Scalar field analysis with aberrant noise

Mickaël Buchet

joint work with F. Chazal, T. Dey, F. Fan,
S. Oudot and Y. Wang

How many peaks do you see?

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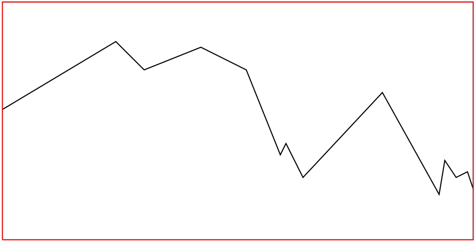
Persistence diagram



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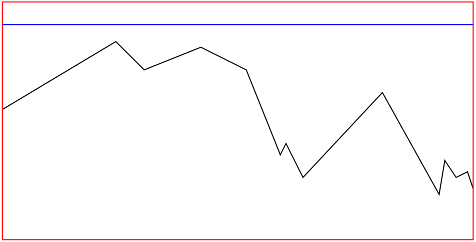

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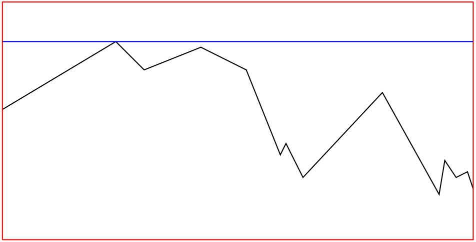

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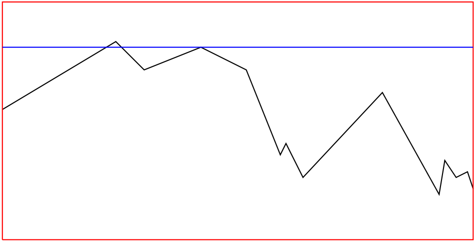

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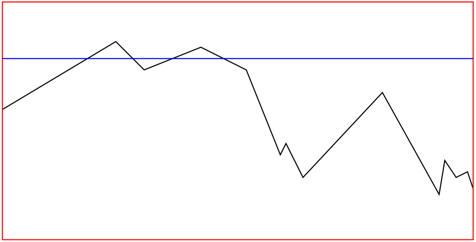

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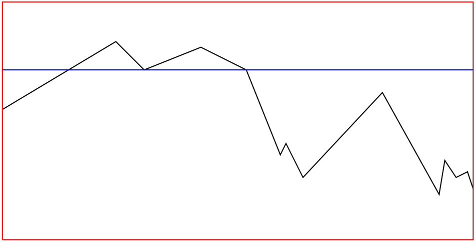

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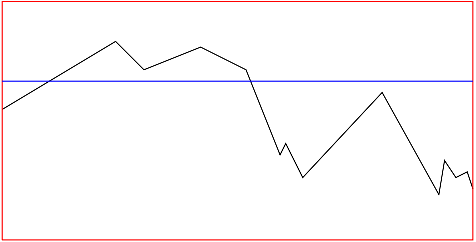

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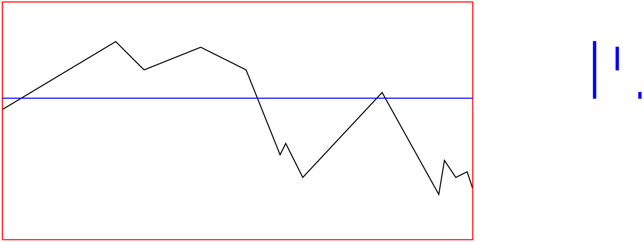

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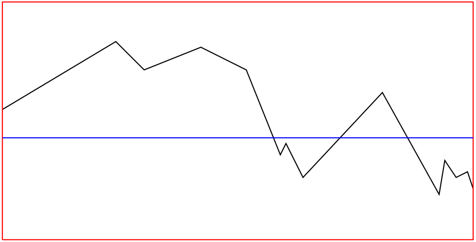

