Convex analysis and optimisation

2022, November 16
9h – 11h30

You may use the courses notes that were given to you and your handwritten notes during the exam, but you shouldn't use any online resource nor communicate with each other. A fraud could lead to a ban to pass university exams in France for up to five years.

The exam has two parts: Part I corresponds to the algorithmic part by J.C. Pesquet and Part II to the convex analysis part by Q. Mérigot. They are graded independently and account each for half of the final grade.

Please use different sheets for Part I and II.

Part I: Article study

Answer the following questions concerning the article: "Playing with Duality: An Overview of Recent Primal-Dual Approaches for Solving Large-Scale Optimization Problems" by N. Komodakis and J.-C. Pesquet [1]. All the provided answers should be properly justified.

1. Prove Property iii in Table I for the conjugation operation.
2. Is Equation (21) correct?
3. How can (22) be derived from first-order optimality conditions for Problem (19)?
4. How would you write the Douglas-Rachford algorithm for solving (19) when $L = 0$?
5. Specify the conditions of convergence of this Douglas-Rachford algorithm.
6. What are the limitations of ADMM?
7. Do Algorithms 2 and 5 solve the same optimization problem?
8. What is the main advantage of Algorithm 7?
9. Explain **one** of the strategies allowing primal-dual methods to be parallelized?
10. What does Figure 7 show?

[1] https://arxiv.org/abs/1406.5429

Part II

The exercises are independent. If you do not find the solution to a question, you can admit the answer and skip it. Given the length of this part, it will be possible to get a good grade without answering all the questions.

Exercise 1. Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be convex functions such that $g \leq f$. Prove that if $f(x_0) = g(x_0)$, then $\partial g(x_0) \subseteq \partial f(x_0)$.

Exercise 2. Let E be a normed space, f, g be lower semicontinous convex functions on E and consider the (nonconvex) minimization problem:

$$P = \inf_{x \in \mathrm{dom}(f)} f(x) - g(x) \tag{1}$$

1. Prove that $P = D$, where $D := \inf_{p \in \mathrm{dom}(g^*)} g^*(p) - f^*(p)$.
   (Hint: prove that any minorant of P is a minorant of D.)
2. Assume that P is finite. Prove that if x is a minimizer of P and $p \in \partial g(x)$, then p is a solution to D.

Exercise 3. For some $N \in \mathbb{N}$, we let $Y = \mathcal{M}_{N,N}(\mathbb{R})$ be the set of $N \times N$ matrices. The entries of a matrix $\Gamma \in E$ are denoted $(\Gamma_{ij})_{1 \leq i \leq j}$. Given $\mu, \nu \in \mathbb{R}_+^N$, we define

$$\Gamma(\mu, \nu) = \left\{ \Gamma \in \mathcal{M}_{N,N}(\mathbb{R}) \mid \Gamma_{ij} \geq 0, \sum_i \Gamma_{ij} = \nu_j, \sum_j \Gamma_{ij} = \mu_i \right\}$$

Note that $\Gamma(\mu, \nu)$ is empty if either μ or ν take negative values, or if $\sum_i \mu_i \neq \sum_j \nu_j$. From now on, we fix a matrix $C \in \mathcal{M}_{N,N}(\mathbb{R})$.

1. Prove that if $\mu, \nu \in \mathbb{R}_+^N$ satisfy $\sum_j \nu_j = \sum_i \mu_i$, the optimization problem

$$\inf_{\Gamma \in \Gamma(\mu,\nu)} \sum_{i,j} C_{ij} \Gamma_{ij} \tag{2}$$

   has a solution, and takes value $+\infty$ otherwise.
2. Prove the weak duality inequality

$$\inf_{\Gamma \in \Gamma(\mu,\nu)} \sum_{i,j} C_{ij} \Gamma_{ij} \geq \sup_{\phi, \psi \in \mathbb{R}^N \mid \forall i,j,\, \phi_i + \psi_j \leq C_{ij}} \sum_i \phi_i \mu_i + \sum_j \psi_j \nu_j \tag{3}$$

   (Hint: no need to apply a theorem of the course, the direct proof is short.)

We consider the value of the problem (2) *as a function of μ, ν:*

$$T_C : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$$
$$(\mu, \nu) \mapsto \begin{cases} \inf_{\Gamma \in \Gamma(\mu,\nu)} \sum_{ij} C_{ij} \Gamma_{ij}, & \textit{if } \mu \geq 0, \nu \geq 0 \\ +\infty & \textit{otherwise.} \end{cases}$$

We will use (without proof) that the function T_C is convex and lower semicontinuous. In the following definition, the canonical basis of $\mathbb{R}^N$ is denoted $(e_i)_{1\le i\le N}$.

$$F_C : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$$
$$(\mu,\nu) \mapsto \begin{cases} mC_{i,j} & \textit{if } \exists m \in \mathbb{R}_+ \textit{ and } i,j \in \{1,\dots,N\} \textit{ s.t. } \mu = me_i, \nu = me_j \\ +\infty & \textit{otherwise} \end{cases}$$

3. Compute F_C^* and F_C^{**}.
4. Prove that $T_C \le F_C$.
5. Deduce from the previous questions and from (3) that $T_C = F_C^{**}$.

Exercise 4. Let$f, g_1, \dots, g_\ell : \mathbb{R}^d \to \mathbb{R}$ be convex. Assume in addition that f is strictly convex and coercive and $g_1, \dots, g_\ell \ge 0$. For any $\lambda \in \mathbb{R}_+^\ell$, consider

$$v(\lambda) = \inf_{x\in\mathbb{R}^d} f(x) + \langle \lambda | G(x) \rangle \text{ with } G(x) = (g_1(x), \dots, g_\ell(x)) \in \mathbb{R}^\ell.$$

1. Prove that the minimum in the definition of $v(\lambda)$ is attained at a unique point x_λ, and that the map $\lambda \in \mathbb{R}_+^\ell \mapsto x_\lambda \in \mathbb{R}^d$ is continuous.
2. Prove that v is concave and continuous on $\mathbb{R}_+^\ell$.
3. Prove that v is $\mathcal{C}^1$ on $(\mathbb{R}_+^*)^\ell$ and that $\nabla v(\lambda) = G(x_\lambda)$

For any $\alpha \in \mathbb{R}^\ell$, we define $K_\alpha = \{x \in \mathbb{R}^d \mid g_i(x) \le \alpha_i \text{ for } 1 \le i \le \ell\}$ and

$$w(\alpha) = \inf_{x\in K_\alpha} f(x) \text{ with } .$$

4. Prove that $w(\alpha) = \sup_{\lambda\in\mathbb{R}_+^\ell} v(\lambda) - \langle \lambda | \alpha \rangle$ and that w is convex and lsc.
5. Deduce that $\lambda \in (\mathbb{R}_+^*)^\ell$ belongs to $\partial w(\alpha)$ if and only $G(x_\lambda) = \alpha$.