reconstruction by Voronoi filtering

– Cohen-Steiner et al., 2007,Voronoi-based Variational Reconstruction

– G. M. O., 2009,Robust Voronoi-based Curvature and Feature Estimation



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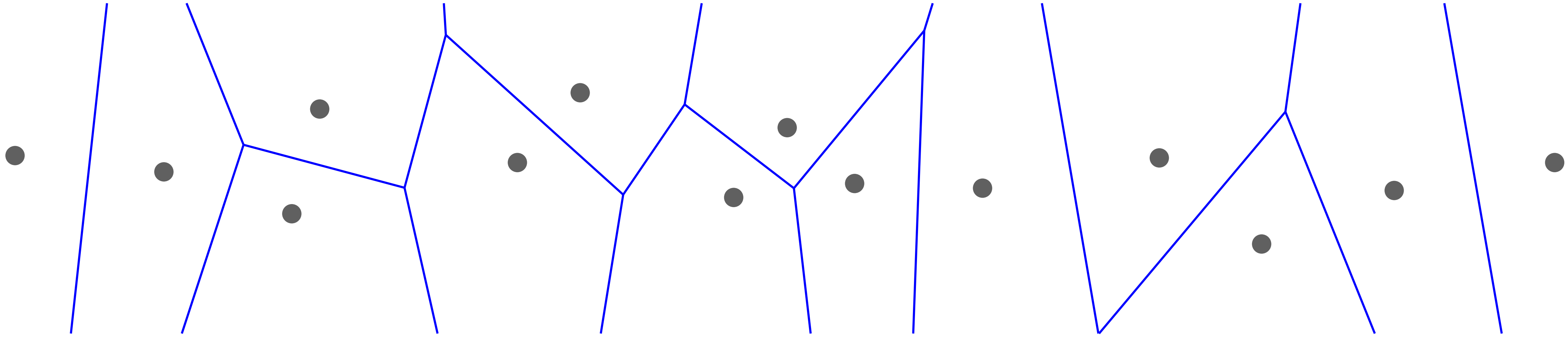

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– Chazal, Cohen-Steiner, Merigot, 2010, Geometric Inference for probability measure

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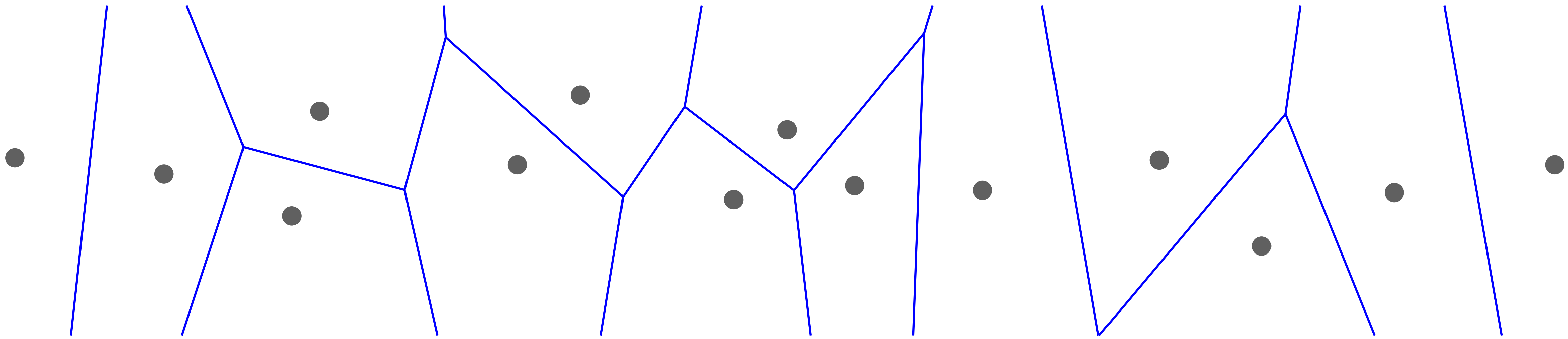

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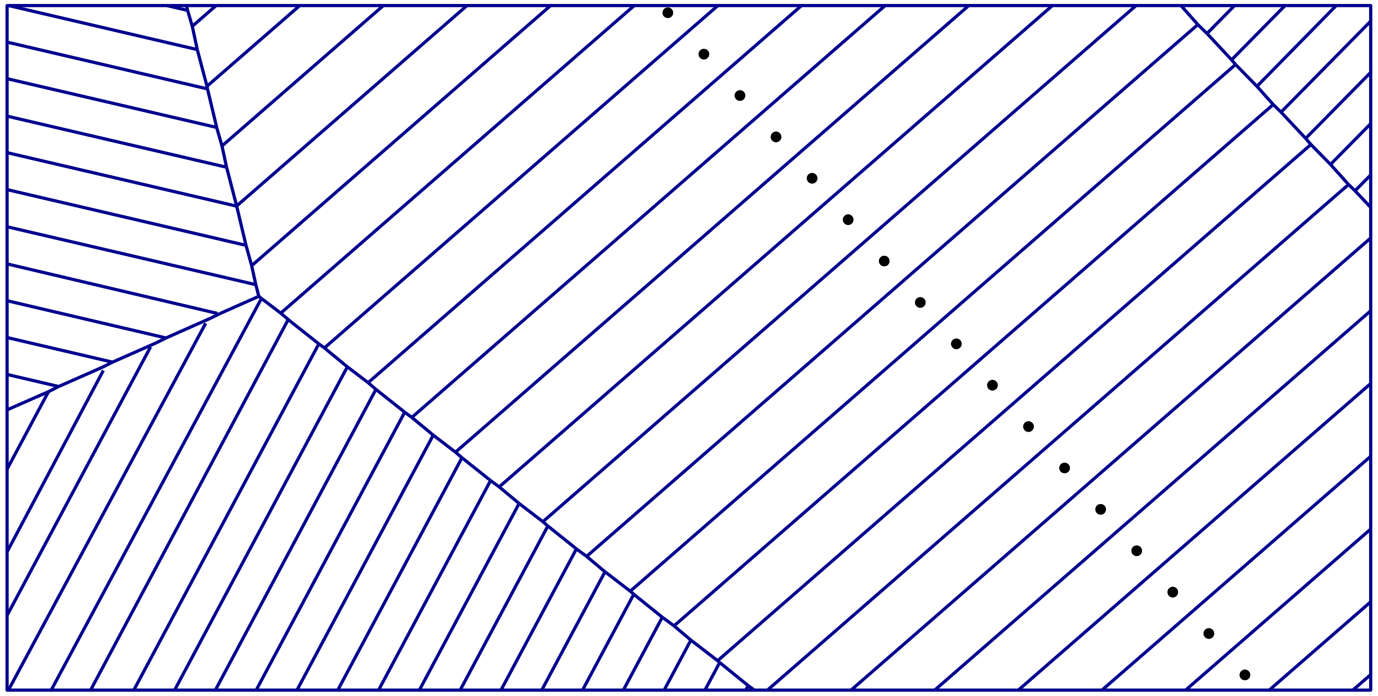
our approach $= 1 + 2$

Normal estimation based on Voronoi



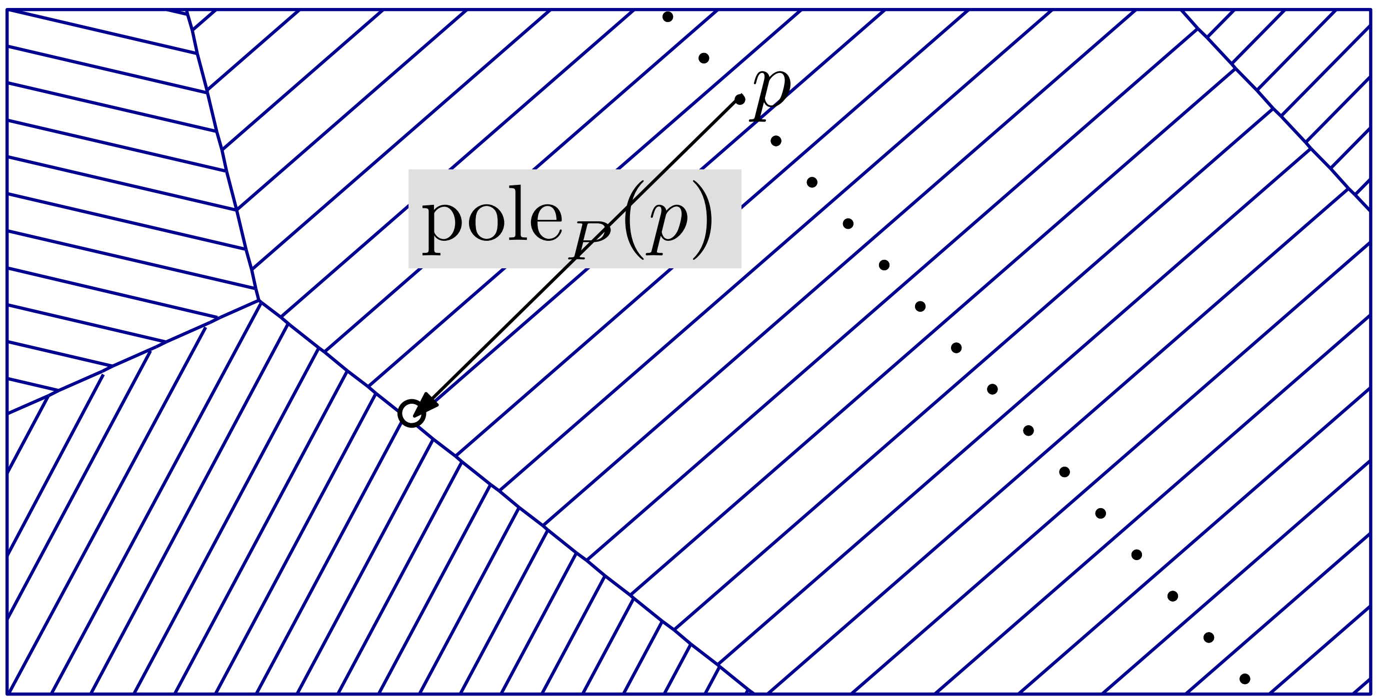
$$P = \{p_1, \ldots, p_N\} \subseteq \mathbb{R}^d$$

Definition :