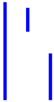

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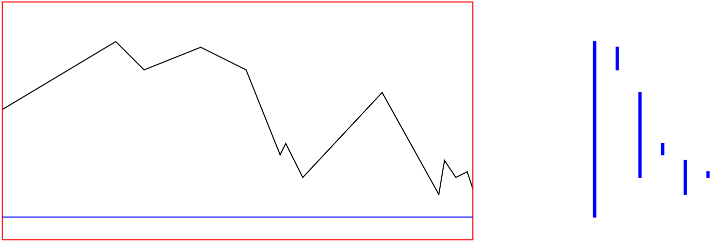

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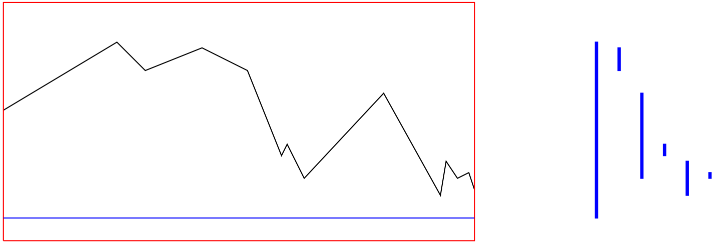

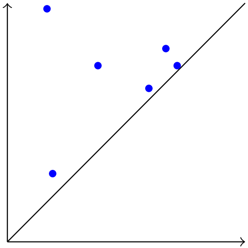

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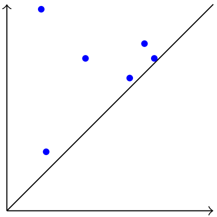

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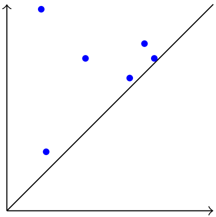

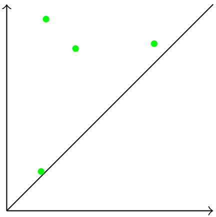

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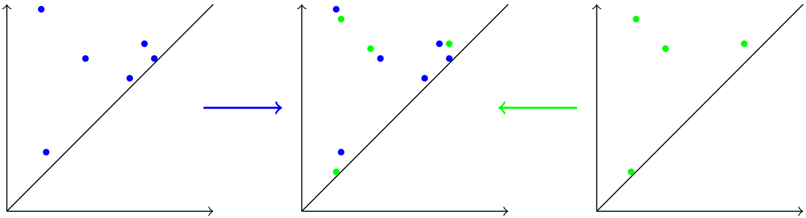

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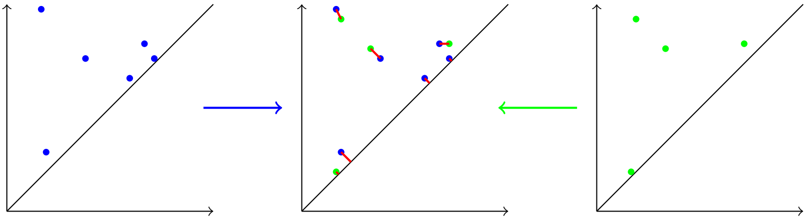
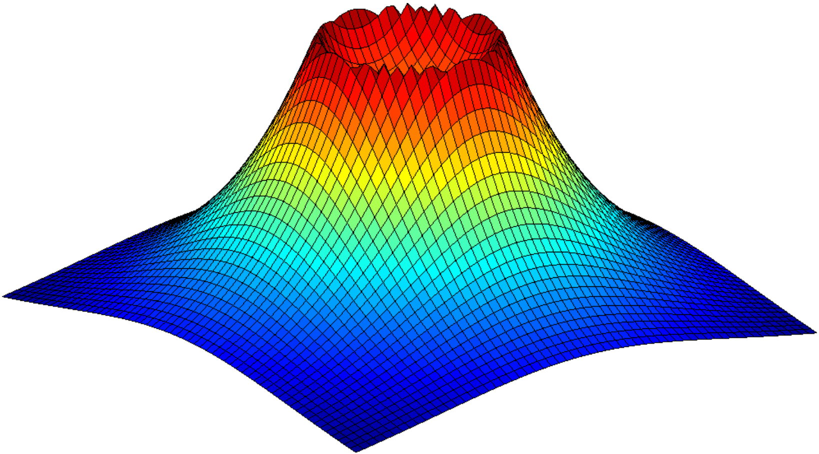


