# Convex analysis and optimisation

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# 1 Convex sets

Most of the statements of this course hold when $\mathcal{X}$ is a normed space i.e. a vector space endowed with a norm $\|\|$ which endows $\mathcal{X}$ with a topology. The closed ball of radius around a point $x$ is then denoted $\mathrm{B}(x, r) = \{y \in \mathcal{X} \mid \|x - y\| \leq r\}$, and the topology is induced by the metric $d(x, y) = \|x - y\|$. In particular, the results of this course can be applied to $\mathbb{R}^d$, to Hilbert spaces, and to Banach spaces.

However, it is often the case that our statement hold in the more setting of (locally convex) topological vector spaces, which is particularly adequate when one wants to consider weak/weak$*$ topologies, which are important in some applications in infinite dimension (e.g. calculus of variations, optimal control, or optimal transport), because they allow to define *weak* topologies, which are useful to prove existence to optimization problems. We propose a short introduction to these notions in Appendix A.

**Definition 1** (Locally convex topological vector space)**.** A *topological vector space* is a vector space $\mathcal{X}$ endowed with a topology which makes addition of vectors $(x, y) \in \mathcal{X}^2 \mapsto x+y \in \mathcal{X}$ and multiplication by a scalar $(\lambda, x) \in \mathbb{R} \times \mathcal{X} \mapsto \lambda x \in \mathcal{X}$ continuous.

A topological vector $\mathcal{X}$ is called *locally convex* if there exists a family $\mathcal{B}$ of convex open sets containing the origin such that for all neighborhood $O$ of the origin, there exists a basis set $\omega \in \mathcal{B}$ such that $\omega \subseteq O$.

Any normed space is a locally convex topological vector space, and most of the course can be understood knowing only normed spaces, by replacing "(locally convex) topological vector spaces" with "normed space" in all statements.

## 1.1 Convex sets

**Definition 2** (Convex set)**.** A subset $K$ of $\mathcal{X}$ is called *convex* if and only if for any $x, y \in K$, the segment $[x, y] = \{(1 - t)x + ty \mid t \in [0, 1]\}$ is included in $K$.

*Example* 1. Elementary examples of convex sets include

- open and closed balls in normed vector spaces,
- hyperplanes, i.e. sets of the form $\{x \in \mathcal{X} \mid \phi(x) = \alpha\}$ for some linear function $\phi$ on $\mathcal{X}$ and some $\alpha \in \mathbb{R}$,
- affine subspaces, i.e. intersection of hyperplanes,
- halfspaces, i.e. sets of the form $\{x \in \mathcal{X} \mid \phi(x) \leq \alpha\}$ for some linear function $\phi$ on $\mathcal{X}$ and some $\alpha \in \mathbb{R}$,
- intersection of halfspaces, which are called *polyhedra* if one take a *finite* intersection of halfspaces,
- the space of symmetric positive definite matrices,
- sublevel sets $\{x \in \mathcal{X} \mid f(x) \geq \alpha\}$ or epigraphs $\{(x, t) \mid t \geq f(x)\}$ where $f : \mathcal{X} \to \mathbb{R}$ is a convex function and $\alpha \in \mathbb{R}$.

**Definition 3** (Convex hull)**.** The *convex hull* of a set $A \subseteq \mathcal{X}$, denoted $\mathrm{conv}(A)$, is the smallest convex set containing $A$.

**Proposition 1.** *The following properties hold*

*(i) If $(K_\alpha)_{\alpha \in A}$ is family of convex sets, $\cap_{\alpha \in A} K_\alpha$ is convex*

*(ii) If $K \subseteq \mathcal{X}$ is convex and $L : \mathcal{X} \to \mathcal{Y}$ is linear, then $L(K)$ is convex.*

*(iii) If $K \subseteq \mathcal{X}$ is convex and $L : \mathcal{Y} \to \mathcal{X}$ is linear, then $L^{-1}(K)$ is convex.*

*(iv) If $K \subseteq \mathcal{X}, L \subseteq \mathcal{Y}$ are convex, then $K \times L$ is convex.*

*(v) If $K, L \subseteq \mathcal{X}$ are convex, then $K \oplus L$ is convex.*

*(vi) $\mathrm{conv}(A)$ is the intersection of all convex sets containing $A$.*

*(vii) $\mathrm{conv}(A) = \left\{ \sum_{i=1}^k \alpha_i x_i \mid k \geq 1, x_i \in A, \ \alpha_i \geq 0, \sum_{i=1}^k \alpha_i = 1 \right\}$.*

*Proof.* Exercise. □

## 1.2 Hahn-Banach theorem

The most important tool of this course is the representation of closed convex sets as intersection of closed half-spaces, which is the basis of subdifferential calculus, of the duality theory of Fenchel and Rockafellar, etc. This representation relies mainly on the Hahn-Banach theorem.

**Definition 4** (Sublinearity)**.** A function $p : \mathcal{X} \to \mathbb{R}$ is *sublinear* if it is

(i) 1-*homogeneous*: for any $x \in \mathcal{X}$ and $\lambda \geq 0$, $p(\lambda x) = \lambda p(x)$;

(ii) *subadditive*: for any $x, y \in \mathcal{X}$, $p(x + y) \leq p(x) + p(y)$.

Example of sublinear functionals include norms over the space, or more generally *gauges* of convex sets.

**Definition 5** (Gauge)**.** The *gauge* — also called Minkowski functional — of a convex $K \subseteq \mathcal{X}$ convex is the function $p_K$ taking value in $\mathbb{R} \cup \{+\infty\}$ defined on $\mathcal{X}$

$$p_K(x) = \inf\{r > 0 \mid x \in rK\}.$$

**Theorem 2** (Hahn-Banach)**.** *Let $\mathcal{X}$ be a topological vector space and let $p$ be a continuous sublinear function on $\mathcal{X}$. If $E$ is a linear subspace of $\mathcal{X}$ and if $f$ is a linear function on $E$ satisfying $f \leq p$, then $f$ can be extended into a continuous linear form $\hat{f} \in \mathcal{X}^*$ which also satisfies $\hat{f} \leq p$.*

We refer to Brézis [Bre10] for a proof of the Hahn-Banach theorem in the case of general vector spaces (which also does not require that $p$ is continuous), while we prove it only in the case where $\mathcal{X}$ is separable (i.e. $\mathcal{X}$ contains a dense subset), which makes the proof more constructive.

*Proof.* Let $(v_i)_{i \geq 1}$ be a countable dense subset of $\mathcal{X}$, and define by induction $E_0 = E$ and $E_i = \mathrm{span}(E_{i-1}, v_i)$.

**Step 1: Extension of $f$ from $E_{i-1}$ to $E_i$:** We first show how to construct a linear extension $f_i$ of $f$ on $E_i$ satisfying $f_i \leq p$, assuming that $f_{i-1}$ is already constructed. If $v_i$ belongs to $E_{i-1}$, one has $E_i = E_{i-1}$ and there is nothing to do. If not, for any $\alpha \in \mathbb{R}$ we can define a linear function $f_{i,\alpha}$ on $E_i$ by setting

$$\forall x \in E_{i-1}, \forall t \in \mathbb{R}, f_{i,\alpha}(x + tv_i) = f_{i-1}(x) + t\alpha.$$

We now need to choose $\alpha$ so that $f_{i,\alpha} \leq p$. Equivalently, we want

$$\begin{aligned}
&\forall x \in E_{i-1}, \forall t \in \mathbb{R}, f_{i-1}(x) + t\alpha \leq p(x + te_i) \\
&\iff \forall x \in E_{i-1}, \forall t \in \mathbb{R} \setminus \{0\}, f_{i-1}\left(\frac{x}{|t|}\right) + \frac{t}{|t|}\alpha \leq p\left(\frac{x}{|t|} + \frac{t}{|t|}e_i\right) \\
&\iff \forall x \in E_{i-1}, f_{i-1}(x) \pm \alpha \leq p(x \pm e_i).
\end{aligned}$$

We used the homogeneity of $p$ to go from the first to the second line. This is again equivalent to $\alpha$ satisfying

$$\sup_{x \in E_{i-1}} f_{i-1}(x) - p(x - e_i) \leq \alpha \leq \inf_{y \in E_{i-1}} p(y + e_i) - f_{i-1}(y).$$

Thus, such an $\alpha$ exists if the supremum is less than the infimum, i.e. if

$$\begin{aligned}
&\forall (x, y) \in E_{i-1}, f_{i-1}(x) - p(x - e_i) \leq p(y + e_i) - f_{i-1}(y) \\
&\iff \forall (x, y) \in E_{i-1}, f_{i-1}(x + y) \leq p(y + e_i) + p(x - e_i) \\
&\impliedby \forall (x, y) \in E_{i-1}, f_{i-1}(x + y) \leq p(x + y),
\end{aligned}$$

where the last implication is deduced from the subadditivity of $p$. Since $f_{i-1} \leq p$, we can backtrack the chain of implications to deduce the existence of $\alpha \in \mathbb{R}$ such that $f_{i,\alpha} \leq p$. In practice, one can set

$$\alpha = \inf_{y \in E_{i-1}} p(y + e_i) - f(y) \geq -p(-e_i) > -\infty.$$

**Step 2: Extension of $f$ to $\mathcal{X}$** The previous constructions allows to construct a extension $\tilde{f} : \hat{E} \to \mathbb{R}$ of $f$ to the linear subspace $\hat{E} = \cup_i E_i$ satisfying $\tilde{f} \leq p$. The subspace $\hat{E}$ contains the vectors $(v_i)_{i \geq 1}$ and is therefore dense in $\mathcal{X}$. To extend $\tilde{f}$ to $\mathcal{X}$, we will use the continuity of $p$. We first notice that by assumption

$$-p(y - x) \leq \tilde{f}(x - y) \leq p(x - y),$$

implying that

$$\left|\tilde{f}(x - y)\right| \leq \max(|p(x - y)|, |p(y - x)|). \tag{1}$$

Using the continuity of $p$, this inequality shows that if $(x_n)$ is a sequence of points of $\hat{E}$ converging to some $x \in \mathcal{X}$, then $(\tilde{f}(x_n))$ is a Cauchy sequence in $\mathbb{R}$, thus converging. The same inequality also proves that if two sequences $(x_n), (y_n)$ of elements of $\hat{E}$ converge to the same point $x \in \mathcal{X}$, then $\lim_{n \to +\infty} \tilde{f}(x_n) = \lim_{n \to +\infty} \tilde{f}(y_n)$. We denote $\hat{f}(x)$ the common limit. The function $\hat{f}$ also satisfies the inequality (1), proving its continuity. $\square$

### 1.3 Linear separation

The *topological dual* of $\mathcal{X}$ is the space of all continuous linear functions on $\mathcal{X}$, and is denoted $\mathcal{X}^*$. The elements of $\mathcal{X}^*$ are denoted with a star, e.g. $x^* \in \mathcal{X}^*$, and we denote the pairing between $\mathcal{X}^*$ and $X$ using the notation for scalar product, i.e. $\langle x^* \mid x \rangle := x^*(x)$, to underline the similarity with the case of Hilbert spaces.