Voronoi cell: $\mathrm{Vor}^P(q) = \{$ points whose closest point in P is $q\}$

Normal estimation based on Voronoi

Poles' method



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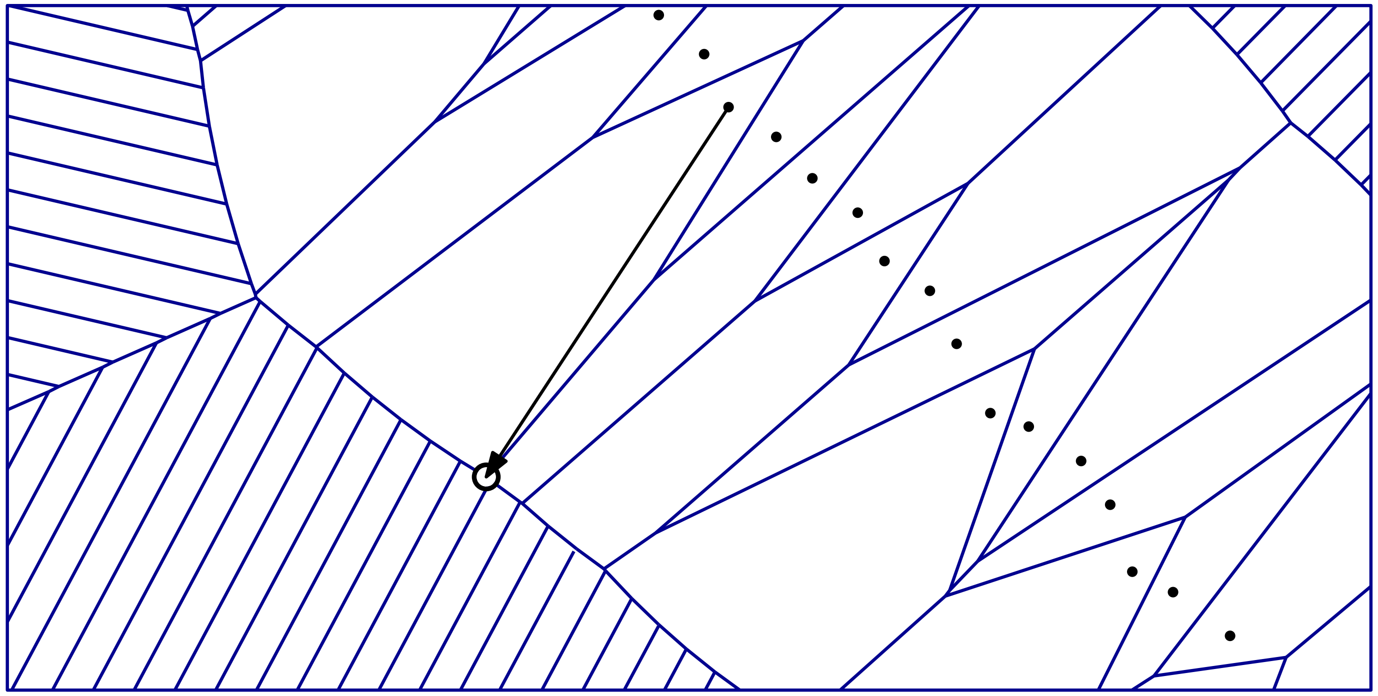

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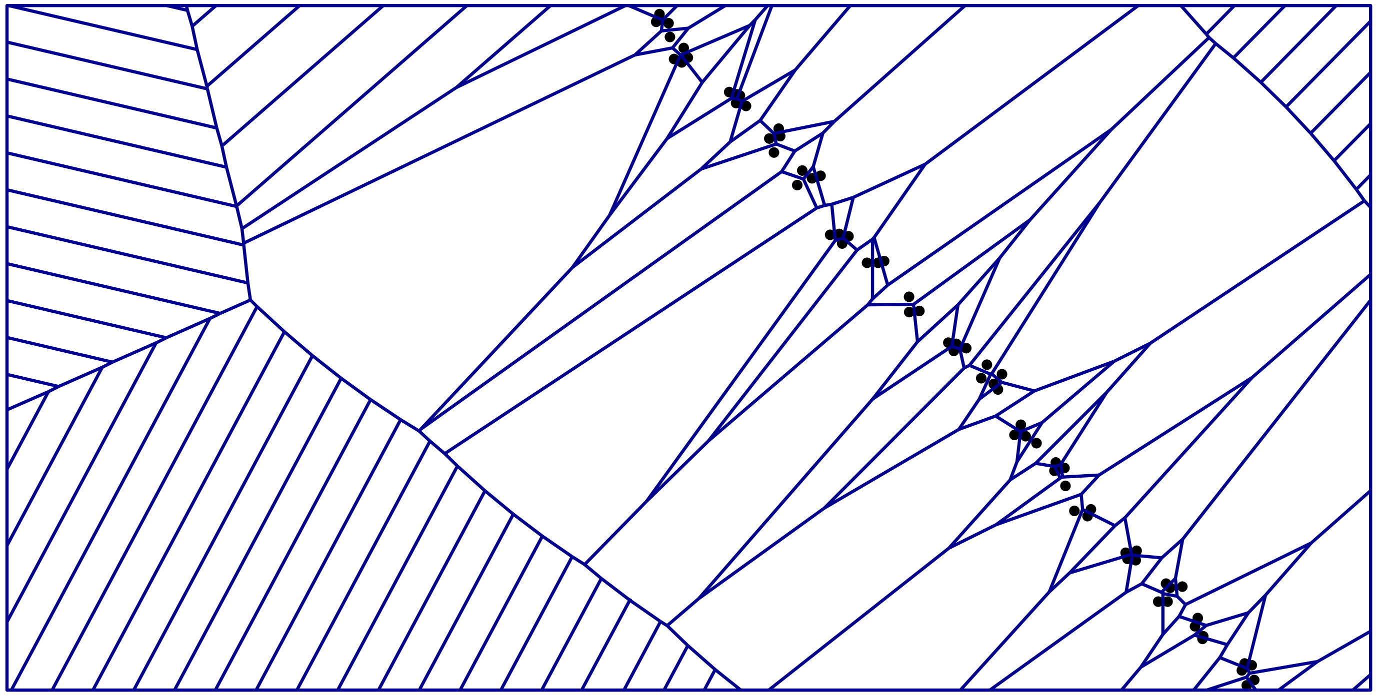

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Idea : integrate to obtain stability.

Voronoi Covariance

Alliez, Cohen-Steiner, Tong, Desbruns, Proc. Symposium Geometry Processing 2007

Covariance Matrix: $\mathrm{cov}_p(\Omega) := \int_\Omega (x-p)(x-p)^{\mathbf{t}} \,\mathrm{d}\, x.$

Eigenvectors of $\mathrm{cov}_p(\Omega)$ are the **principal axes** of Ω (viewed from p).

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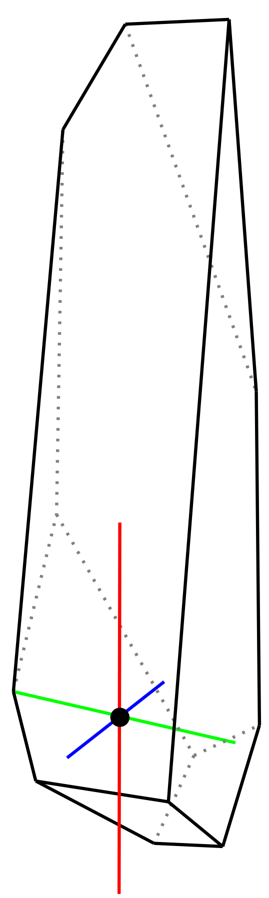
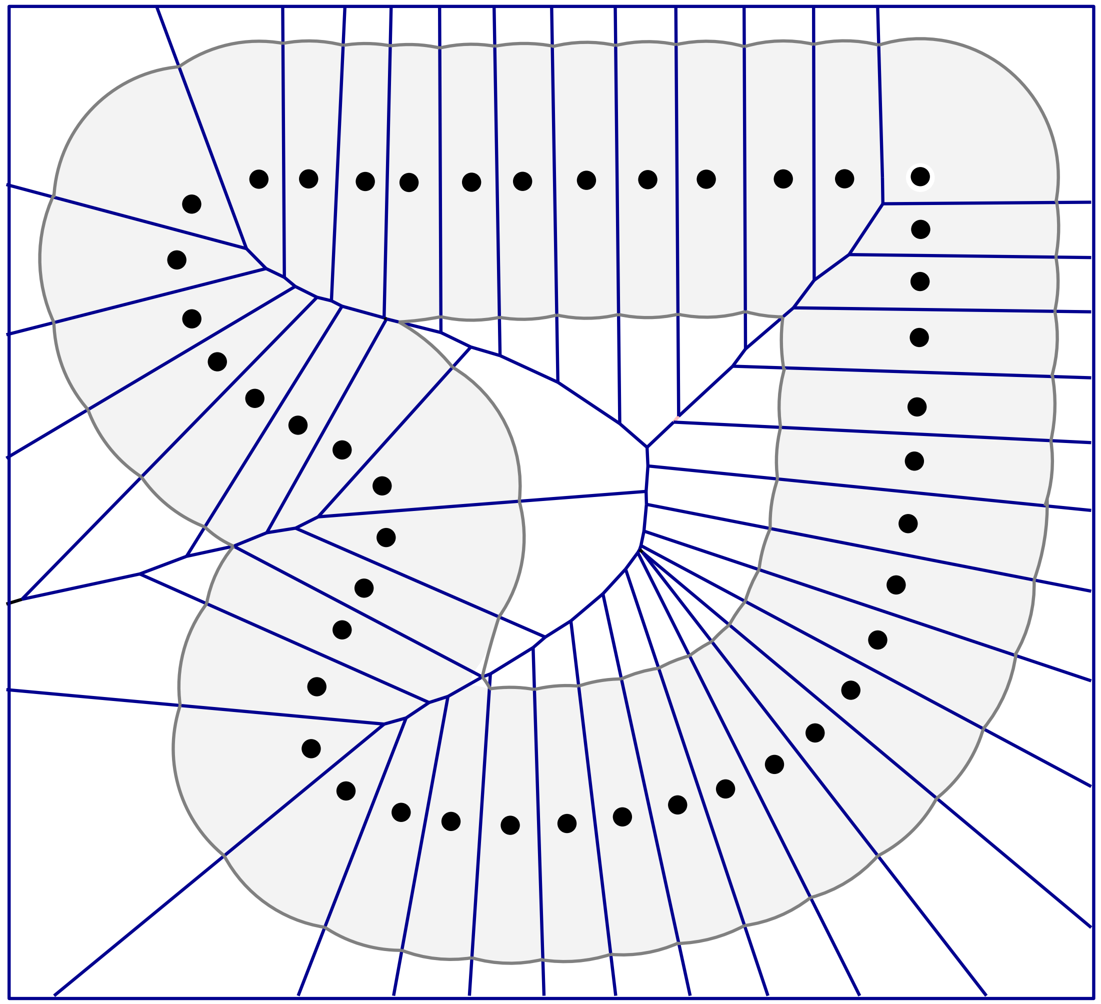
Algorithm:



- They take : $\Omega = \mathrm{Vor}_P(p_i) \cap E$
- The normal is estimated by the eigenvector corresponding to the largest eigenvalue (in red).
- Stability to noise : Sum matrices over a neighborhod.

Voronoi covariance measure



Definition: Offset of P of radius R : $P^R = \cup_{p \in P} \mathrm{B}(p, R)$.