Bottleneck distance:

$$d_B(D, E) = \inf_{b \in \mathcal{B}} \max_{x \in D} ||x - b(x)||_\infty$$

Higher dimensions



Applicative setting

The *ground truth*:

- a sub-manifold M of $\mathbb{R}^d$
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Approximate the persistence diagram D of f by a diagram $\hat{D}$

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From Chazal, Guibas, Oudot and Skraba (DCG'11)
Under some conditions on ϵ and the geometry of M,

Theorem

If P is an ϵ Riemannian sample of M *and* $||f|_P - \tilde{f}||_\infty \leq \xi$, *then:*

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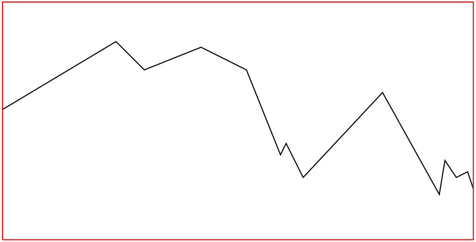
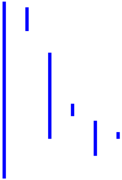
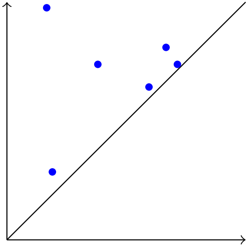
$$d_B(D, \hat{D}) \leq (4\epsilon + 2\nu)c$$

A bad example







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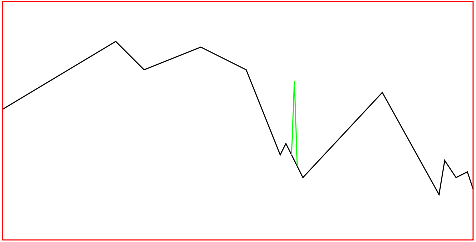

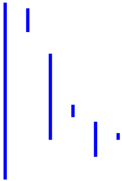

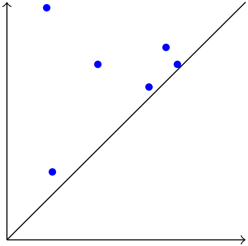

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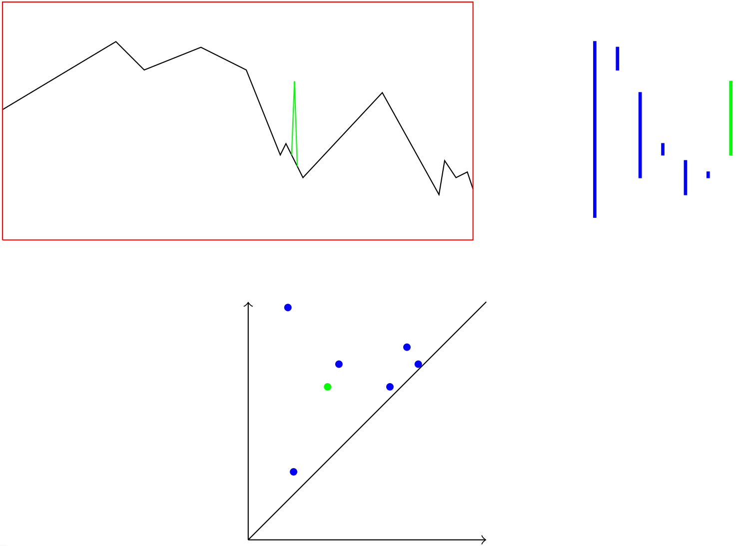

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A simplicial complex is a set X of simplices such that for any simplex $\sigma \in X$, all facets of σ are also in X.