**Definition 6** (Separation)**.** Two sets $K, L \subseteq X$ are *continuously linearly separated*, or separated for short, if there exists a non-trivial continuous linear form $x^* \in X^* \backslash \{0\}$ such that $\sup_K \langle x^* \mid \cdot \rangle \leq \inf_L \langle x^* \mid \cdot \rangle$. If the inequality is strict, we will say that the sets are *strongly separated*.

**Corollary 3.** *Let $\mathcal{X}$ be a topological vector space, $K$ an open convex subset of $\mathcal{X}$ and $z \in \mathcal{X} \setminus K$. Then $K$ and $\{z\}$ are separated.*

**Lemma 4.** *Let $K \subseteq \mathcal{X}$ be a convex neighborhood of the origin. Then the gauge $p_K$ is sublinear and continuous, and $K \subseteq \{p_K \leq 1\}$.*

*Proof.* The sublinearity of $p_K$ is left as an exercise, we only show the continuity. By definition of the gauge, we easily see that $|p_K(x)| \leq 1$ for all $x$ in $K$. Then, for any $\varepsilon > 0$, we have $|p_K| \leq \varepsilon$ on $\varepsilon K$, implying the continuity of $p_K$ at the origin. We now use the sublinearity of $p_K$ to extend the continuity to the whole space $\mathcal{X}$. Given $x, y \in \mathcal{X}$, we have

$$\begin{cases} p_K(x) = p_K(x - y + y) \leq p_K(x - y) + p_K(y) \\ p_K(y) = p_K(y - x + x) \leq p_K(y - x) + p_K(x) \end{cases}$$

implying that $|p_K(y) - p_K(x)| \leq \max(p_K(x - y), p_K(y - x))$. Let $\varepsilon > 0$, and let $K' = x + \varepsilon(K \cap -K)$. Then $K'$ is a neighborhood of $x$ and $|p_K(\cdot) - p_K(x)| \leq \varepsilon$ on $K'$, thus showing the continuity of $p_K$ at $x$. □

*Proof of Corollary 3.* Translating if necessary, we assume that $K$ contains the origin in its interior, and we consider its gauge $p_K$. Let $E = \mathbb{R}z$, and define $f(tz) := t$ on $E$. Since $z \notin K$, $p_K(z) \geq 1 = f(z)$, from which we deduce by positive homogeneity that $f(tz) \leq p_K(tz)$ for $t \geq 0$. For $t \leq 0$ we have $f(tz) = t \leq 0 \leq p_K(tz)$. Thus, $f \leq p_K$ on $E$. By the Hahn-Banach theorem, we get the existence of a continuous linear extension $\hat{f}$ of $f$ such that $\hat{f} \leq p_K$ on $\mathcal{X}$. Moreover, for all $x \in K$, we have $\hat{f}(x) \leq p_K(x) \leq 1 = \hat{f}(z)$, thus showing that $K$ is separated from $\{z\}$ by a continuous linear form, which moreover is nontrivial since $\hat{f}(z) = 1$. □

A simple argument using the Minkowski sum allows to separate two convex sets, one of which is open. We recall that the Minkowski sum $A \oplus B$ between two sets is

$$A \oplus B = \{x + y \mid x \in A, y \in B\}.$$

The Minkowski difference is defined similarly as $A \ominus B := \{x + y \mid x \in A, y \in B\}$. The Minkowski sum of two convex sets is convex (exercise).

**Corollary 5.** *Let $\mathcal{X}$ be a topological vector space and let $K$ and $L$ be two disjoint convex subsets of $\mathcal{X}$, such that $L$ is open. Then, $K$ and $L$ are separated.*

*Proof.* We consider the Minkowski difference between $K$ and $L$, $M = K \ominus L$,

$$M := \{x - y \mid x \in K, y \in L\} = \bigcup_{y \in L} (K - y),$$

which is open (as a union of open sets) and convex as already noted. Since $K$ and $L$ are disjoint, $M$ does not contain $z := 0$ in its interior. By the previous corollary, we get the existence of $x^* \in \mathcal{X}^*$ such that

$$\forall x \in M = K \oplus (-L), \langle x^* \mid x \rangle \leq \langle x^* \mid z \rangle = 0$$

This directly implies that $K$ and $L$ are separated by $x^*$. □

**Corollary 6.** *Let $\mathcal{X}$ be a topological vector space and let $K$ and $L$ be two convex subsets of $\mathcal{X}$. Assume that there exists an open convex neighborhood $V$ of 0 such that $(K \oplus V) \cap L = \emptyset$. Then, $K$ and $L$ are strongly separated.*

*Proof.* We first note that the set $K \oplus V$ is convex (at the sum of convex sets) and open (as in the proof of the previous corollary). Thus, by the previous corollary, we can separate $K \oplus V$ from $L$, i.e. there exists $x^* \in \mathcal{X}^* \setminus \{0\}$ so that

$$\sup_{x \in K \oplus V} \langle x^* \mid x \rangle \leq \inf_{y \in L} \langle x^* \mid y \rangle.$$

In addition, one easily sees that

$$\sup_{x \in K \oplus V} \langle x^* \mid x \rangle = \sup_{x \in K} \langle x^* \mid x \rangle + \sup_{y \in V} \langle x^* \mid y \rangle.$$

Since $x^* \neq 0$, there exists $v \in \mathcal{X}$ such that $\langle x^* \mid v \rangle \neq 0$, and replacing $v$ by $-v$ if necessary, we can assume that $\langle x^* \mid v \rangle > 0$. Since the set $V$ is open and since 0 belongs to $V$, for $t > 0$ sufficiently small, $tv$ must also belongs to $V$. Thus, $\sup_{y \in V} \langle x^* \mid y \rangle > t \langle x^* \mid v \rangle > 0$: this shows the strong separation. □

**Corollary 7.** *Let $\mathcal{X}$ be a locally convex topological vector space and let $K$ and $L$ be two convex subsets of $\mathcal{X}$, and assume that $K$ is compact and $L$ is closed. Then, $K$ and $L$ are strongly separated.*

**Lemma 8.** *In a topological vector space, if $X$ is compact and $Y$ is closed, then the Minkowski sum $X \oplus Y$ is closed.*

*Proof of Lemma 8, Normed spaces.* Let $z_n$ be a converging sequence of points in $Z = X \oplus Y$, so that $z_n = x_n + y_n$ with $x_n \in X$ and $y_n \in Y$. We denote $z$ the limit of $(z_n)_{n \geq 1}$. By compactness of $X$, taking a subsequence if necessary, we can assume that the sequence $(x_n)_{n \geq 1}$ converges to some $x \in X$. The relation $z_n = x_n + y_n$ then implies that the sequence $(y_n)_{n \geq 1}$ converges to some point $y = z - x$, and by closedness of $L$, the point $y$ belongs to $L$. In conclusion, $z = x + y$ belongs to $Z = X \oplus Y \oplus (-L)$ and $M$ is closed. □

*Proof of Lemma 8, Topological vector spaces.* Let $z \notin X \oplus Y$. Then, for every $x \in X$ we have $z - x \notin Y$, implying by closedness of $Y$ that there exists an open set $O_x$ so that $z - x \in O_x$ and $O_x \cap Y = \emptyset$. By continuity of the substraction, there exists opens set $B_x$ containing $x$ and $U_z$ containing $z$ such that $(U_z \ominus B_x) \subseteq O_x$, so that $(U_z \ominus B_x) \cap Y = \emptyset$. The sets $(B_x)_{x \in X}$ cover the compact set $X$, so that by compactness, one may extract a finite family $x_1, \dots, x_N \in X$ s.t. $\bigcup_i B_{x_i}$ contains $X$. Then $U = \cap_i U_{x_i}$ is an open set containing $z$ and such that

$$(U \ominus X) \cap Y \subseteq \left( U \ominus \bigcup_i B_{x_i} \right) \cap Y \subseteq \bigcup_i (U_{x_i} \ominus B_{x_i}) \cap Y = \emptyset,$$

thus implying that $U \cap (X \oplus Y) = \emptyset$. This shows that $(X \oplus Y)$ is closed. □

*Proof of Corollary 7.* We first notice that thanks to the previous lemma, $K \ominus L = K \oplus (-L)$ is closed, and by hypothesis it does not contain the origin. Therefore, there exists an open set $V$ such that $V \cap (K \ominus L) = \emptyset$. Moreover, since we assumed that $\mathcal{X}$ is locally convex, we may assume that $V$ is convex. Thus, we can apply Corollary 6 to conclude that $K$ and $L$ are strongly separated. □

*Remark* 1 (Separation of points). The previous corollary implies that if $\mathcal{X}$ is a separated (Hausdorff) locally convex topological vector space, then for every distinct points $x, y \in \mathcal{X}$, there exists $x^* \in \mathcal{X}^*$ such that $\langle x^* \mid x \rangle > 0 > \langle x^* \mid y \rangle$. In particular, this shows that $\mathcal{X}$ endowed with the *weak topology* $\sigma(\mathcal{X}, \mathcal{X}^*)$ is separated (see Definition 38).

## 1.4 Closed convex sets

We recall that thanks to Lemma 71, in a topological vector $\mathcal{X}$ a *closed half-space* is a set of the form $H = \{x \in \mathcal{X} \mid \langle x^* \mid x \rangle \leq \alpha\}$ where $x^* \in \mathcal{X}^*$ and $\alpha \in \mathbb{R}$.