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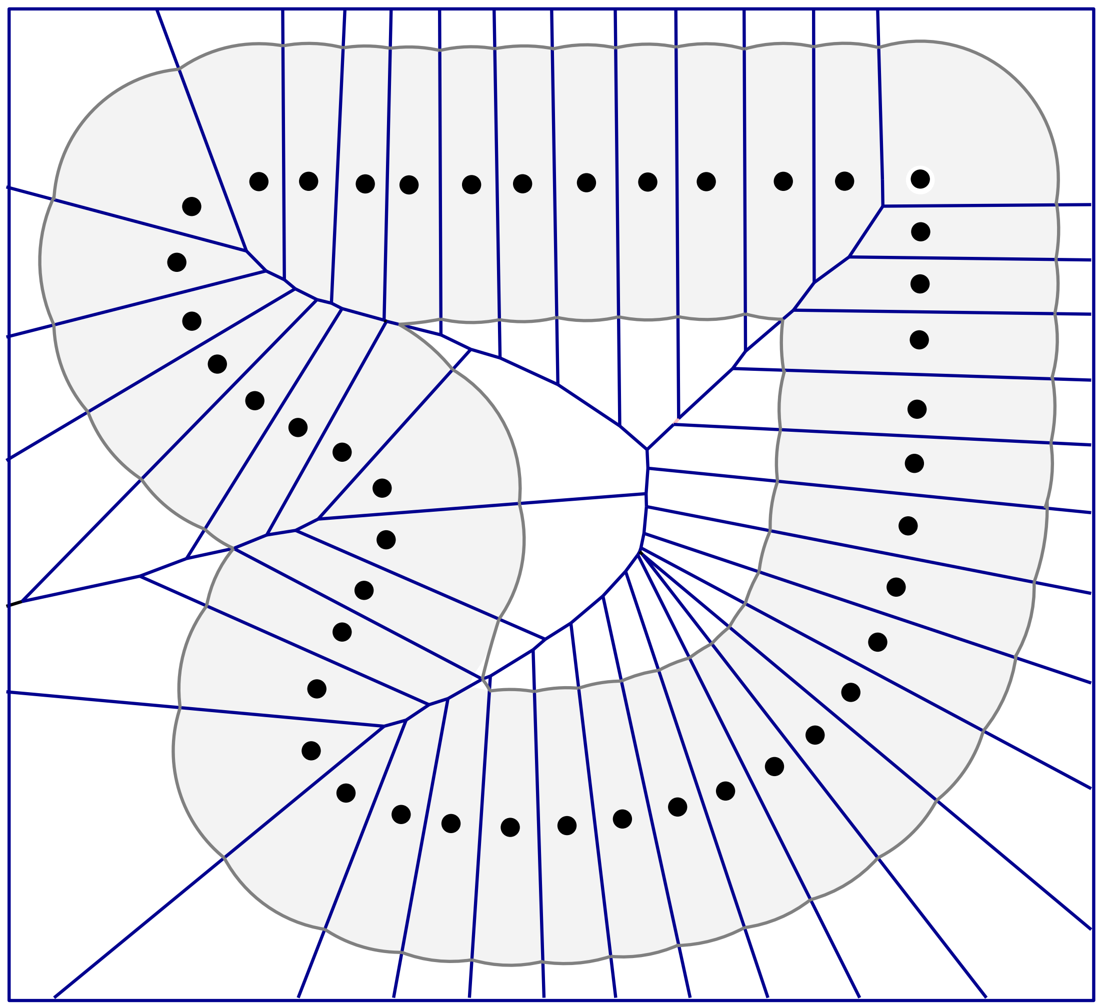

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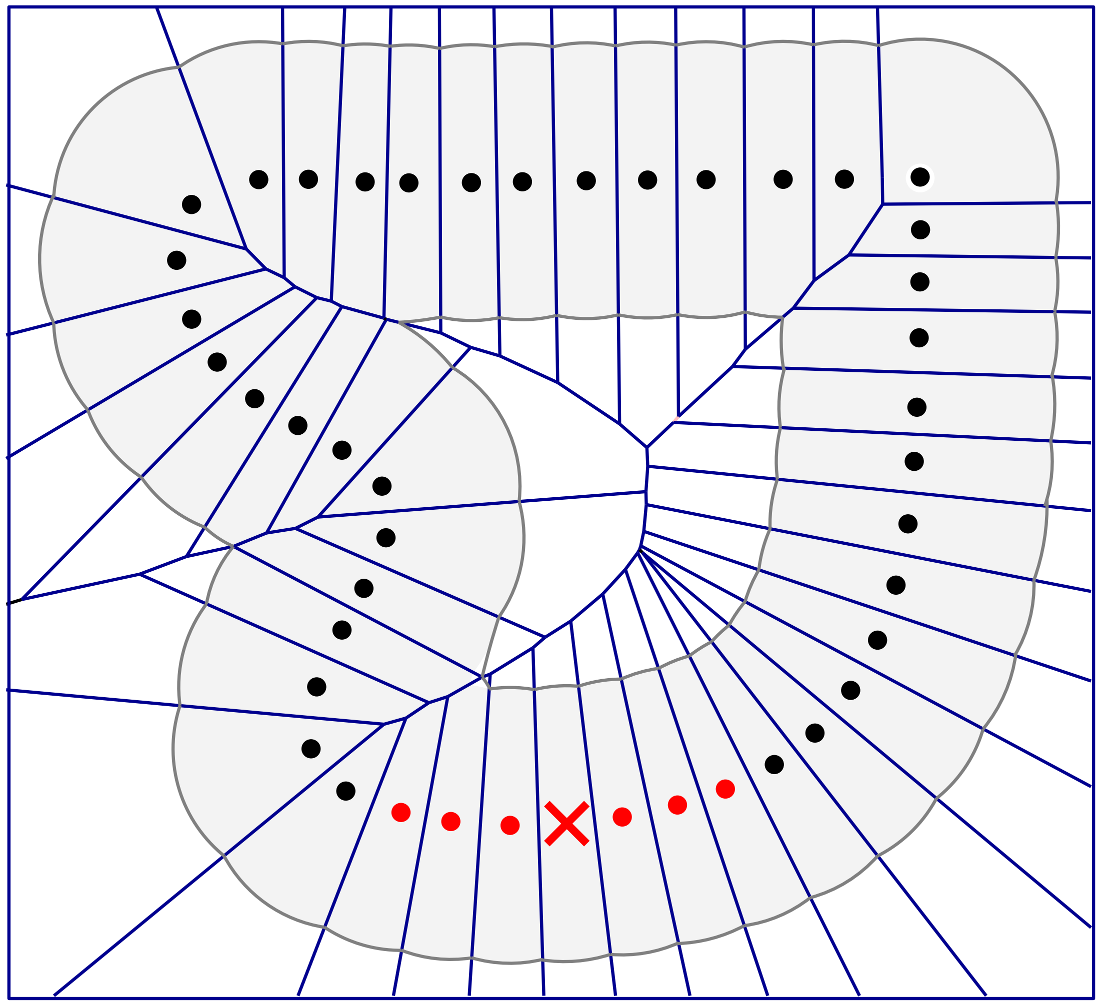
$$A(p) := \mathrm{cov}_p(\mathrm{Vor}_P(p_i) \cap P^R)$$

Definition: The *Voronoi covariance measure* of P of offset radius R is :

$$\mathcal{V}(P, R) := \sum_{i=1}^{N} A(p)\delta_p$$

Voronoi covariance measure



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The VCM is defined for all compacts.

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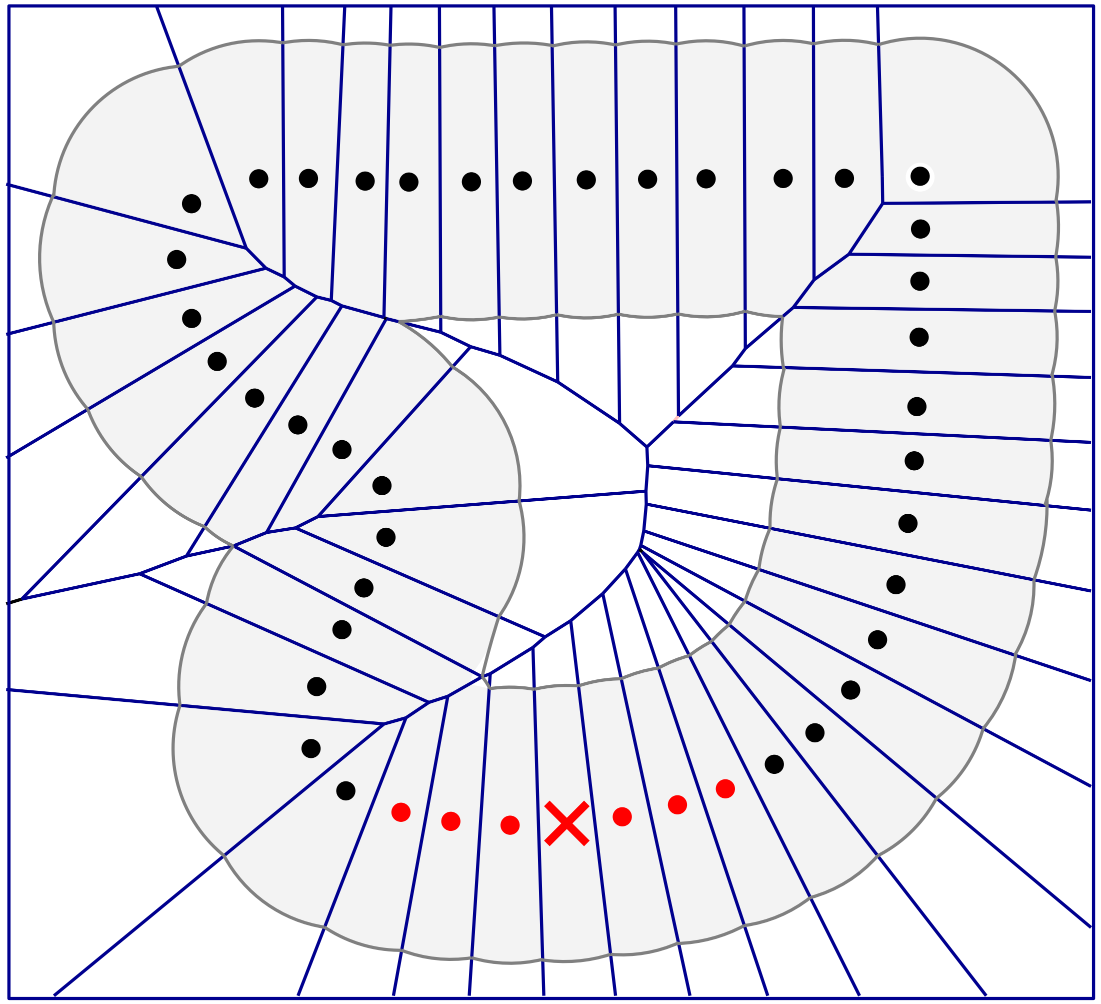

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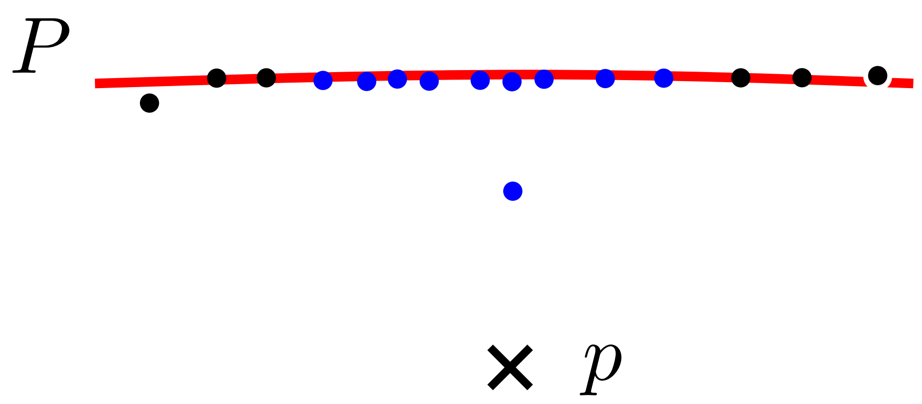
The VCM is defined for all compacts.