We say that X is filtered when there exists a family of simplicial complexes $\{X_\alpha\}_{\alpha\in\mathbb{R}}$ such that for all $\alpha < \beta$, $X_\alpha \subset X_\beta \subset X$.

The Rips complex

Topologies of $\{f^{-1}(]-\infty,\alpha)\}$ and $\{\tilde{f}^{-1}(]-\infty,\alpha])\}$ are completely different:

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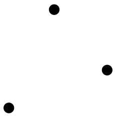

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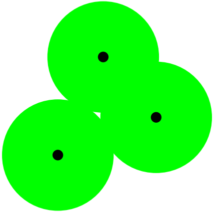

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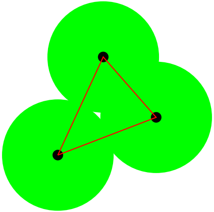

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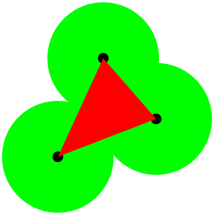

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Effect of "cleaning" the persistence diagram.

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$$\{R_\delta(P_\alpha) \hookrightarrow R_{\delta'}(P_\alpha)\}_{\alpha \in \mathbb{R}}$$

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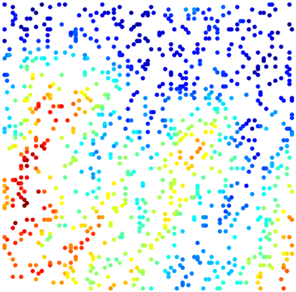
Theorem

If P is an ϵ Riemannian sample of M *and the distance between points of P are given by a function $\tilde{d}$ such that* $\frac{d_\mathrm{M}(x,y)}{\lambda} \leq \tilde{d}(x,y) \leq \nu + \mu\frac{d_\mathrm{M}(x,y)}{\lambda}$*, then:*

$$\forall \delta \geq \nu + 2\mu\frac{\epsilon}{\lambda},\ \delta' \in [\nu + 2\mu\delta, \frac{1}{\lambda}\varrho(\mathrm{M})[,$$

$$d_B(\mathrm{Dgm}(f), \mathrm{Dgm}(R_\delta(P_\alpha) \hookrightarrow R_{\delta'}(P_\alpha)) \leq c\lambda\delta'$$

Sources of noise



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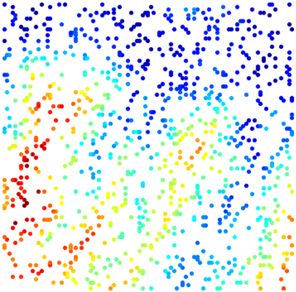

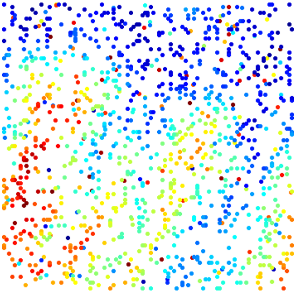

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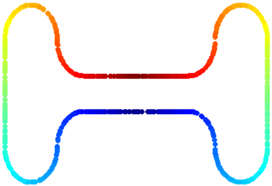

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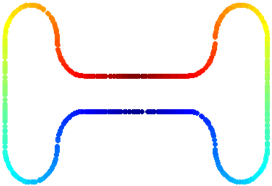