**Proposition 9.** *Let $K$ be a closed convex set of a locally convex topological vector space $\mathcal{X}$. Then, $K$ is the intersection of all the closed half-spaces containing $K$ and more precisely,*

$$K = \bigcap_{x^* \in \mathcal{X}^*} \{x \in \mathcal{X} \mid \langle x^* \mid x \rangle \leq \sigma_K(x^*)\}, \tag{2}$$

*where $\sigma_K(x^*) = \sup_{x \in K} \langle x^* \mid x \rangle$.*

*Proof.* Denote $L$ the intersection of all the closed half-spaces containing $K$. The inclusion $K \subseteq L$ is obvious, so let us prove that $(\mathcal{X} \setminus K) \subseteq (\mathcal{X} \setminus L)$. To do that, we consider some point $x \in \mathcal{X} \setminus K$. By the strong separation theorem (Corollary 7) applied to the closed convex set $K$ and the compact convex set $\{x\}$, there exists $x^* \in \mathcal{X}^*$ and $\varepsilon > 0$ so that

$$\sigma_K(x^*) = \sup_{z \in K} \langle x^* \mid z \rangle < \langle x^* \mid x \rangle.$$

The closed half-space $H_{x^*} = \{z \in E \mid \langle x^* \mid z \rangle \leq \sigma_K(x^*)\}$ contains $K$, so that by definition $L \subseteq H$. Since $x$ does not belong to $H$, this shows that $x \notin L$. □

**Definition 7** (Closed convex hull)**.** The *closed convex hull* of a subset $A$ of a normed space is the smallest closed convex set containing $A$, or equivalently the intersection of all the closed convex sets containing $A$. We denote it $\overline{\mathrm{conv}} A$.

**Corollary 10.** *Let $\mathcal{X}$ be a locally convex topological vector space. Then, for all subsete $A \subseteq \mathcal{X}$, the closed convex hull of $A$ equals the intersection of all the half-spaces containing $A$.*

*Example* 2. Let $A = \{(x, y) \in \mathbb{R}^2 \mid y \geq \frac{1}{1+x^2}\}$. Then, $\overline{\text{conv}} A = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$.

**Definition 8** (Weak convergence)**.** A sequence $(x_n)$ of points in $\mathcal{X}$ is called *weakly convergent* to $x \in \mathcal{X}$ if for any continuous linear form $x^*$ one has

$$\lim_{n \to +\infty} \langle x^* \mid x_n \rangle = \langle x^* \mid x \rangle.$$

As a consequence of the proposition, we see that all closed convex sets are also closed under weak limits. This is useful to show existence to optimization problems posed in infinite dimensions, e.g. in optimal control, calculus of variations or optimal transport. We can actually prove that closed convex sets are also closed under the *weak topology* on the space. More details are provided in Appendix A.4.

**Corollary 11.** *Let $K \subseteq \mathcal{X}$ be a closed convex set in a topological vector space $\mathcal{X}$, and let $(x_n)$ be a weakly converging sequence, with limit in $x$. Then, $x$ belongs to $K$.*

*Proof.* By Proposition 9, one can write $K = \cap_{i \in I} H_i$, where $H_i = \{\langle x_i^* \mid \cdot \rangle \leq \alpha_i\}$ are closed half-spaces. In particular, for all $i \in I$ and $n$, one has $\langle x_i^* \mid x_n \rangle \leq \alpha_i$. By weak convergence, we deduce that

$$\langle x_i^* \mid x \rangle = \lim_{n \to +\infty} \langle x_i^* \mid x_n \rangle \leq \alpha_i,$$

thus proving that $x$ also belongs to $K$. □

**Proposition 12.** *Let $\mathcal{X}$ be a locally convex topological vector space, and let $K \subseteq \mathcal{X}$ be convex. Then, $K$ is weakly closed if and only if it is closed.*

*Proof.* We already know that (Remark 28) that if $K$ is weakly closed, then $K$ is closed. Conversely, assume that $K$ is closed and convex. By the Proposition 9, $K$ is a (possibly uncountable) intersection of closed halfspaces. Since every closed halfspace is weakly closed, we deduce that $K$ is also weakly closed. □

## 1.5 Support function and normal cone

**Definition 9** (Support function, support hyperplane)**.** The *support function* of a non-empty set $A \subseteq \mathcal{X}$ is the fonction $\sigma_A$ on $\mathcal{X}^*$ defined by

$$\sigma_A : x^* \in \mathcal{X}^* \mapsto \sup_{x \in A} \langle x^* \mid x \rangle \in \mathbb{R} \cup \{+\infty\} \tag{3}$$

A (closed) hyperplane $H = \{\langle x^* \mid \cdot \rangle = \alpha\}$ is a *support hyperplane* to $A$ at at a point $x \in A$, or *supports $A$ at $x$* if $\langle x^* \mid x \rangle = \alpha$ and if $A \subseteq \{\langle x^* \mid \cdot \rangle \leq \alpha\}$. The linear form $x^*$ is then called an *(exterior) normal* to $A$ at $x$.

*Remark* 2. Note that $H$ supports $A$ at $x$ if and only if $\sigma_A(x^*) = \langle x^* \mid x \rangle$, explaining why $\sigma_A$ is called the support function of $A$.

The support function of a singleton ($A = \{x\}$) is linear, while in general the support function of a set is sublinear (as in Definition 4). Support functions were introduced by Minkowski in the finite-dimensional setting. A nice feature of the embedding $K \mapsto \sigma_K$ is that it preserves much of the structure of the set of closed convex subsets of $\mathcal{X}$. We refer to Hörmander [Hör55] for a short summary of the properties of support functions of closed convex sets. Support function can be used to get a geometric interpretation of some rules of subdifferential calculus.

**Proposition 13.** *Let $\mathcal{X}$ be a locally convex topological vector space. Then,*

*(i) Let $K, L \subseteq \mathcal{X}$ be closed and convex. Then, $K \subseteq L$ iff $\sigma_K \leq \sigma_L$.*
*(In particular $\sigma_K = \sigma_L$ if and only if $K = L$.)*

*(ii) Let $K_1, \dots, K_\ell$ be closed and convex and let $\lambda_1, \dots \lambda_\ell > 0$. Then*

$$\sigma_{\bigoplus_i \lambda_i K_i} = \sum_i \lambda_i \sigma_{K_i};$$

*(iii) Let $K_1, \dots, K_\ell$ be closed and convex. Then, $\sigma_{\bigcup_i K_i} = \max_i \sigma_{K_i}$. Then,*

$$\sigma_{\bigcup_i K_i} = \max_i \sigma_{K_i}.$$

*Remark* 3. This proposition is sometimes used to construct algorithm for computing the Minkowski sums of convex sets and especially convex polyhedra.

*Proof.* (i) The direct implication (if $K \subseteq L$, then $\sigma_K \leq \sigma_L$) is obvious from the definition, and the converse follows from Eq. (2).
(ii) We have for every $x^* \in \mathcal{X}$,

$$= \sup_{x \in \bigoplus_i \lambda_i K_i} \langle x^* \mid x \rangle \qquad = \sup_{(x_1,\dots,x_\ell) \in K_1 \times \dots \times K_\ell} \langle x^* \mid \sum_i \lambda_i x_i \rangle = \sum_i \lambda_i \sigma_{K_i}(x^*).$$

(iii) is proven similarly. □

**Definition 10** (Normal cone)**.** Let $K \subseteq \mathcal{X}$ be a convex set. The *normal cone* to $K$ at a point $x$ in $K$ is defined by $\operatorname{Nor}_x K = \{x^* \in \mathcal{X}^* \mid \forall y \in K, \langle x^* \mid x - y \rangle \geq 0\}$. When $x$ is outside of $K$, we define $\operatorname{Nor}_x K = \emptyset$.

Informally, the normal cone $\operatorname{Nor}_x K$ is the set of "slopes" $x^* \in \mathcal{X}^*$ of all support hyperplanes $\{\langle x^* \mid \cdot \rangle = \alpha\}$ to $K$ at the point $x$.

*Example* 3 (Smooth sublevel set). Let $\mathcal{X} = \mathbb{R}^d$, let $g : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^1$ convex function such that $\inf_{\mathbb{R}^d} g < 0$, and let $K = g^{-1}((-\infty, 0])$. Note in particular that $K$ has non-empty interior. Then,

$$\operatorname{Nor}_x K = \begin{cases} \mathbb{R}_+ \nabla g(x) & \text{if } x \in \partial K \\ 0 & \text{if } x \in \operatorname{int}(K) \\ \emptyset & \text{if } x \notin K \end{cases}$$

The notion of normal cone allows to generalize this construction for convex sets that are not differentiable, e.g. polyhedra.

*Example* 4 (Hilbertian setting). In $\mathcal{X}$ is a Hilbert space, the dual space $\mathcal{X}^*$ can be identified by $\mathcal{X}$ itself thanks to Riesz' theorem (i.e. any continuous linear form on $\mathcal{X}$ is of the form $\langle x \mid \cdot \rangle$, where $x \in \mathcal{X}$). Then, the normal cone can be characterized by

$$\mathrm{Nor}_x K = \{v \in \mathcal{X} \mid \exists t > 0,\ \mathrm{p}_K(x + tv) = v\},$$

where $\mathrm{p}_K$ is the orthogonal projection on $K$.

The following statement refers to the *weak*∗ topology on the dual space $\mathcal{X}^*$, which is the coarsest topology on $\mathcal{X}^*$ making the (evaluation) maps $x^* \in \mathcal{X}^* \mapsto \langle x^* \mid x \rangle$ continuous. See §A.4 for more details about this topology, and especially Theorem 77, which shows that if $\mathcal{X}^*$ is a separable normed space, then any bounded sequence in $\mathcal{X}^*$ admits a weak∗ converging subsequence.