Theorem: Let P, K be two compacts and $p \in \mathbb{R}^d$.

$$\|\mathcal{V}(P, R) * \chi_r(p) - \mathcal{V}(K, R) * \chi_r(p)\| = \mathrm{O}(\mathrm{d_H}(K, P)^{\frac{1}{2}})$$

Guibas, Mérigot, Ovsjanikov, 2009

Distance to a measure



Definition: The distance to a measure of parameter k (or k-distance) :

$$d^2_{k,P}(p) = \frac{1}{k} \sum_{p_i \in NN_k(p)} ||p - p_i||^2$$

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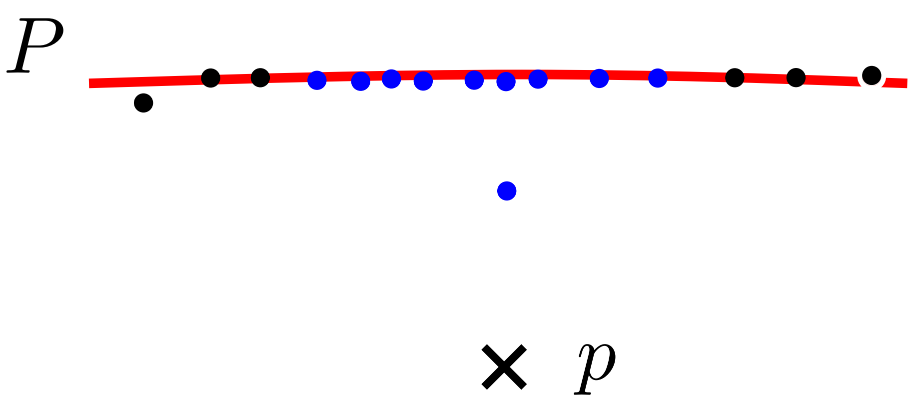

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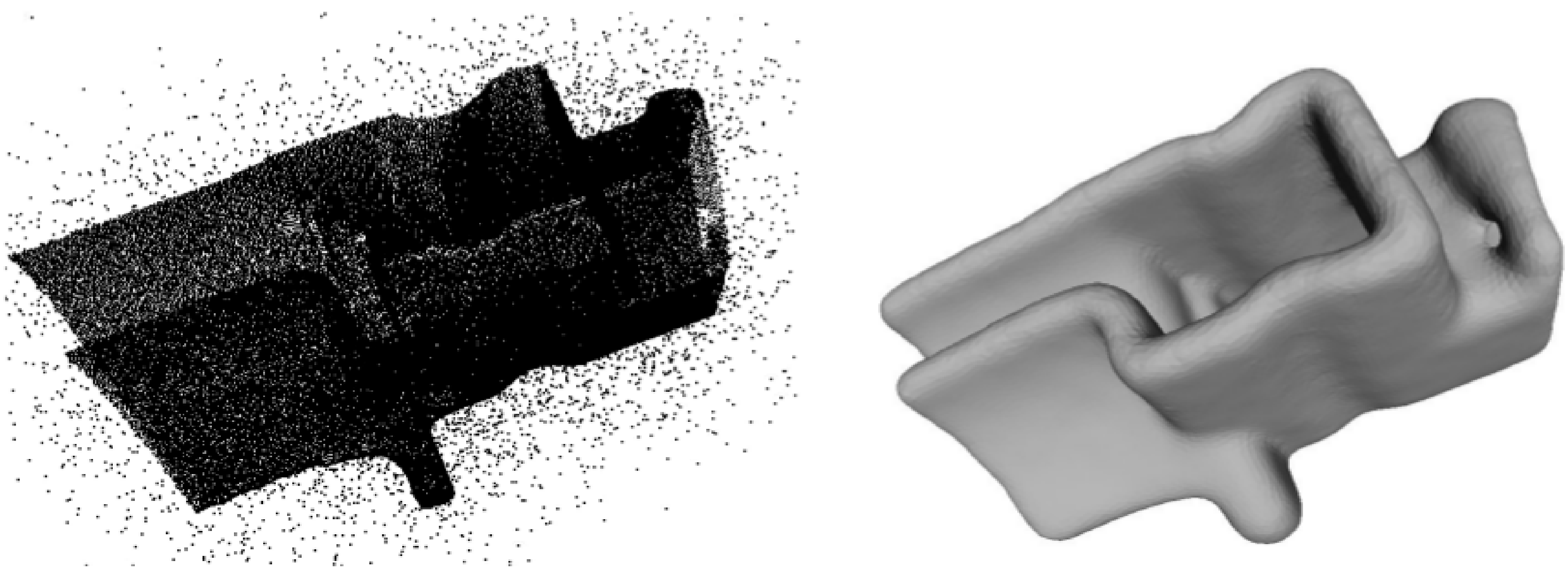

Theorem : Let P, K be two compact sets and $\mathbb{R}^d$.

$$\|d_{P,k} - d_{K,k}\| \leq \frac{1}{\sqrt{k}} W_2(\mu_P, \mu_K)$$

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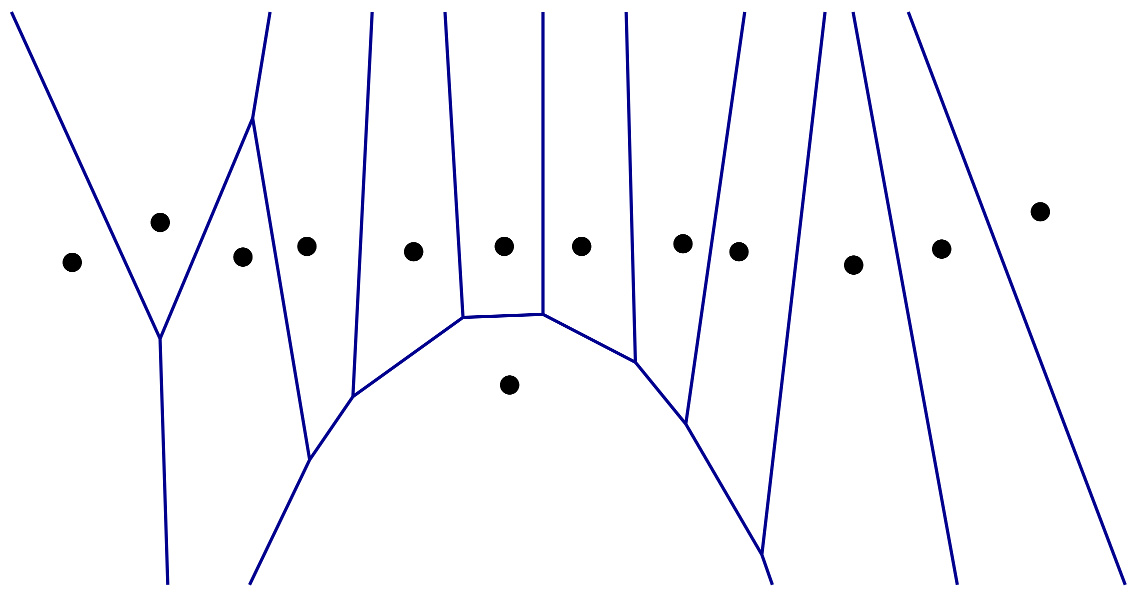
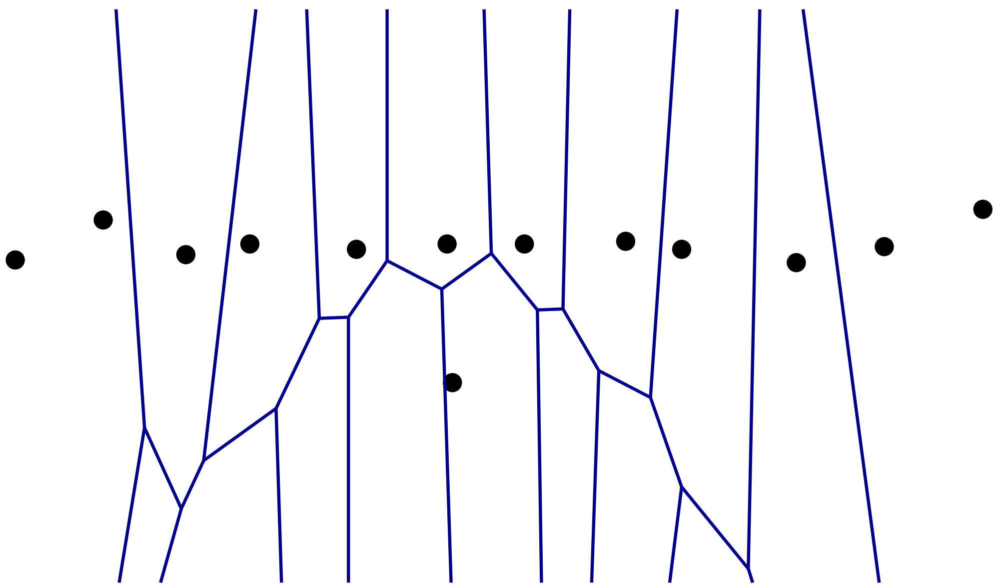
k-Voronoi covariance measure