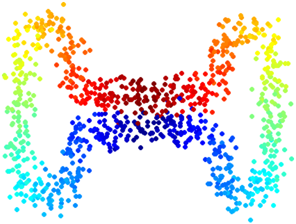

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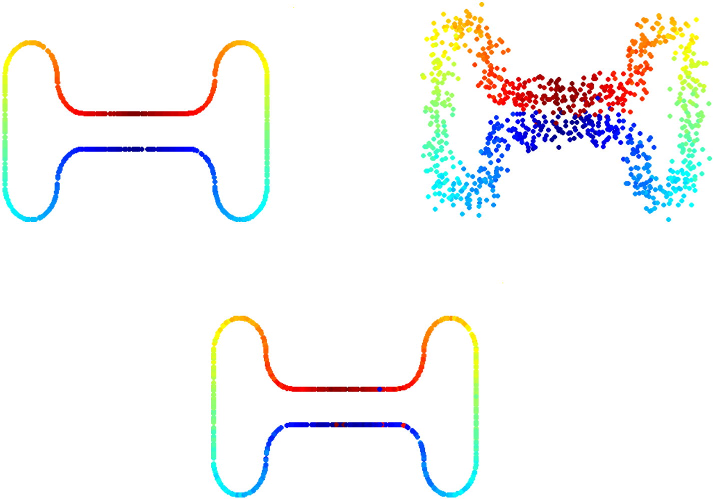

Discrepancy

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3. Fix the new function value $\hat{f}(p)$ as the barycenter of Y.

A variant using the median

We compute new functionnal values for points :

For every point p in P:

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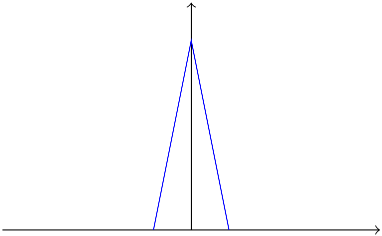

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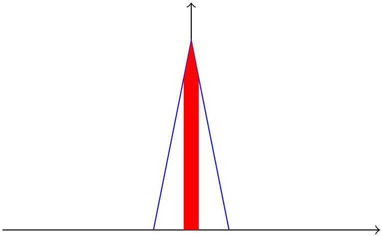

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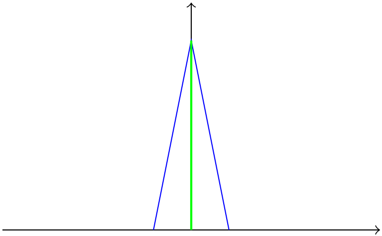

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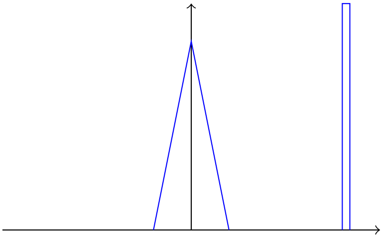

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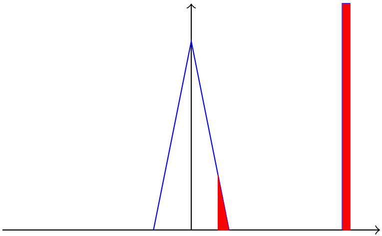

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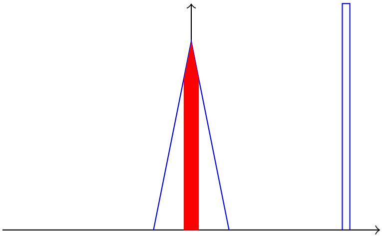

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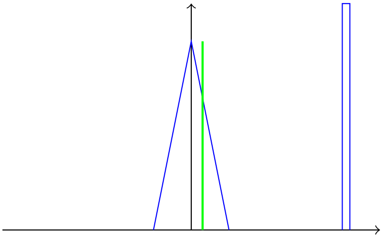

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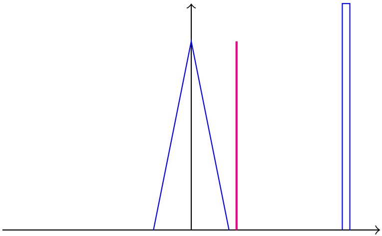
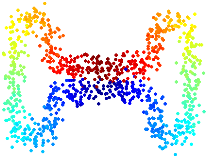
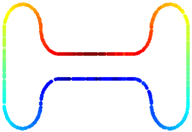
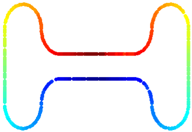
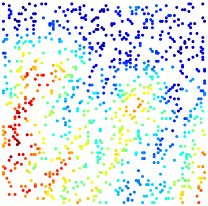