**Proposition 14.** *Let $\mathcal{X}$ be a topological vector space and let $K \subseteq \mathcal{X}$ be closed and convex. Then,*

*(i) The normal cone $\mathrm{Nor}_x K$ is convex and weak∗ closed for any $x \in K$, and can be characterized by $x^* \in \mathrm{Nor}_x K \Longleftrightarrow \sigma_K(x^*) = \langle x^* \mid x \rangle$.*

*(ii) If $K$ has non-empty interior, then for any $x \in \partial K$, $\mathrm{Nor}_x K \neq \{0\}$.*

*Proof.* (i) The convexity and weak∗ closedness of $\mathrm{Nor}_x K$ follows from the fact that this set can be written as an intersection of weak∗ closed halfspaces:

$$\mathrm{Nor}_x K = \{x^* \in \mathcal{X}^* \mid \forall y \in K, \langle x^* \mid x - y \rangle \geq 0\} = \bigcap_{y \in K} \{\langle x^* \mid \cdot \rangle \leq \langle x^* \mid x \rangle\}.$$

The characterization of the normal cone follows from the definitions.
(ii) Let $L = \mathrm{int}\, K$ and let $x \in \partial K$. Since $x \notin L$, the open convex set $L$ and the point $x$ can be linearly separated (Corollary 3): there exists $x^* \neq 0$ such that

$$\sup_{y \in L} \langle x^* \mid y \rangle \leq \langle x^* \mid x \rangle.$$

Using that $K$ is the closure of its interior (exercise!), this implies that $x^*$ belongs to $\mathrm{Nor}_x K$. □

# 2 Convex functions

In convex analysis it is very common to encounter functions taking values in the (half) extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. They arise for instance when one includes constraints directly in the minimized functional, i.e. if one replaces the problem

$$\min_K f,$$

where $K \subseteq \mathcal{X}$ is convex, by the unconstrained minimization problem

$$\min_{\mathcal{X}} f + \mathrm{i}_K,$$

where $\mathrm{i}_K$ is the convex indicator function of $K$, i.e.

$$\mathrm{i}_K(x) = \begin{cases} 0 \text{ if } x \in K \\ +\infty \text{ if not.} \end{cases}$$

This may seem like a formal trick, but this formulation is often of practical interest, because it allows to solve constrained convex optimization problem using general algorithms for minimizing the sum of two convex functions. More importantly perhaps, convex functions taking the value $+\infty$ appear naturally one when considers the Legendre-Fenchel conjugate. For instance, we will see that the Legendre-Fenchel conjugate of a norm is the $0/+\infty$-valued indicator of the dual unit ball.

The set $\mathbb{R} \cup \{+\infty\}$ comes with the intuitive calculus rules such $a + (+\infty) = +\infty$ for $a \in \mathbb{R}$, $a \times (+\infty) = +\infty$ for $a > 0$. Often, one adopts the convention $0 \times +\infty = 0$, so that $0 \times \mathrm{i}_K$ is the zero function.

## 2.1 Definition and first properties

**Definition 11** (Domain and epigraph)**.** Let $f : \mathcal{X} \to \overline{\mathbb{R}}$. We call

(i) *domain of* $f$, denoted $\mathrm{dom}(f)$, the subset of $\mathcal{X}$ where $f$ take finite values:

$$\mathrm{dom}(f) = \{x \in \mathcal{X},\ f(x) \neq +\infty\};$$

(ii) *epigraph of* $f$ the subset of $\mathcal{X} \times \mathbb{R}$ above the graph of $f$, i.e.

$$\mathrm{epi}(f) = \{(x,t) \in \mathcal{X} \times \mathbb{R};\, t \geq f(x)\}.$$

A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is *proper* iff $\mathrm{dom}(f) \neq \emptyset$, i.e. if $f \not\equiv +\infty$.

**Definition 12.** A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is *convex* if $\mathrm{epi}(f)$ is a convex subset of $\mathcal{X} \times \mathbb{R}$.

The relationship between this definition and the usual definition of convexity is given in the next proposition.

**Proposition 15.** *A function* $f : \mathcal{X} \to \overline{\mathbb{R}}$ *is convex iff* $\mathrm{dom}(f)$ *is convex and if for all* $x, y$ *in* $\mathrm{dom}(f)$ *and all* $\alpha \in [0,1]$,

$$f\left((1-\alpha)x + \alpha y\right) \leq (1-\alpha) f(x) + \alpha f(y). \tag{4}$$

*Proof.* Assume $f$ is convex. Defining $\Pi_{\mathcal{X}} : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ as the (linear) application $\Pi_{\mathcal{X}}(x,t) = x$, one has

$$\mathrm{dom}(f) = \{x \in \mathbb{R} \mid \exists t \in \mathbb{R}, (x,t) \in \mathrm{epi}(f)\} = \Pi_{\mathcal{X}}(\mathrm{epi}(f)),$$

Thus, $\mathrm{dom}(f)$ is convex as the image of a convex set under an affine application. Moreover, for all $x, y \in \mathrm{dom}(f)$, the points $x' := (x, f(x))$ and $y' := (y, f(y))$ belong to $\mathrm{epi}(f)$. Thus, by convexity of $\mathrm{epi}(f)$, one gets that for all $\alpha \in [0,1]$,

$$z' = (1-\alpha)x' + \alpha y' = ((1-\alpha)x + \alpha y, (1-\alpha)f(x) + \alpha f(y))$$

belongs to the epigraph of $f$, giving (4). The converse is left as an exercise. □

The proposition belows give some examples of how to construct convex functions.

**Proposition 16.** *(i) If $(f_i)_{i\in I}$ is a (possibily uncountable) family of convex functions, the function $x \mapsto \sup_{i\in I} f_i(x)$ is also convex.*
*(ii) Let $A : \mathcal{X} \to \mathcal{Y}$ be affine, and let $f$ be convex on $\mathcal{X}$. Then, $f \circ A$ is convex.*
*(iii) If $f_1, \ldots, f_N$ are convex and if $\lambda_1, \ldots, \lambda_N \geq 0$, then $\sum_{i=1}^N \lambda_i f_i$ is convex.*

*Example* 5. a. Let $K \subseteq \mathcal{X}$ and let $\mathrm{i}_K : E \to \mathbb{R}$ be its indicatrix function, defined by $\mathrm{i}_K(x) = 0$ if $x \in K$ and $+\infty$ if $x \notin K$. Then, $\mathrm{i}_K$ is convex iff $K$ is convex.
b. Linear forms (even discontinuous) are convex.
c. Sublinear functions (e.g. norms, gauges, support functions) are convex.
d. The sublevel sets of a convex functions are convex sets. The reciprocal is false: there exists non-convex functions whose sublevel sets are convex.
e. If $f$ is convex and if $K$ is a convex set, the function $g = f + \mathrm{i}_K$ is convex.
f. The composition of convex functions is not necessarily convex: e.g. $f(x) = x^2$ and $g(x) = -x$ are convex on $\mathbb{R}$, but $g \circ f$ is not.

**Definition 13** (Strict, strong and semi-convexity)**.** A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is *strictly convex* if it is convex and if

$$\forall x \neq y \in \mathrm{dom}\, f,\ \forall t \in (0,1), \quad f\left((1-t)x + ty\right) < (1-t)f(x) + tf(y) \tag{5}$$

On a normed vector space, a function $f$ has *convexity modulus* $\omega : \mathbb{R} \to \mathbb{R}$ if

$$\begin{aligned} &\forall x, y \in \mathrm{dom}\, f,\ \forall t \in [0,1], \\ &\quad f((1-t)x + ty) + t(1-t)\omega(\|x-y\|) \leq (1-t)f(x) + tf(y) \end{aligned} \tag{6}$$

When $\omega(t) = \frac{\alpha}{2}t^2$ with $\alpha > 0$, $f$ is called *$\alpha$-strongly convex.* If $\alpha < 0$, $f$ is called *$\alpha$-semiconvex.* Note that a semi-convex function is usually not convex.

## 2.2 Continuity of convex functions

**Proposition 17.** *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be convex on a topological vector space, and assume that $f$ is finite at $x$ and upper bounded on a neighborhood of $x$. Then, $f$ is continuous at $x$.*

*Remark* 4. Proposition 17 implies that if $f$ is linear, then it is continuous at $x$ if and only if it is upper bounded (or lower bounded) on a neighborhood of $x$. Proposition 17 may be seen as a generalization of the well-know characterization of continuity for linear forms (Lemma 70).

*Proof.* Without loss of generality, we assume that $x = 0$ and $f(x) = 0$, and we let $O$ be a neighborhood of the origin and $M \geq 0$ so that $f \leq M$ on $O$. Replacing $O$ by the intersection $O \cap (-O)$ if necessary, we may assume that $O$ is symmetric, i.e. $O = -O$. Let $\varepsilon \in (0, 1)$ and let $x \in O_\varepsilon := \varepsilon O$. Then, since $x = (1-\varepsilon)0 + \varepsilon \frac{1}{\varepsilon} x$ we get by convexity of $f$

$$f(x) \leq (1-\varepsilon) f(0) + \varepsilon f(x/\varepsilon) \leq \varepsilon M.$$

Using $0 = 1/(1+\varepsilon)x + \varepsilon/(1+\varepsilon)(-x/\varepsilon)$ we get by convexity of $f$

$$f(0) \leq \frac{1}{1+\varepsilon} f(x) + \frac{\varepsilon}{1+\varepsilon} f(-x/\varepsilon),$$

so that, using $-x/\varepsilon \in O$ and $f \leq M$ on $O$ we get

$$f(x) \geq (1+\varepsilon) f(0) - \varepsilon f(-x/\varepsilon) \geq -M\varepsilon.$$

We have proven that $|f| \leq M\varepsilon$ on $O_\varepsilon$, thus showing continuity of $f$ at 0. □

From now on, we denote

$$\operatorname{cont} f = \{x \in \operatorname{dom} f \mid f \text{ is continuous at } x\}.$$

**Proposition 18.** *Let $f : \mathcal{X} \to \mathbb{R}$ be convex on a topological vector space. If there exists an open set on which $f$ is upper bounded, then* $\operatorname{cont} f = \operatorname{int} \operatorname{dom} f$.