VCM



k-VCM



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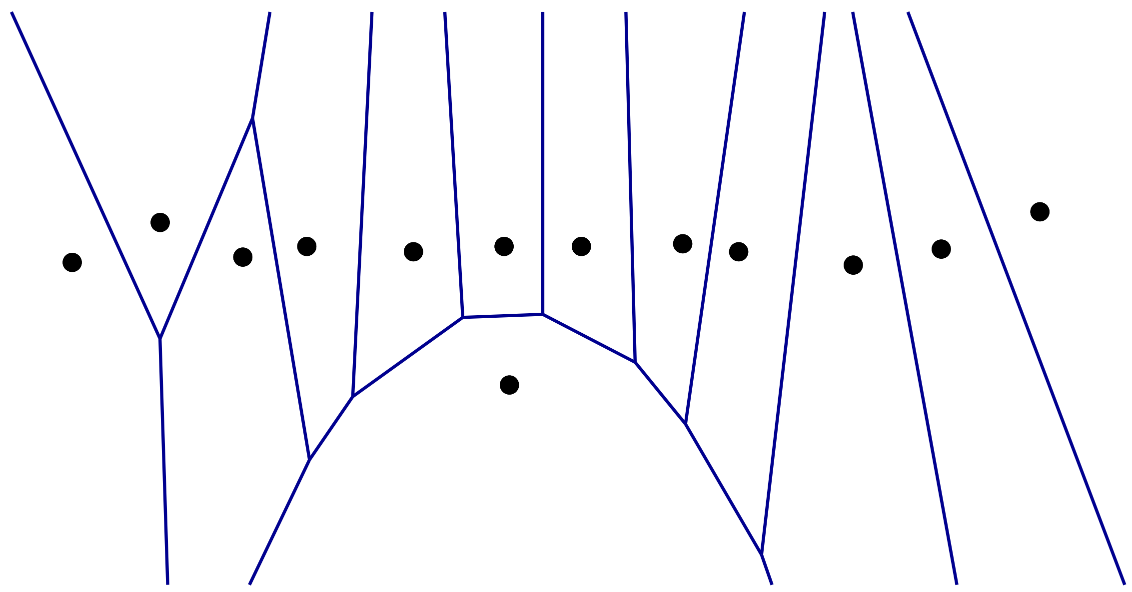

A(p):=

$$\int_{\mathrm{Vor}_P(p)\cap P^R} (x-\pi(x))(x-\pi(x))^{\mathbf{t}}\,\mathrm{d}\,x.$$

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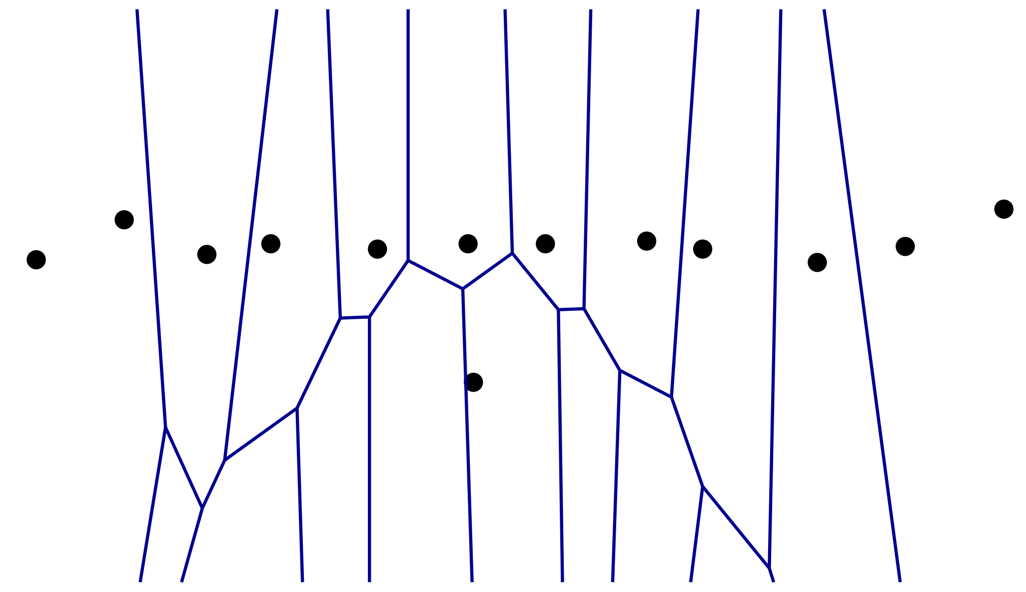

$A_k(p) :=$

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k-Voronoi covariance measure

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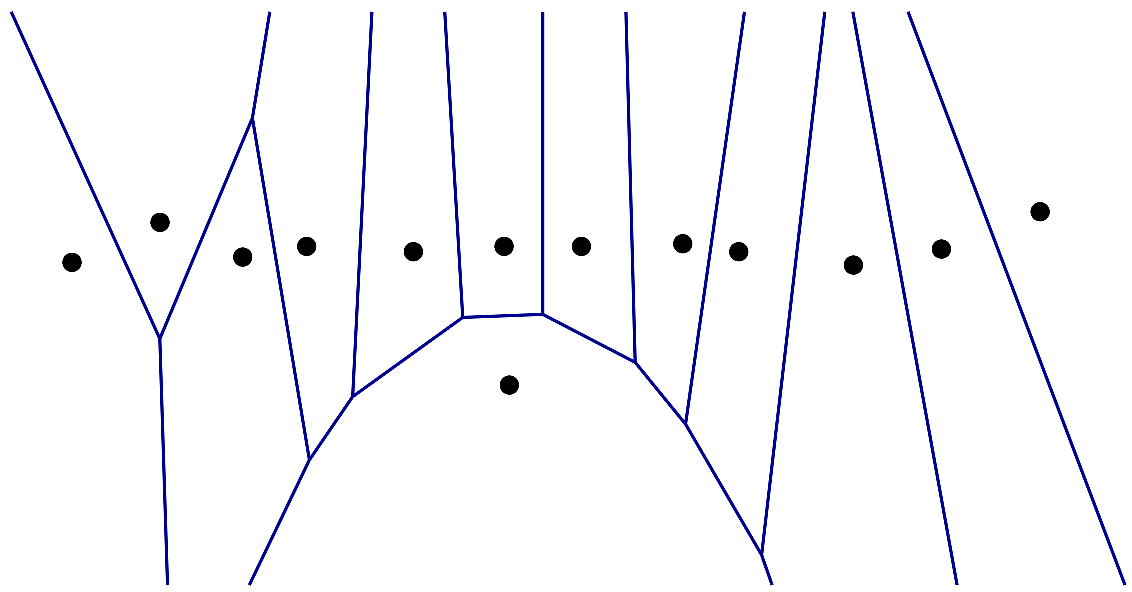

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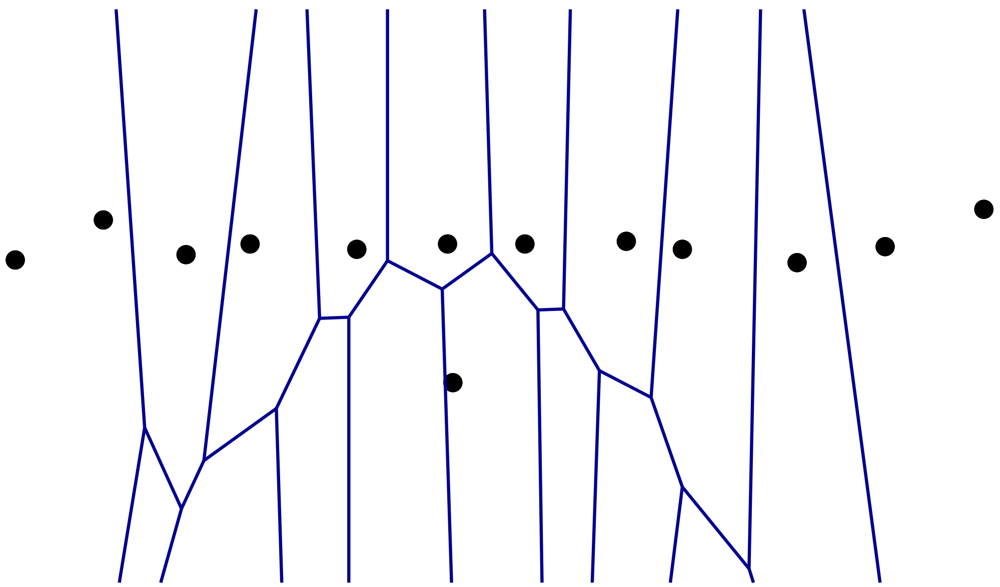

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Idea of the proof

$$\int_{\mathrm{Vor}_k^P(p)\cap d_{P,k}^{-1}([0,R])}(x-\pi_k^P(x))(x-\pi_k^P(x))^{\mathbf{t}}\,\mathrm{d}\,x-\int_{\mathrm{Vor}^K(p)\cap K^R}(x-\pi^K(x))(x-\pi^K(x))^{\mathbf{t}}\,\mathrm{d}\,x$$

- Control the symetric difference of the two integration domains :

$$\left[\mathrm{Vor}_k^P(p)\cap d_{P,k}^{-1}([0,R])\right]\triangle\left[\mathrm{Vor}^K(p)\cap K^R\right]=O(\|d_{P,k}-d_K\|_\infty)$$

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- Control of $A_P(x)-A_K(x)$:

$$\|A_P(x)-A_K(x)\|\leq(kR^2+2R)\|\pi_k^P(x)-\pi^K(x)\|$$

- **Theorem :** f and g locally convex on E s.t. $diam(\nabla f(E)\cup\nabla g(E))$ is bounded, then

$$\|\nabla f-\nabla g\|_{L_1(E)}=O(\|f-g\|_\infty^{\frac{1}{2}})$$

Chazal, Cohen-Steiner, Mérigot, (2010)

Federer, Curvature measure (1959)

Generalization : δ-VCM

Definition: Let $\delta : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a 1-lipschitz function.
δ is a distance-like function if $\psi_\delta(x) = ||x||^2 - \delta^2(x)$ is convexe.

Exemple: d_K and $d_{k,K}$ are distance-like.