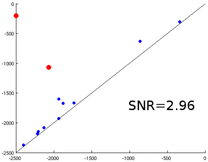
Bone

	Noisy input	k-NN regression	Discrepancy
			
Max	16.23	3.18	0.37
Mean	0.349	.204	.097

Persistence of topographic map

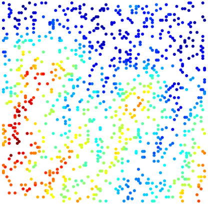


Topographic map

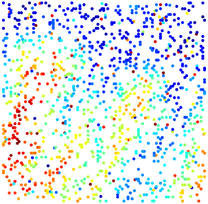


Original persistence diagram

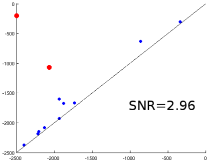
Persistence of topographic map



Topographic map



Noisy topographic map



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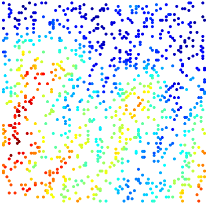

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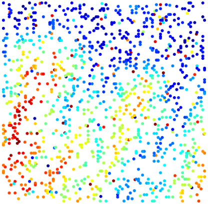

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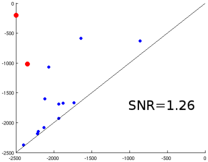

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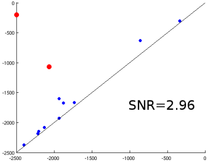

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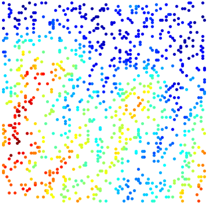

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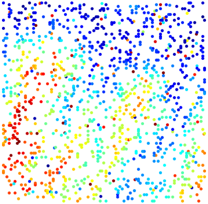

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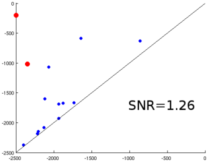

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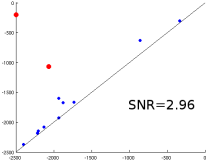

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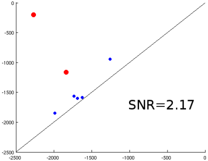
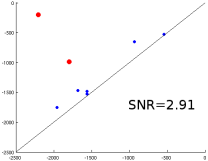