*Proof.* Let $O$ be an open set on which $f \leq M$, and let $x \in O$. We will use this hypothesis to prove that $f$ is locally upper bounded near any point $y \in \Omega = \operatorname{int} \operatorname{dom}(f)$. By openness of $\Omega$, there exists $t > 0$ such that $z := y + t(y - x)$ belongs to $\Omega$. The point $y$ belongs to the segment $[x, z]$, and more precisely

$$y = (1-\alpha)x + \alpha z$$

with $\alpha = 1/(1+t)$. The set $O_y = (1-\alpha)O + \alpha z$ is therefore open (as the Minkowski sum of an open set with a non-empty set) and it contains $y$. We now prove that $f \leq \max(M, f(z))$ on $B := \mathrm{B}(y, (1-\alpha)\delta)$. By definition of the Minkowski sum, for all point $w_y \in O_y$, there exists $w_x \in O$ such that $w_y = (1-\alpha)w_x + \alpha z$. Thus, using $f \leq M$ on $O$,

$$f(w_y) \leq (1-\alpha) f(w_x) + \alpha f(z) \leq \max(M, f(z)).$$

The function is then upper bounded in a neighborhood of $y$, and by the previous proposition, it is continuous at $y$. □

*Remark* 5. The hypothesis that $f$ is upper bounded is crucial. For instance, take $f$ a discontinuous linear form on an infinite-dimensional space $\mathcal{X}$. Then, $\mathrm{dom}(f) = \mathcal{X}$, $f$ is convex, but $f$ is discontinuous at every point of $\mathcal{X}$.

*Example* 6. Let $\mathcal{X}$ be a normed space, let $K \subseteq \mathcal{X}$ and assume that $K$ contains the ball $B(0, r)$ for some $r > 0$. Let $f = j_K$ be the gauge of $K$. Then, $|f| \leq 1/r$ on the unit ball. The previous proposition implies that $f$ is continuous on $\mathcal{X}$.

**Corollary 19.** *If $f : \mathcal{X} \to \overline{\mathbb{R}}$ is convex and* $\mathrm{cont}\, f \neq \emptyset$*, then* $\mathrm{cont}\, f = \mathrm{int}\,\mathrm{dom}\, f$.

**Corollary 20.** *If $\mathcal{X}$ is finite dimensional, and $f : \mathcal{X} \to \overline{\mathbb{R}}$ is convex, then* $\mathrm{cont}\, f = \mathrm{int}\,\mathrm{dom}\, f$.

*Proof.* Take $f : \mathcal{X} \to \overline{\mathbb{R}}$ convex. If $\mathrm{dom}(f)$ has non-empty interior, then there exists points $x_1, \ldots, x_n$ in the interior so that $K = \mathrm{conv}(\{x_1, \ldots, x_n\})$ also has non-empty interior (take e.g. $x_1 \in \mathrm{int}\,\mathrm{dom}(f)$ and $x_2, \ldots, x_n \in \mathrm{int}\,\mathrm{dom}(f)$ so that $\mathrm{vect}(\{x_2 - x_1, \ldots, x_n - x_1\}) = \mathcal{X}$). Moreover, since every element of $K$ is a convex combination of the points $x_1, \ldots, x_n$,

$$\forall x \in K, f(x) \leq M = \max_i f(x_i).$$

We conclude using the previous corollary. □

In the remainder of this paragraph, we assume that $\mathcal{X}$ is a normed space. Then, the continuity of convex functions can be improved into local Lipschitzness.

**Definition 14** (Locally Lipschitz)**.** A function $f : \Omega \subseteq \mathcal{X} \to \mathbb{R}$ is *locally Lipschitz* on the open set $\Omega$ if every point in $\Omega$ has a neighborhood on which $f$ is Lipschitz:

$$\forall x_0 \in \Omega, \exists \delta > 0, \exists M \in \mathbb{R}, \quad \forall x, y \in \mathrm{B}(x_0, \delta), |f(x) - f(y)| \leq M \|x - y\|.$$

We start with the following intermediary result.

**Lemma 21.** *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be convex and assume that $|f| \leq M$ on* $\mathrm{B}(x_0, 2\delta)$*. Then, $f$ is $\frac{2M}{\delta}$-Lipschitz on* $\mathrm{B}(x_0, \delta)$.

*Proof.* Let $x, y \in \mathrm{B}(x_0, \delta)$, and define $\alpha = \|x - y\|$ and $z := y + \frac{\delta}{\alpha}(y - x)$ which by construction belongs to $\mathrm{B}(x_0, 2\delta)$. The point $y$ is a convex combination of $z$ and $x$,

$$y = \frac{\delta/\alpha}{1 + \delta/\alpha} x + \frac{1}{1 + \delta/\alpha} z,$$

so that by convexity of $f$,

$$\begin{aligned} f(y) &\leq \frac{\delta/\alpha}{1 + \delta/\alpha} f(x) + \frac{1}{1 + \delta/\alpha} f(z) \\ \text{i.e.} \quad f(y) - f(x) &\leq \frac{-1}{1 + \delta/\alpha} f(x) + \frac{1}{1 + \delta/\alpha} f(z) \end{aligned}$$

We now use the upper bound on $|f|$:

$$f(y) - f(x) \leq \frac{2M}{1 + \delta/\alpha} \leq \frac{2M}{\delta}\alpha = \frac{2M}{\delta} \|x - y\|.$$

We conclude by inversing the role of $x$ et $y$ to get the desired Lipschitz property. □

Combining Lemma 21 and the previous continuity results, we get:

**Proposition 22.** *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be convex on a normed space. If* $\operatorname{cont} f \neq \emptyset$*, then $f$ is locally Lipschitz in* $\operatorname{int}\operatorname{dom}(f)$*.*

*Remark* 6. The assumption that $\Omega$ is open cannot be removed. Take for instance $f : x \in [0,1] \mapsto -\sqrt{x}$. Then, $\lim_{x\to 0, x\neq 0} f'(x) = -\infty$, so that $f$ is not locally Lipschitz at 0.

## 2.3 Lower semicontinuous convex functions

**Definition 15** (Lower semi-continuity)**.** Let $\mathcal{X}$ be a topological vector space. A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is called *lower semicontinuous* or *lsc* if for all $a \in \mathbb{R}$ the sublevel set $\operatorname{lev}_{\leq a} f$ is closed, where

$$\operatorname{lev}_{\leq a} f = \{x \in \mathcal{X} \mid f(x) \leq a\}.$$

Equivalently, $f$ is lower semi-continuous if the strict superlevel set $\operatorname{lev}_{>a} f$ is open for all $a \in \mathbb{R}$. Lower semi-continuous functions are sometimes called *closed* in the convex analysis literature.

*Remark* 7 (Relation to continuity). A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is continuous if the sublevel sets $\operatorname{lev}_{\leq a}$ and superlevel sets $\operatorname{lev}_{\geq a}$ are closed for all $a \in \mathbb{R}$, or equivalently if the strict sublevel sets $\operatorname{lev}_{<a}$ and strict superlevel sets $\operatorname{lev}_{>a}$ are open. Indeed, this openness assumption implies that for all $a \leq b$,

$$f^{-1}((a,b)) = \operatorname{lev}_{>a} f \cap \operatorname{lev}_{<b} f$$

is open, as the intersection of a finite number of open sets. Recalling that every open set $O$ of $\mathbb{R}$ is a (possibly uncountable) union of open intervals, we deduce that the inverse image $f^{-1}(O)$ is open, proving the continuity of $f$.

*Example* 7 (Indicator function). The $0/+\infty$-valued indicator function $\mathrm{i}_A$ of a set $A \subseteq \mathcal{X}$ is lower semi-continuous iff $A$ is closed.

**Proposition 23.** *Let $\mathcal{X}$ be a topological space, let $f_i : \mathcal{X} \to \overline{\mathbb{R}}$ be lower semicontinuous functions indexed by some set $I$. Then,*

*(i) the function $\sup_{i\in I} f_i$ is lower semicontinuous;*

*(ii) if $I$ is finite and if $(\lambda_i)_{i\in I}$ are positive, then $\sum_{i\in I} \lambda_i f_i$ is lower semi-continuous.*

*Remark* 8 (Lsc envelope). Let $\mathcal{X}$ be a topological space and let $f : \mathcal{X} \to \overline{\mathbb{R}}$. The previous propositions show that the function $\hat{f}$ defined by

$$\hat{f}(x) = \sup\{g(x) \mid g : \mathcal{X} \to \overline{\mathbb{R}} \text{ is lsc and } g \leq f\}.$$

is lower semicontinuous, and is in fact the largest lower semicontinuous function that lies below $f$. It is called the lower semi-continuous envelope of $f$.

*Proof.* (i) Let $f = \sup_{i \in I} f_i$. Then its sublevel sets

$$\mathrm{lev}_{\leq a} f = \{x \in \mathcal{X} \mid \forall i, f_i(x) \leq a\} = \bigcap_{i \in I} \mathrm{lev}_{\leq a} f_i,$$

are closed as intersections of closed sets.

(ii) Let $f = \sum_{i \in I} \lambda_i f_i$. Then,

$$\begin{aligned} \mathrm{lev}_{>\alpha} f &= \{x \in \mathcal{X} \mid \sum_{i \in I} \lambda_i f_i(x) > \alpha\} \\ &= \bigcup_{t \in \mathbb{R}^I \text{ s.t. } \sum_{i \in I} \lambda_i t_i > \alpha} \left[ \bigcap_{i \in I} \mathrm{lev}_{>t_i} f_i \right], \end{aligned}$$

is open as an (arbitrary) union of finite intersections of open sets. □

We have the following alternative characterizations of lower semicontinuity. In the following the product space $\mathcal{X} \times \mathbb{R}$ is always endowed with the product topology (Definition 34). This means that a subset $O \subseteq \mathcal{X} \times \mathbb{R}$ is open if for all point $(x, t) \in O$, there exists an neighborhood $N_x$ of $x$ in $\mathcal{X}$ and $\varepsilon > 0$ such that $N_x \times (t - \varepsilon, t + \varepsilon) \subseteq O$.

**Proposition 24.** *The following properties are equivalent:*

*(i) the function $f$ is lower-semicontinuous ;*

*(ii) the set* $\mathrm{epi}(f) \subseteq \mathcal{X} \times \mathbb{R}$ *is closed ;*

The limit inferior of a real-valued sequence $(f_n) \in \mathbb{R}^{\mathbb{N}}$, denoted $\liminf_{n \to +\infty} f_n$, is the smallest cluster point of the sequence. In a metric space $\mathcal{X}$, a function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is lower semi-continuous if for all $x \in \mathcal{X}$ and every sequence $(x_n)_{n \geq 1}$ converging to $x$ one has

$$f(x) \leq \liminf_{n \to +\infty} f(x_n).$$

This characterization in the metric setting is left as an exercise.