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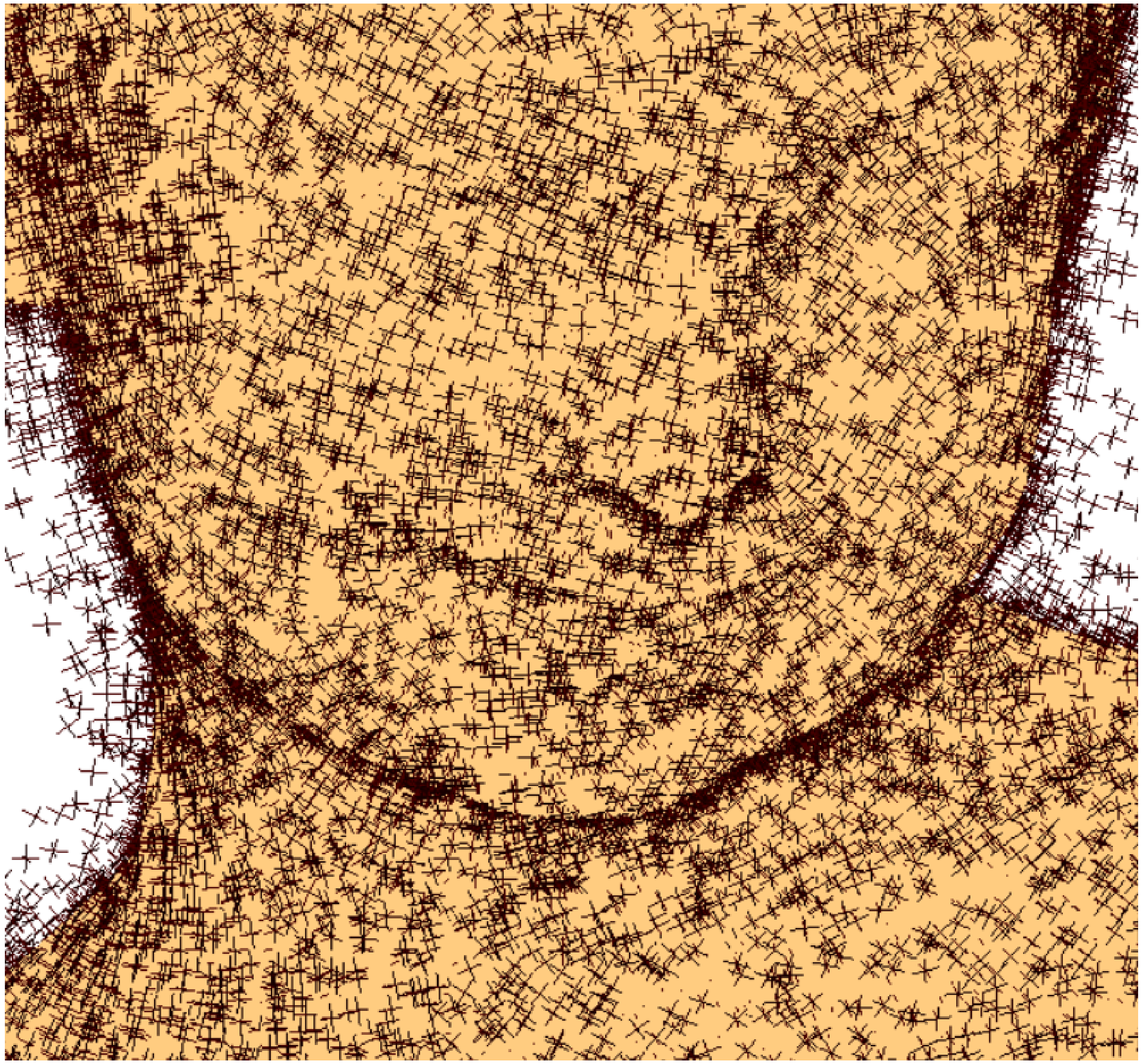
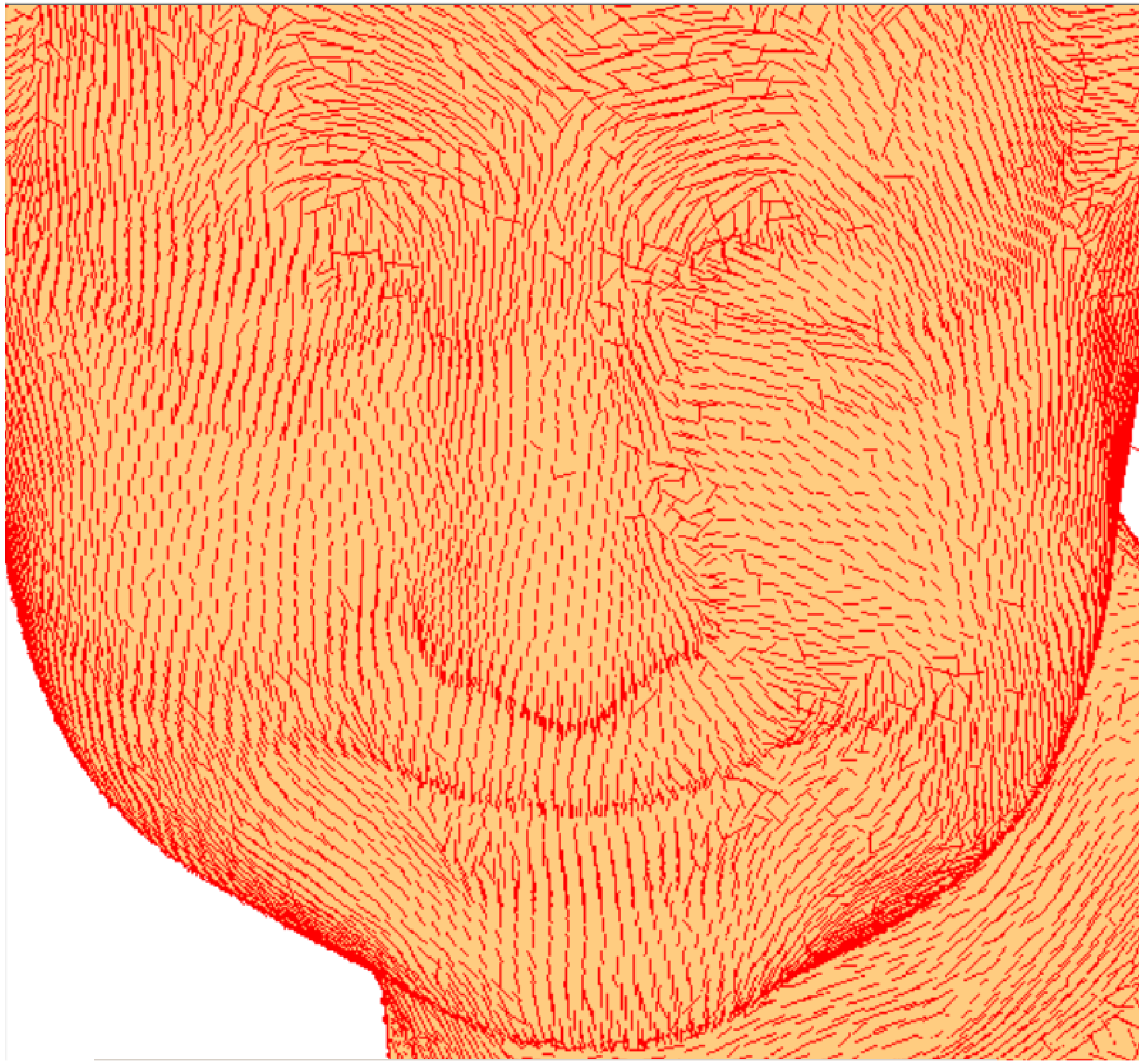
curvature direction estimation