k-NN persistence diagram



Discrepancy persistence diagram

Images

No noise

Images

No noise

40% outliers

Images

No noise

40% outliers

kNN

Images

No noise

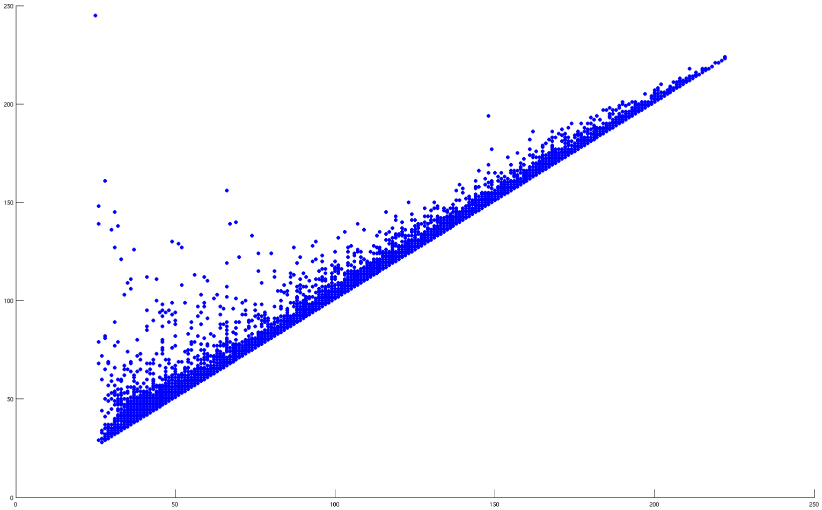
40% outliers

kNN

Discrepancy

Median

Lena's diagrams



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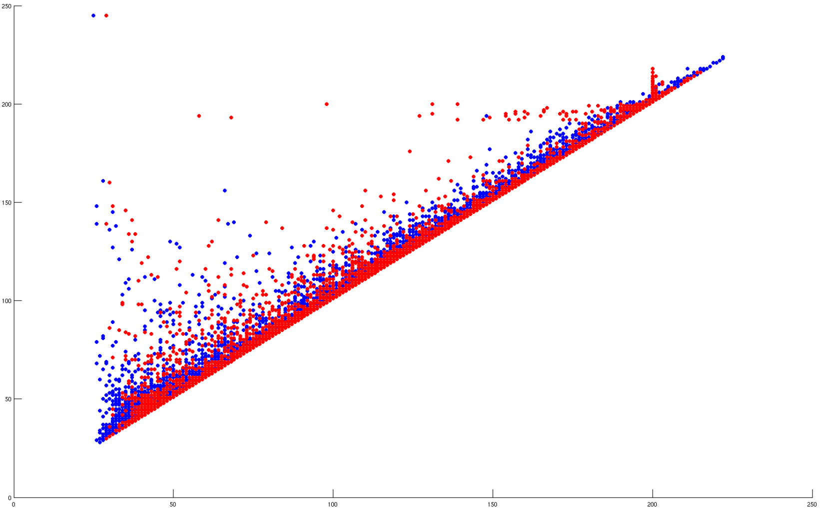

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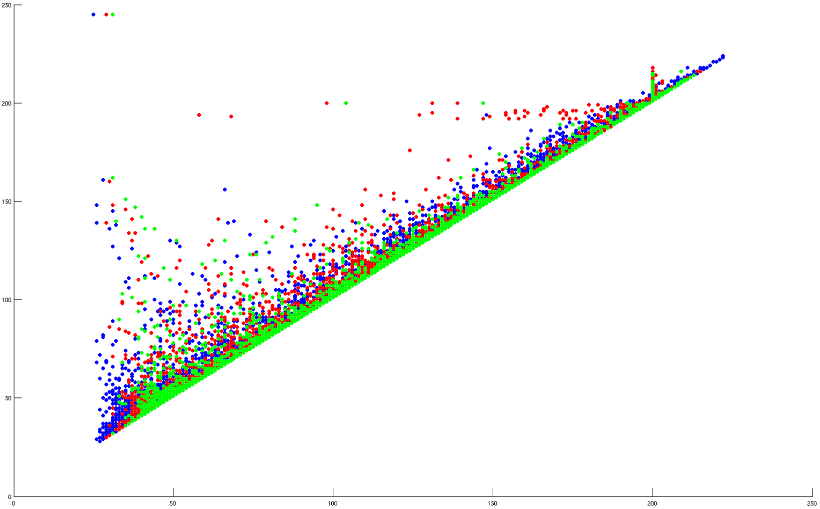