*Proof.* (ii) $\implies$ (i): Assume that $\mathrm{epi}(f)$ is closed. Then, for all $a \in \mathbb{R}$, the sublevel set of $f$ can be written as

$$\mathrm{lev}_{\leq a} f = \Pi_{\mathcal{X}}(\mathrm{epi}(f) \cap \mathcal{X} \times [a, +\infty)),$$

i.e. the image under the continuous projection $\Pi_{\mathcal{X}}$ of the intersection between two closed subsets of $\mathcal{X} \times \mathbb{R}$. Thus, $\mathrm{lev}_{\leq a} f$ is closed for every $a$ and $f$ is lsc.

(i) $\implies$ (ii): We will show that the complement $(\mathcal{X} \times \mathbb{R}) \setminus \mathrm{epi}(f)$ is open. Consider a point $(x, t) \notin \mathrm{epi}(f)$, i.e. such that $t < f(x)$. There exists $\alpha \in \mathbb{R}$ such that $t < \alpha < f(x)$, so that $x$ does not belong to the sublevel set $\mathrm{lev}_{\leq \alpha} f$. By closedness of this set, there exists a small neighborhood $N_x$ around $x$ so that $N_x \cap \mathrm{lev}_{\leq \alpha} f = \emptyset$. The set $N_x \times (-\infty, \alpha)$ is a neighborhood of $(x, t)$ in $\mathcal{X} \times \mathbb{R}$ and it does not intersect $\mathrm{epi}(f)$, proving that $(\mathcal{X} \times \mathbb{R}) \setminus \mathrm{epi}(f)$ is closed. □

The next proposition asserts that for convex functions, weak and strong lower-semicontinuity coincide. This is useful because typically much easier to prove that a function is lower semi-continuous than to prove that it is weakly lower-semicontinuous.

**Proposition 25.** *Let $\mathcal{X}$ be a locally convex topological vector space, and let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be convex. Then $f$ is lower semi-continuous if and only if $f$ is weakly lower semi-continuous.*

*Proof.* This follows from Proposition 12: since the sublevel sets of a convex function are convex, they are closed iff they are weakly closed. □

**Definition 16** (Convex lsc functions)**.** The set of lower-continuous convex functions is denoted $\Gamma(\mathcal{X})$, and the set of *proper* lsc convex functions is denoted

$$\Gamma_0(\mathcal{X}) = \{f \in \Gamma(\mathcal{X}) \mid \mathrm{dom}(f) \neq \emptyset\}.$$

**Proposition 26.** *Let $\mathcal{X}$ be a locally convex topological vector space. If $f \in \Gamma(\mathcal{X})$ is convex lsc, then $f$ is equal to the supremum of its affine minorant, i.e.*

$$\forall x \in \mathcal{X}, \quad f(x) = \sup\{\langle x^* \mid x\rangle + b \mid x^* \in \mathcal{X}, b \in \mathbb{R} \textit{ s.t. } f \geq \langle x^* \mid \cdot\rangle + b\}.$$

*Proof.* If $\mathrm{dom} f = \emptyset$, $f$ is constant and equal to $+\infty$, so that there is nothing to prove. Let us suppose that $\mathrm{dom} f \neq \emptyset$, or equivalently that $K = \mathrm{epi}(f)$ is a non-empty (closed convex) subset of $\mathcal{X} \times \mathbb{R}$. Define

$$g(x) = \sup\{\langle x^* \mid x\rangle + b \mid x^* \in \mathcal{X}^*, b \in \mathbb{R} \text{ s.t. } f \geq \langle x^* \mid \cdot\rangle + b\}.$$

The function $g$ is a supremum of functions below $f$, so that $g \leq f$. We now prove that $g \geq f$. For this purpose, we will prove that for all $x_0$ in $\mathcal{X}$ and all $t_0 < f(x_0)$, one has $g(x_0) \geq t_0$. Since $t_0 < f(x_0)$, the point $(x_0, t_0)$ does not belong to $K$, so that we can strongly separate the closed convex set $K$ from the compact convex set $\{(x_0, t_0)\}$ (Corollary 7). This means that there exists a continuous linear form on $\mathcal{X} \times \mathbb{R}$, which can be described by $(x^*, v) \in \mathcal{X}^* \times \mathbb{R}$ such that

$$\inf_{(x,t)\in\mathrm{epi}(f)} \langle x^* \mid x\rangle + tv > \langle x^* \mid x_0\rangle + t_0 v$$

Taking $(x, t) = (x_0, f(x_0))$, we get in particular $vf(x_0) > t_0 v$, i.e. $v(f(x_0) - t_0) > 0$. Since $f(x_0) > t_0$, we deduce that $v > 0$. Therefore, taking $(x, f(x)) \in \mathrm{epi}(f)$ in the previous inequality, we deduce

$$\forall x \in \mathcal{X}, \quad f(x) \geq \langle \frac{x^*}{v} \mid x_0 - x\rangle + t_0.$$

By definition of $g$ as the supremum of affine minorant of $f$, we get

$$g(x) \geq \langle \frac{x^*}{v} \mid x_0 - x\rangle + t_0,$$

In particular, $g(x_0) \geq t_0$ for any $t_0 < f(x_0)$ so that $g(x_0) \geq f(x_0)$. □

Since an intersection of closed convex sets is convex and closed, we deduce that a supremum of convex lower semi-continuous functions is convex lower semi-continuous. This leads to the following definition.

**Definition 17** (Lsc convex envelope)**.** Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a function. Its convex lower-semicontinuous envelope is defined by

$$\overline{\operatorname{conv}} f = \sup\{g \mid g \leq f \text{ and } g \in \Gamma(\mathcal{X})\}.$$

By the previous proposition, we can also characterize $\overline{\operatorname{conv}} f$ by

$$\overline{\operatorname{conv}} f(x) = \sup\{\langle x^* \mid x\rangle + b \mid x^* \in \mathcal{X}, b \in \mathbb{R} \text{ s.t. } f \geq \langle x^* \mid \cdot\rangle + b\}.$$

*Example* 8 (Norm). Let $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ be a convex non-decreasing function, which we assume to be lower semi-continuous. Then, $f(x) := \phi(\|x\|)$ is convex and strongly lower semi-continuous, and therefore weakly lower-semicontinuous.

We now show how to represent this function $f$ as a supremum of affine functions for two choices of $\phi$ (and refer to Section 2.4 for the general picture). When $\phi(t) = t$,

$$f(x) = \sup_{\|x^*\| \leq 1} \langle x^* \mid x\rangle,$$

where $\|x^*\|_* = \sup_{x \in \mathrm{B}(0,1)} \langle x^* \mid x\rangle$ is the dual norm. If $\phi(t) = \frac{1}{2}t^2$, then

$$f(x) = \sup_{x^* \in \mathcal{X}^*} \langle x^* \mid x\rangle - \phi(\|x_*\|).$$

## 2.4 Convex conjugate

The convex conjugate is an analogue, for convex functions, of the notion of support function for convex sets. In particular, it allows to represent explicitly a lower semi-continuous convex function as a supremum of affine functions (Theorem 28) in a similar way the support function may be used to represent a closed convex set as an intersection of closed halfspaces (Equation (2)).

**Definition 18** (Convex conjugate)**.** Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a proper function on a space $\mathcal{X}$. Its *convex conjugate* or *Legendre-Fenchel transform* is the function $f^* : \mathcal{X}^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in \mathcal{X}} \langle x^* \mid x\rangle - f(x).$$

Note that since $f$ is proper, there exists $x \in \operatorname{dom}(f)$ implying that $f^* \geq \langle \cdot \mid x\rangle - f(x)$. In particular, $f^*$ never takes the value $-\infty$.

*Example* 9. We start of a few examples of conjugate functions.

a. Let $A$ be a non-empty subset of $\mathcal{X}$, and let $\mathrm{i}_A$ be its indicator function. Then,

$$\forall x^* \in \mathcal{X}^*, \quad \mathrm{i}_A^*(x^*) = \sup_{x \in \mathcal{X}} \langle x^* \mid x\rangle - \mathrm{i}_A(x) = \sup_{x \in A} \langle x^* \mid x\rangle = \sigma_A$$

One can therefore think of the Legendre-Fenchel transform as a generalization of the support function.

b. Let $f = \|.\|$ be the norm on $\mathcal{X}$. Then,

$$\forall x^* \in \mathcal{X}^*, f^*(x^*) = \sup_{x \in \mathcal{X}} \langle x^* \mid x \rangle - \|x\|$$

If $\|x^*\| > 1$, by definition there exists $x \in \mathcal{X}$ such that $\langle x^* \mid x \rangle - \|x\| > 0$. Multiplying $x$ by $\lambda$, one can see that $f^*(x^*) = +\infty$. On the other hand, if $\|x^*\|_* \leq 1$, $\langle x^* \mid x \rangle - \|x\| \leq 0$ with equality when $x = 0$. Thus, $f^*(x^*) = 0$. Finally, $f^*$ is the indicator function of the unit ball in $\mathcal{X}^*$.

c. Let $f = \langle z^* \mid \cdot \rangle$ be a continuous linear form. Then,

$$f^*(x^*) = \sup_{x \in E} \langle x^* - z^* \mid x \rangle = \begin{cases} 0 & \text{if } x^* = z^* \\ +\infty & \text{if not.} \end{cases}$$

Thus, $f^* = i_{\{z^*\}}$.

d. If $f(x) = \frac{1}{p} |x|^p$ on $\mathbb{R}$ with $p \in (1, +\infty)$, and if we identify the dual space with $\mathbb{R}$, one can verify that $f^*(y) = \frac{1}{q} |y|^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

**Proposition 27** (Basic properties)**.** *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be proper. Then, the following properties hold*

*(i)* [Fenchel-Young]*:* $\forall (x, x^*) \in \mathcal{X} \times \mathcal{X}^*, f(x) + f^*(x^*) \geq \langle x^* \mid x \rangle$,

*(ii)* $f^*(0) = -\inf f$,

*(iii) $f^*$ is convex and weak* lower semicontinuous,*

*(iv) if in addition $f \in \Gamma_0(\mathcal{X})$, then $f^* \in \Gamma_0(\mathcal{X}^*)$.*

*Example* 10. Hölder inequality: with $f(x) = \frac{1}{p} |x|^p$ on $\mathbb{R}$, we have :

$$xy \leq f(x) + f^*(y) = \frac{1}{p} |x|^p + \frac{1}{q} |x|^q$$

Summing, we get for all $x, y \in \mathbb{R}^n$, such that $\|x\|_p = \|y\|_q = 1$,

$$\langle x \mid y \rangle = \sum_{i=1}^{n} x_i y_i \leq \frac{1}{p} \|x\|_p + \frac{1}{q} \|y\|_q = \|x\|_p \|y\|_q .$$

By homogeneity, this inequality remains true for all $x, y \in \mathbb{R}^n$.