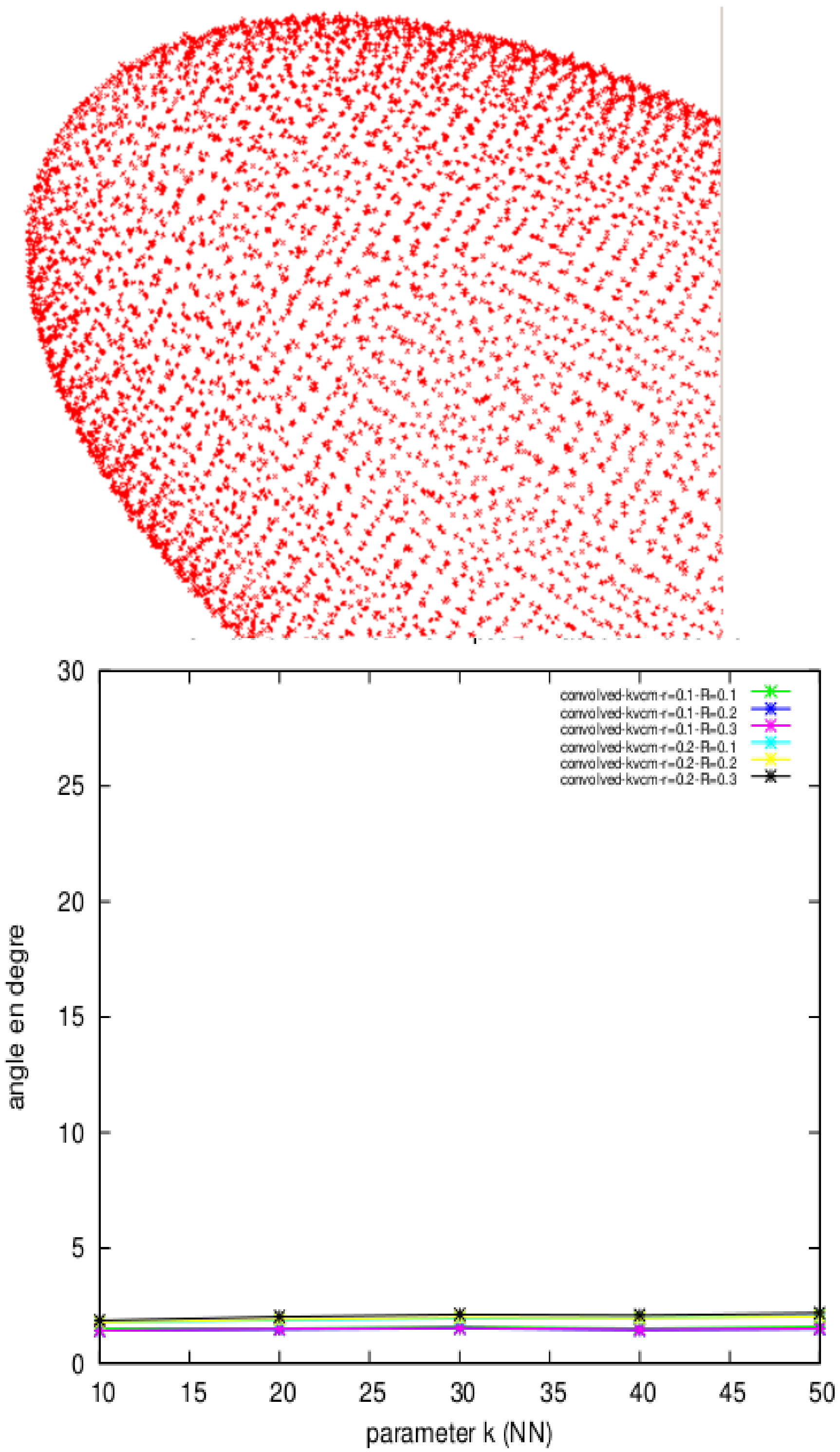




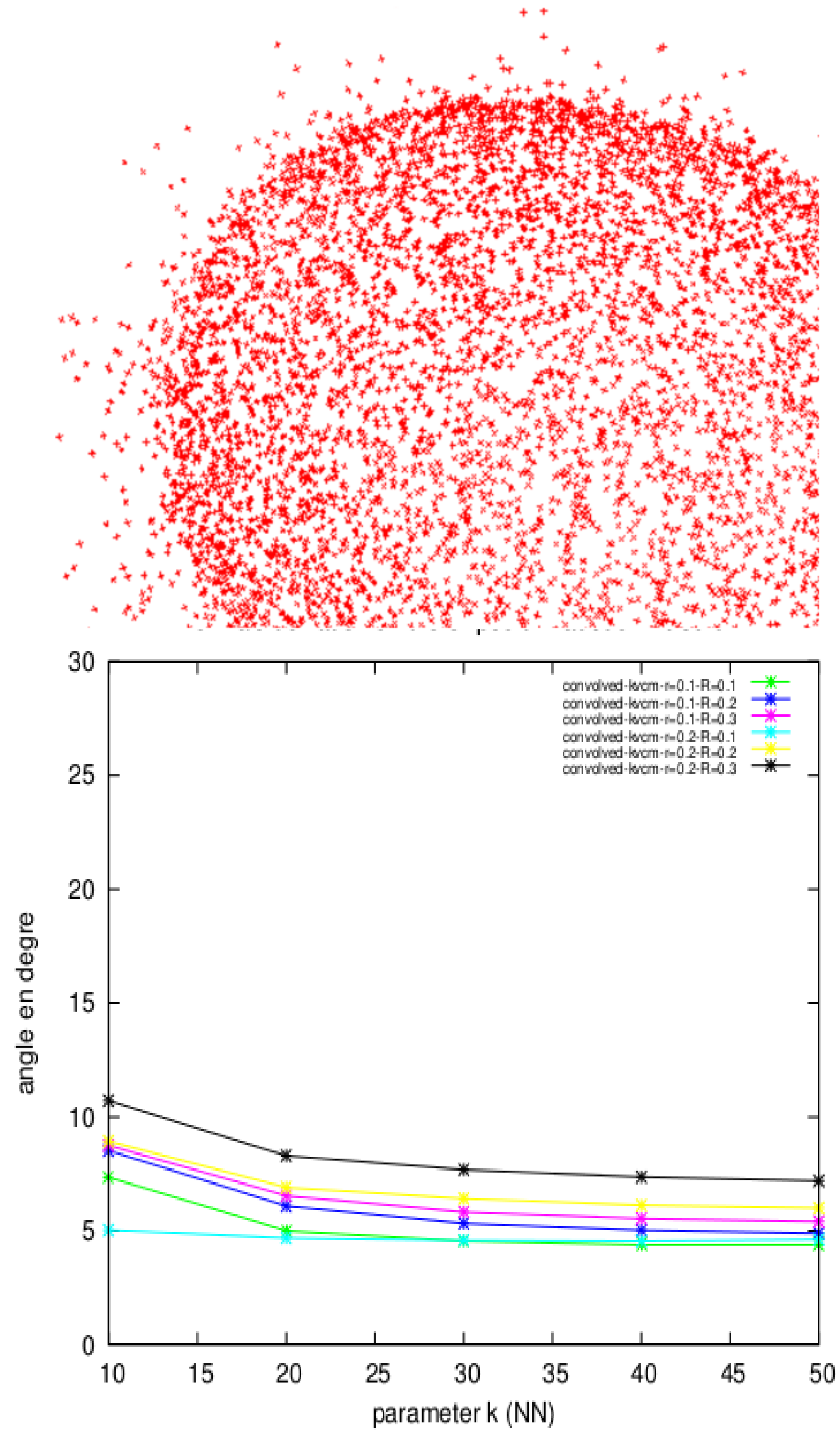
Bimba 100k points

Normal estimation : Ellipsoid

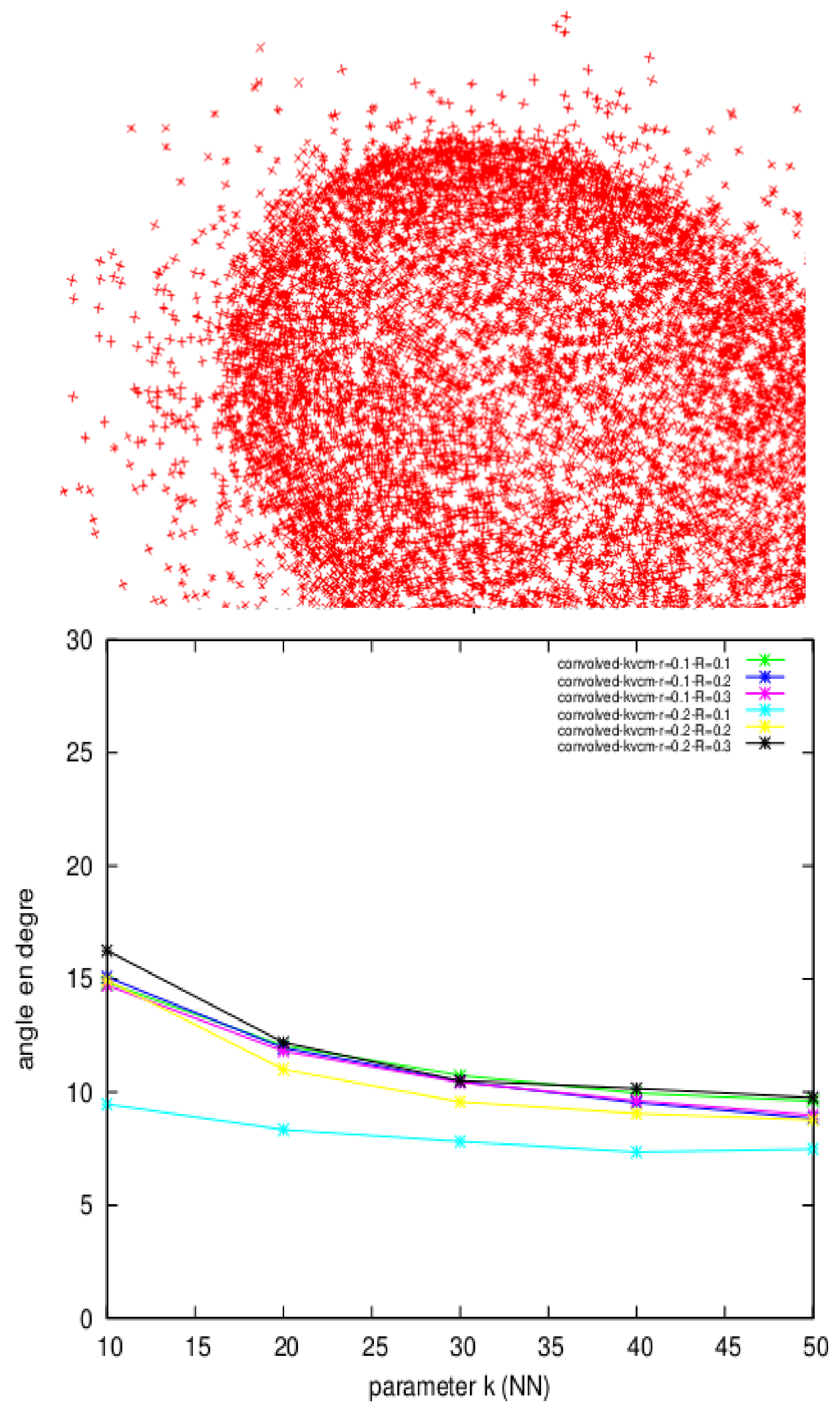
Sensibility to parameters.