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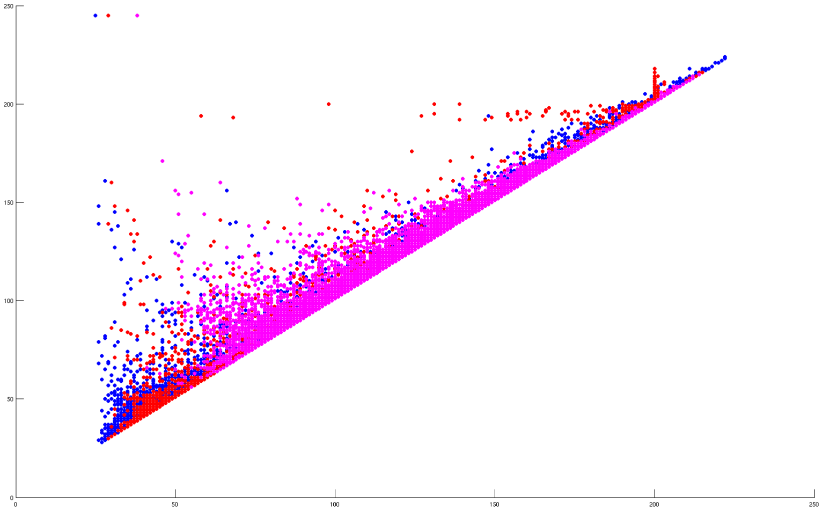
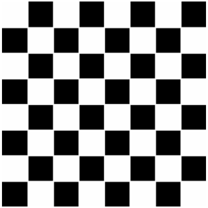
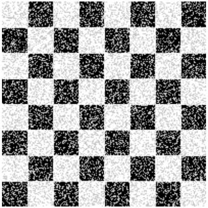


Chessboard

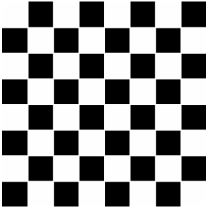


No noise



30% outliers

Chessboard



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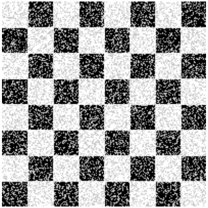

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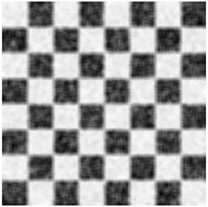
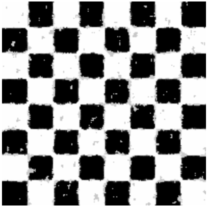
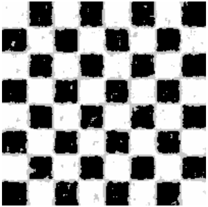


kNN



Discrepancy



Median

Building a complex with good properties

We need a complex that has the correct geometric structure to analyze the scalar field.

Building a complex with good properties

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- Use of the super-level sets of a density estimator
- Use of the sub-level sets of a distance-like function.

Distance to measure in a nutshell

The distance to a measure is a distance-like function designed to cope with outliers.

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  [Chazal, Cohen-Steiner, Mérigot, 2011]
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  [Biau, Chazal, Cohen-Steiner, Devroye, Rodrigues, 2011]

A complete noise model

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3. For any point close to M, most of the neighboring values are good:

$$\forall p \in d_{\mu,m}^{-1}(]-\infty, \eta]),\ |\{q \in NN_k(p) |\, |\tilde{f}(q) - f(\pi(p))| \leq s\} \geq k'$$

Theoretical results

Theorem

If P is a set verifying the previous conditions anf f is a c-Lipschitz fucntion then:

$$\forall \delta \in [2\eta + 6\epsilon, \frac{\varrho(\mathrm{M})}{2}],\ \delta' \in [2\eta + 2\epsilon + \frac{2R_M}{R_M - (\eta + \epsilon)}\delta, \frac{R_M - (\eta + \epsilon)}{R_M}\varrho(M)],$$

$$d_B(\mathrm{Dgm}(f), \hat{D}) \leq \left(\frac{cR_M\delta'}{R_M - (\eta + \epsilon)} + \xi s \right)$$

with $\xi = 1$ for the median and $\xi = 1 + 2\sqrt{\frac{k-k'}{2k'-k}}$ for the discrepancy.

Take home

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- A versatile and model free algorithm for functional denoising

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- The algorithm needs some parameters.
- Heuristics exist but there is no general method to choose their value.