*Proof.* (iii) These properties hold because $f^*$ is a supremum of functions of the form $x^* \mapsto \langle x^* \mid x \rangle - f(x)$, which are convex and weak* continuous.
(iv) If $f \in \Gamma_0(\mathcal{X})$ then by Proposition 26 it is equal to the supremum of its affine minorants. In particular, there exists $x^* \in \mathcal{X}^*$ and $\alpha \in \mathbb{R}$ s.t. $f \geq \langle x^* \mid \cdot \rangle + \alpha$. Therefore, $f^*(x^*) = \sup_{x \in E} \langle x^* \mid x \rangle - f(x) \leq \sup_{x \in E} \langle x^* - x^* \mid x \rangle - \alpha \leq -\alpha$, so that $f^*$ is indeed proper. ☐

**Definition 19** (Biconjugate)**.** The *convex biconjugate* of a function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is the function $f^{**}$ on $\mathcal{X}$ defined by

$$f^{**}(x) = \sup_{x^* \in \mathcal{X}^*} \langle x^* \mid x \rangle - f^*(x^*).$$

**Theorem 28** (Fenchel-Moreau). *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a proper function. Then, $f^{**}$ is the lower semicontinuous convex envelope of $f$ (Definition 17):*

$$f^{**} = \overline{\text{conv}} f,$$

*In particular one always has $f^{**} \leq f$, with equality if and only $f \in \Gamma_0(\mathcal{X})$.*

*Proof.* We first prove $f^{**} \leq f$. By Fenchel-Young, for all $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$ we have $f(x) + f^*(x^*) \geq \langle x^* \mid x \rangle$. Applying this inequality in the definition of $f^{**}$ we obtain

$$f^{**}(x) = \sup_{x^* \in \mathcal{X}^*} \langle x^* \mid x \rangle - f^*(x^*) \leq f(x).$$

Since in addition is convex and lower semicontinuous, we deduce from the definition of $\overline{\text{conv}} f$ that $f^{**} \leq \overline{\text{conv}}(f)$. To prove the converse inequality, we use that $\overline{\text{conv}} f$ is equal to the supremum of the affine minorants of $f$. It is therefore sufficient to prove that $f^{**}$ is larger than any affine minorant of $f$. Consider $x^* \in \mathcal{X}^*$ and $\alpha \in \mathbb{R}$ s.t. $f \geq \langle x_0^* \mid \cdot \rangle + \alpha$. Then,

$$f^*(x^*) = \sup_{x \in \mathcal{X}} \langle x^* \mid x \rangle - f(x) \leq \sup_{x \in \mathcal{X}} \langle x^* \mid x \rangle - (\langle x^* \mid x \rangle + \alpha) = -\alpha.$$

Thus,

$$\forall x \in \mathcal{X}, \quad f^{**}(x) = \sup_{z^* \in \mathcal{X}^*} \langle z^* \mid x \rangle - f^*(z^*) \geq \langle x^* \mid x \rangle - f^*(x^*) = \langle x^* \mid x \rangle + \alpha. \quad \square$$

## 2.5 Application: Existence of minimizers

**Examples in functional spaces** For the next two examples, we will use the Sobolev space $W^{1,p}(\Omega)$, where $\Omega$ is a *bounded* open set of $\mathbb{R}^d$ and $p > 1$. Roughly speaking, an element of $W^{1,p}(\Omega)$ is a function $u \in \mathrm{L}^p(\Omega)$ such that for all $1 \leq i \leq d$, the partial derivative $\partial_i u$ belongs to $\mathrm{L}^p(\Omega)$ in the sense of distributions.[1] This Sobolev space is a Banach space, when endowed with the norm

$$\|u\|_{W^{1,p}}^p = \|u\|_{\mathrm{L}^p(\Omega)}^p + \|\nabla u\|_{\mathrm{L}^p(\Omega,\mathbb{R}^d)}^p,$$

and a reflexive space when $1 < p < +\infty$. We consider $W_0^{1,p}(\Omega)$ to be the space of elements of $\Omega$ who vanish on $\partial\Omega$, formally defined as the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$. When $p = 2$, these spaces are Hilbert spaces, and one denotes $H^1(\Omega) = W^{1,2}(\Omega)$ and $H_0^1(\Omega) = W_0^{1,2}(\Omega)$. We will use the following results:

- **Poincaré's inequality:** For a bounded open set $\Omega$, there exists a constant $C = C(\Omega)$ such that

$$\forall u \in W_0^{1,p}(\Omega), \|u\|_{W^{1,p}(\Omega)} \leq C \|\nabla u\|_{\mathrm{L}^p(\Omega,\mathbb{R}^d)}.$$

- **Banach-Alaoglu's theorem:** let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $W^{1,p}(\Omega)$ (for $p \in (1, +\infty)$). Then, $(u_n)$ admits a weakly converging subsequence.

[1] This means that there exists a function $v_i \in \mathrm{L}^p(\Omega)$ such that for all smooth and compactly supported test function $\phi \in \mathcal{C}_c^\infty(\Omega)$ one has $\int_\Omega (\partial_i \phi) u = -\int_\Omega u v_i$. One then defines $\partial_i u := v_i$.

*Example* 11 (Obstacle problem). Let $\mathcal{X} = H_0^1(\Omega)$, $K = \{u \in \mathcal{X} \mid u \geq \Psi \text{ a.e.}\}$ for some $\Psi \in \mathcal{C}_c^\infty(\Omega)$ and consider the minimization problem

$$\min_{u \in K} f(u) \text{ with } f(u) := \int_\Omega \|\nabla u(x)\|^2 \,\mathrm{d}x.$$

Following Exercise 5, one can prove that $K$ is closed, and this set is obviously convex. In addition, the function $f$ is strictly convex, so that the problem $\min_K f$ admits *at most* one solution. Let $(u_k)_{k\geq 1}$ be a minimizing sequence, that is a sequence of elements of $K$ such that

$$\inf_K f = \lim_{k\to+\infty} f(u_k).$$

We can assume, without loss of generality that $(f(u_k))_{k\geq 1}$ is decreasing. Then, Poincaré's inequality implies that $(u_k)_{k\geq 1}$ is bounded, so that by Banach-Alaoglu, this sequence admits a weakly converging subsequence, with limit $u$. Since $K$ is convex and closed, it is weakly closed, so that $u$ belongs. Finally, since $f$ is weakly lower semicontinuous we have

$$f(u) \leq \liminf_{k\to+\infty} f(u_k) = \inf_K f,$$

thus showing the existence of a minimizer

**Proposition 29.** *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set, $p \geq 1$, and $\phi \in \mathcal{C}^0(\mathbb{R}^k)$ be a convex function and assume that $\phi(x) \geq a\|x\|^p$ for some $a \in \mathbb{R}$. Then the function*

$$f : v \in \mathrm{L}^p(\Omega, \mathbb{R}^k) \mapsto \int_\Omega \phi(v(x))\mathrm{d}x \in \overline{\mathbb{R}}.$$

*is convex and weakly lower-semicontinuous.*

*Proof.* We first prove that $f$ is (strongly) lower semi-continuous. To do that, we consider a sequence $(v_n)$ converging (in norm) to $v \in \mathrm{L}^p(\Omega, \mathbb{R}^k)$, and we wish to prove that $f(v) \leq \liminf_{n\to+\infty} f(v_n)$. Taking a subsequence if necessary, we may assume without loss of generality that $(v_n)$ converges Lebesgue-almost everywhere to $v$. Applying Fatou's lemma to $x \mapsto \phi(v_n(x)) - a\|v_n(x)\|^p \geq 0$ we get

$$\begin{aligned} f(v) - a\|v\|^p_{\mathrm{L}^p(\Omega,\mathbb{R}^k)} &= \int_\Omega \lim_{n\to+\infty} \phi(v_n(x)) - a\|v_n(x)\|^p \,\mathrm{d}x \\ &\leq \liminf_{n\to+\infty} \int_\Omega \phi(\|v_n(x)\|) - a\|v_n(x)\|^p \,\mathrm{d}x \\ &= \liminf_{n\to+\infty} f(v_n) - a \lim_{n\to+\infty} \|v_n\|^p_{\mathrm{L}^p(\Omega,\mathbb{R}^k)}, \end{aligned}$$

thus showing that $f(v) \leq \liminf_{n\to+\infty} f(v_n)$, and establishing that $f$ is strongly lower semi-continuous. If one assumes $\phi$ to be convex, then $f$ is also convex, and by the previous proposition we deduce that $f$ is weakly lower-semicontinuous. □

*Remark* 9 (Weakly lsc integral functionals are convex!). For simplicity, we consider an integral functional defined on the unit interval $\Omega = [0,1] \subseteq \mathbb{R}$. Let $\phi \in \mathcal{C}^0(\mathbb{R})$ and define $f : \mathrm{L}^2(\Omega, \mathbb{R}^d) \to \mathbb{R}$ by

$$f(v) = \int_\Omega \phi(v(x))\mathrm{d}x.$$

We assume that $f$ is weakly lower semi-continuous. Given $A, B \in \mathbb{R}^d$ and for any $N \geq 2$, we consider the piecewise-constant function $v_N : \Omega \to \mathbb{R}$ defined by

$$v_N|_{[\frac{k}{2N}, \frac{k+1}{2N})} = \begin{cases} A & \text{if } k \text{ is even} \\ B & \text{if } k \text{ is odd} \end{cases}$$

Then, the sequence $v_N$ weakly converges to the constant function $v \equiv \frac{1}{2}(A+B)$, so that by lower semi-continuity of $f$ we have $f(v) \leq \liminf_{N\to+\infty} f(v_N)$. This directly implies $\phi(\frac{A+B}{2}) \leq \frac{1}{2}\phi(A) + \frac{1}{2}\phi(B)$, so that $\phi$ and $f$ must be convex.