$\varepsilon = 0$



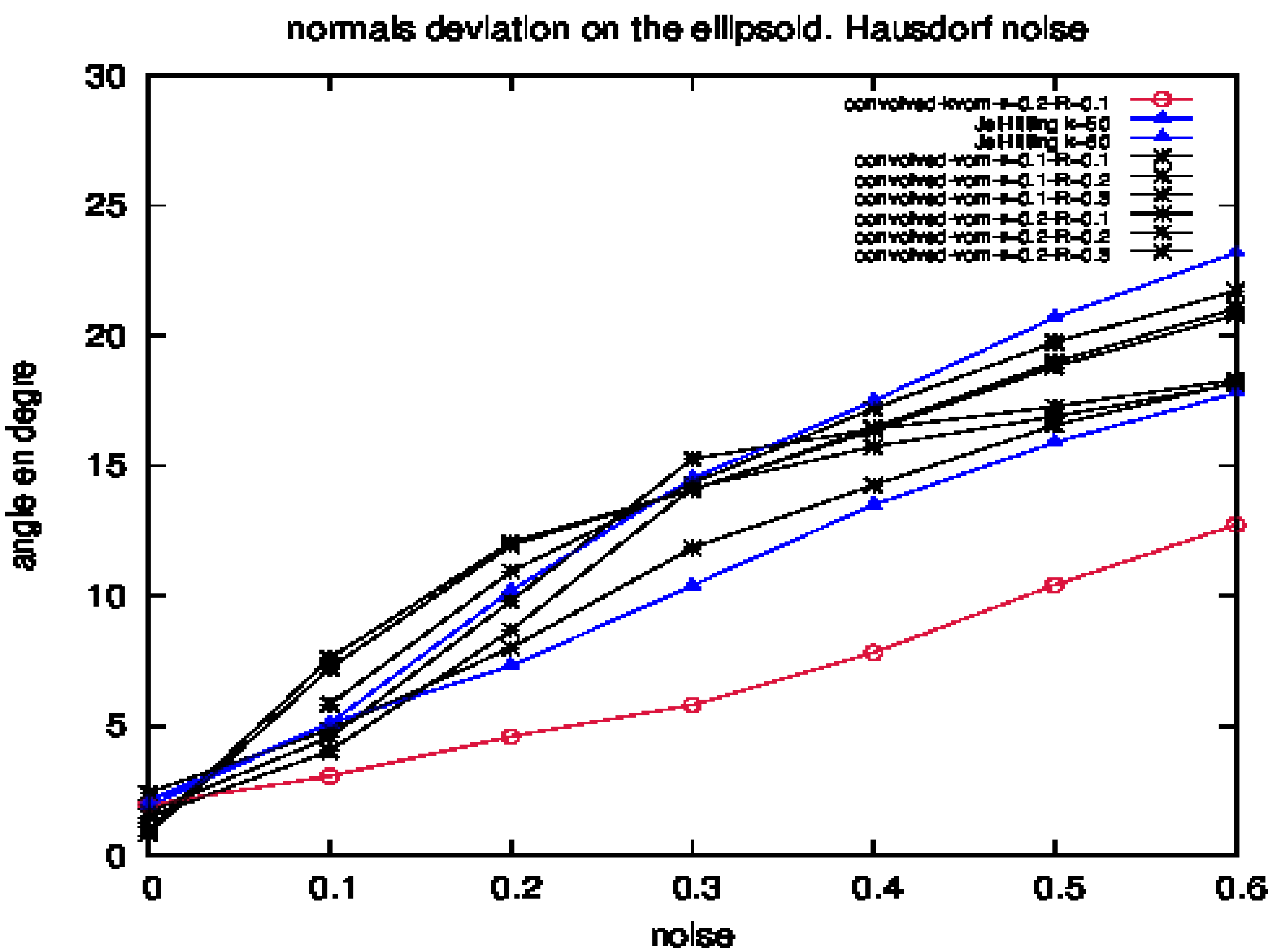
$\varepsilon = 0.2$



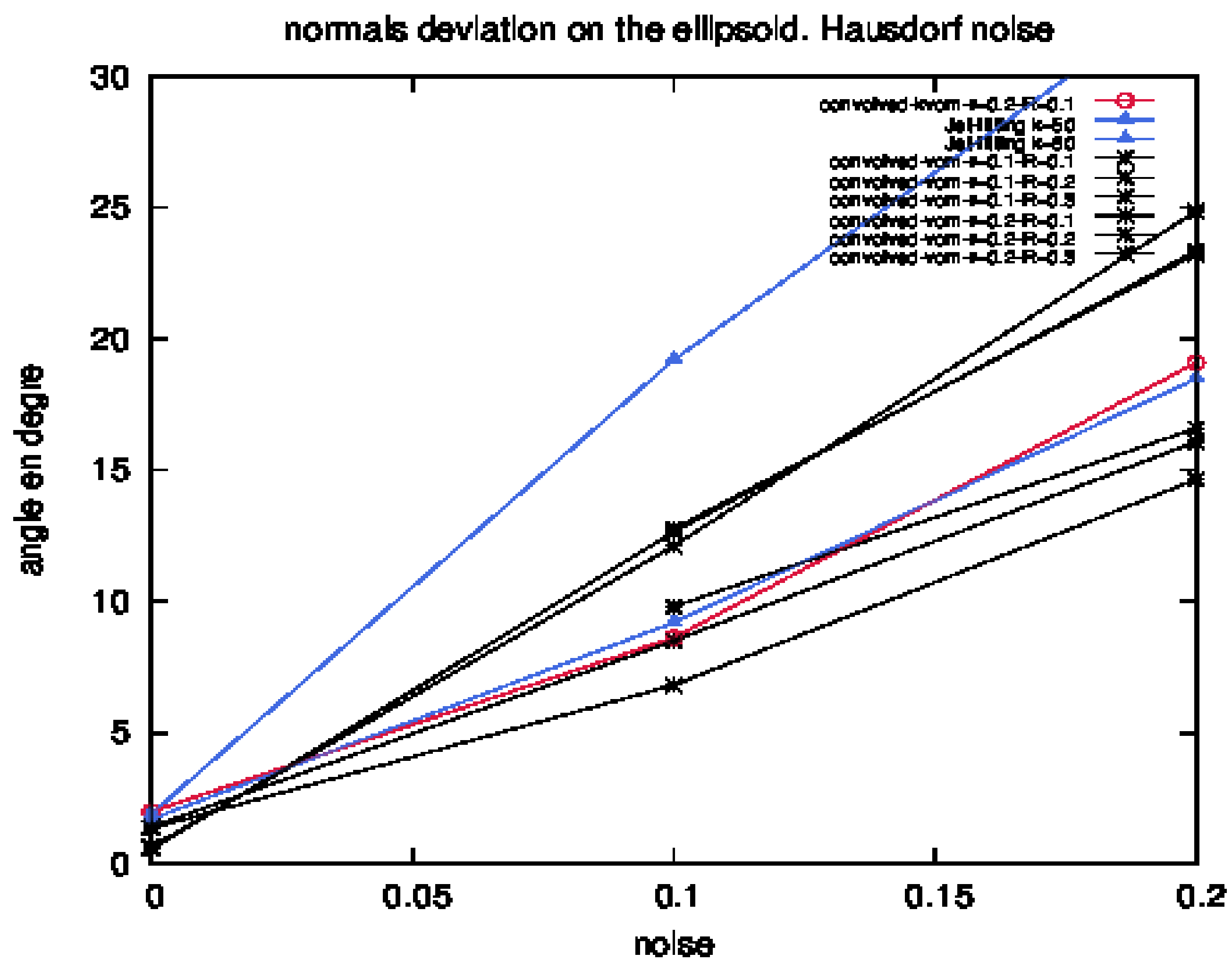
$\varepsilon = 0.4$

Normal Estimation : Ellipsoid

Normale deviation in function of noise.
Comparison with the Jet fitting and the VCM.



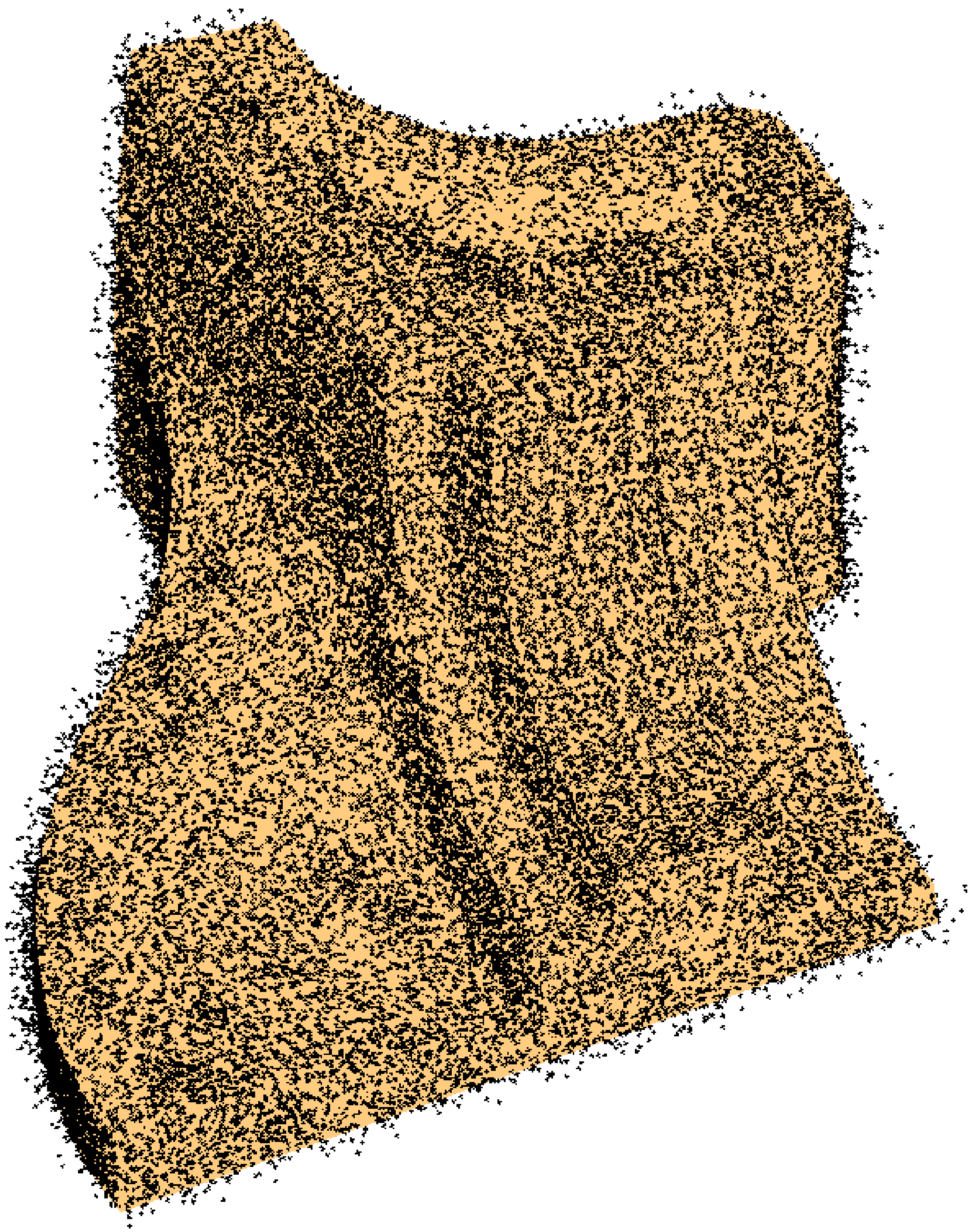
Hausdorff noise + outliers



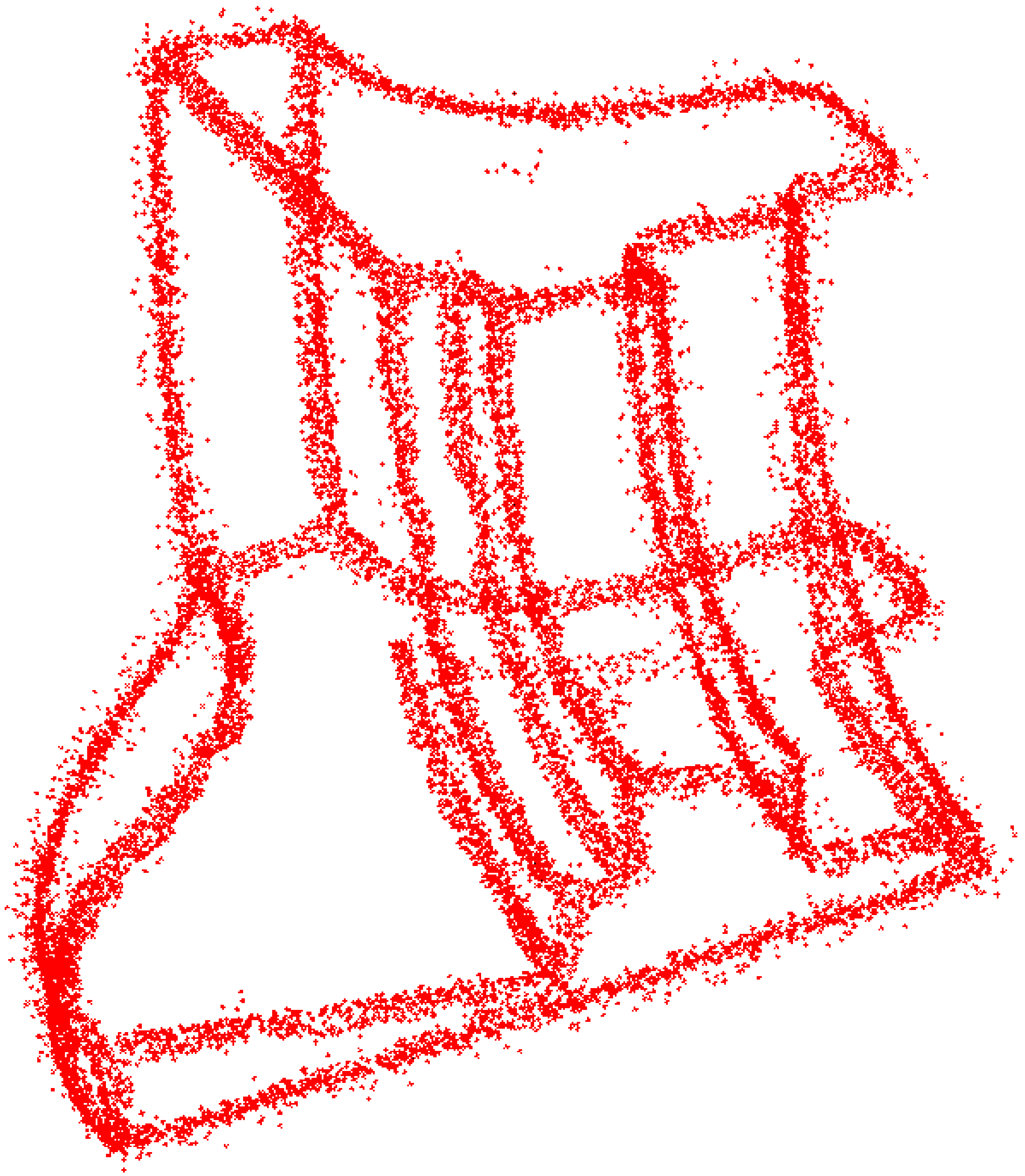
Gaussian noise

– $10k$ points

Sharp feature estimation



Input



Output

fandisk 50k points

Conclusion

Summary:

- k-VCM is a tool to estimate normal and curvature direction on a point cloud.
- Resilient to Hausdorff noise and outliers.

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Next work:

- parameter choice k, R et r.
- Applying k-VCM to pixels/voxels sets.