*Example* 12 (A variational problem). On the space $\mathcal{X} := W_0^{1,p}(\Omega)$ we consider the integral functional

$$f(u) = \int_\Omega \phi(\nabla u(x))\mathrm{d}x,$$

with $\phi : \mathbb{R} \to \mathbb{R}$ a convex continuous function satisfying $\phi(v) \geq a\|v\|^p$ and $a > 0$. Then, the minimization problem

$$\inf_{u\in\mathcal{X}} f(u),$$

admits a solution. To see this, consider a minimizing sequence, i.e. a sequence $(u_n)$ of elements of $\mathcal{X}$ such that $\lim_{n\to+\infty} f(u_n) = \inf_{\mathcal{X}} f$. In particular, $f(u_n) \leq M$ for some constant $M > 0$. By the assumption on $f$ and Poincaré's inequality, we deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded,

$$\|u_n\|^p_{W^{1,p}(\Omega)} \leq C^p \|\nabla u_n\|^p_{\mathrm{L}^p(\Omega,\mathbb{R}^d)} \leq \frac{C^p}{a} f(u_n) \leq \frac{MC^p}{a}$$

By Banach-Alaoglu's theorem, taking a subsequence if necessary, we may assume that the sequence $(u_n)_{n\in\mathbb{N}}$ admits a weakly converging subsequence, with weak limit $u$. Then, by Proposition 29, we know that the functional $f$ is weakly lower semi-continuous on $\mathcal{X}$, so that

$$f(u) \leq \liminf_{n\to+\infty} f(u_n) = \lim_{n\to+\infty} f(u_n) = \inf_{\mathcal{X}} f.$$

This shows that $u$ minimizes $f$ over $\mathcal{X}$.

These two examples can be generalized as follows.

**Definition 20** (Coercivity)**.** A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is *coercive* on a subset $K$ of a normed vector space $\mathcal{X}$ if for every sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $K$ satisfying $\lim_{n\to+\infty} \|x_n\| = +\infty$, one has $\lim_{n\to+\infty} f(x_n) = +\infty$.

**Proposition 30.** *Let $\mathcal{X}$ be a separable and reflexive space, let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be convex, proper, lsc, let $K \subseteq \mathcal{X}$ be closed and convex, and consider the minimization problem*

$$\inf_{x \in K} f(x).$$

*Then the minimum is attained, provided that one of the following conditions hold*

*a. the set $K$ is bounded ;*

*b. or the function $f$ is coercive on $K$.*

*The minimizer is unique if $f$ is strictly convex on $K$.*

*Proof.* Exercise. □

**Example in spaces of measures** In the next example, we work in the space of Radon measures over a compact subset $\Omega \subseteq \mathbb{R}^d$, i.e.

$$\mathcal{M}(\Omega) := \mathcal{C}^0(\Omega)^*.$$

*Example* 13 (Sparse spikes deconvolution). Consider a measure on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ made of finitely many Dirac mass (called “spikes” in this setting), i.e.

$$\mu = \sum_{1 \leq i \leq N} \alpha_i \delta_{x_i}.$$

The measure $\mu$ is not known, but one has access to a blurry version of it, $y = \Phi\mu \in \mathrm{L}^2(\mathbb{T})$. More precisely, we are given a linear operator (typically a convolution operator) $\Phi : \mathcal{M}(\Omega) \to \mathrm{L}^2(\mathbb{T})$, which we assume to be sequentially weak*-continuous. The Beurling LASSO problem is the following optimization problem, for $\lambda > 0$:

$$\inf_{\nu \in \mathcal{M}(\mathbb{T})} \|\Phi\nu - y\|^2_{\mathrm{L}^2(\mathbb{T})} + \lambda \|\nu\|_{TV},$$

We now show that this problem admits a solution. First note that $g = \|\cdot\|_{TV}$ and $h = \|\Phi(\cdot) - y\|^2_{\mathrm{L}^2(\mathbb{T})}$ are sequentially weak* lsc : for $g$ this follows from the definition of the dual norm as a supremum, and for $h$ this is by assumption. In addition, any minimizing sequence $(\nu_n)_{n\in\mathbb{N}}$ is bounded, ensuring by Banach-Alaoglu the existence of a (not relabeled) weak*-converging subsequence, with limit $\nu$. Thus,

$$(g+h)(\nu) \leq \liminf_{n\to+\infty} (f+g)(\nu_n) = \inf_{\nu} g + h.$$

Note also that $g + h$ is a convex function, which is crucial for being able to solve the minimization problem numerically, but we did not use this fact for proving existence. In fact, since the space $\mathcal{M}(X)$ is not reflexive, we cannot apply Proposition 30.

# 3 Subdifferential

## 3.1 Directional derivatives

In this section we study the "algebraic" properties of the directional derivatives of convex functions. All the properties hold for any vector space $\mathcal{X}$, even without a topology. Since we are only using the linear properties of the space, one shouldn't hope to get any information about regularity (even continuity) of the functions.

**Definition 21.** Given a function $f : \mathcal{X} \to \overline{\mathbb{R}}$, $x \in \operatorname{dom}(f)$ and $v \in \mathcal{X}$, we define the directional derivative as the following limit (if it exists):

$$f^+(x; v) = \lim_{\varepsilon \to 0^+} \frac{f(x + \varepsilon v) - f(x)}{\varepsilon} \tag{7}$$

**Proposition 31.** *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be convex and let $x \in \operatorname{dom}(f)$. Then, the directional derivative $f^+(x; v) \in \mathbb{R} \cup \{\pm\infty\}$ is well defined for any $v \in \mathcal{X}$ and moreover,*

$$f^+(x; v) = \inf_{\varepsilon > 0} \frac{f(x + \varepsilon v) - f(x)}{\varepsilon} \tag{8}$$

*Remark* 10. The limit defining $f^+(x; v)$ can take the values $\pm\infty$.

(i) Since the limit (7) can be replaced by an infimum (8), one has $f^+(x; v) = +\infty$ if and only if the half-line $\{x + tv \mid t > 0\}$ does not intersect $\operatorname{dom} f$. As a consequence, if $x$ belongs to the interior of $\operatorname{dom}(f)$, then $f^+(x; v) < +\infty$.

(ii) It is easy to build examples of convex functions on $\mathbb{R}$ such that $f^+(x; v) = -\infty$ for some $x \in \operatorname{dom}(f)$. (Exercise: find one.)

The proof is deduced directly from the following lemma, which is a well-known property of 1D convex functions (slopes are increasing).

**Lemma 32.** *The function $\varepsilon \in \mathbb{R}_+ \mapsto \frac{1}{\varepsilon}(f(x + \varepsilon v) - f(x))$ is increasing.*

*Proof.* Let $\varepsilon_2 > \varepsilon_1 \geq 0$. Since $x + \varepsilon_1 v = (1 - \varepsilon_1/\varepsilon_2)x + \varepsilon_1/\varepsilon_2(x + \varepsilon_2 v)$, one has using the convexity of $f$,

$$f(x + \varepsilon_1 v) \leq (1 - \varepsilon_1/\varepsilon_2) f(x) + \varepsilon_1/\varepsilon_2 f(x + \varepsilon_2 v),$$

which gives the desired inequality. □

**Proposition 33.** *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be convex. Then,*

*(i) If $x, y \in \operatorname{dom}(f)$, then the following monotonicity property holds*

$$\forall x, y \in \operatorname{dom}(f), \quad f^+(x; y - x) \leq f^+(y; y - x). \tag{9}$$

*(ii) If $x \in \operatorname{int}(\operatorname{dom}(f))$, then $f^+(x, \cdot)$ takes finite values only and is sublinear.*

*(iii) If $x \in \operatorname{cont}(f)$, then $f^+(x, \cdot)$ is a continuous sublinear function.*

*Proof.* (i) This properties directly corresponds to the fact that the slopes of convex function on $\mathbb{R}$ are increasing.
(ii) We first prove finiteness. Let $v \in \mathcal{X}$. Since $x \in \mathrm{int}(\mathrm{dom}(f))$, there exists $\varepsilon > 0$ such that $x \pm \varepsilon v$ belong to $\mathrm{dom}(f)$. Using (8), we deduce that $f^+(x, v) \leq \frac{1}{\varepsilon}(f(x+\varepsilon v) - f(x)) < +\infty$. Similarly, we have $f^+(x; -v) < +\infty$. Now, by convexity of $f$ we have

$$f(x) = f\left(\frac{x+\varepsilon v}{2} + \frac{x-\varepsilon v}{2}\right) \leq \frac{1}{2}f(x+\varepsilon v) + \frac{1}{2}f(x-\varepsilon v),$$

and all terms are finite. Therefore,

$$\frac{f(x+\varepsilon v) - f(x)}{\varepsilon} \geq -\frac{f(x-\varepsilon v) - f(x)}{\varepsilon},$$

Taking the limit $\varepsilon \to 0$, we get $f^+(x; v) \geq -f^+(x; -v) > +\infty$, i.e $f^+(x; \cdot)$ is finite.

We now prove sublinearity of $f^+(x; \cdot)$. The definition directly implies 1-homogeneity of $f^+(x; \cdot)$, we therefore show subadditivity. Let $u, v \in \mathcal{X}$, and $\varepsilon > 0$. Then,

$$x + \varepsilon(u + v) = \frac{x + 2\varepsilon u}{2} + \frac{x + 2\varepsilon v}{2}$$

so that by convexity

$$\frac{1}{\varepsilon}(f(x+\varepsilon(u+v)) - f(x)) \leq \frac{1}{2\varepsilon}(f(x+2\varepsilon u) - f(x)) + \frac{1}{2\varepsilon}(f(x+2\varepsilon v) - f(x)).$$

Taking the limit as $\varepsilon \to 0$ gives the desired inequality.

(iii) As $f$ is continuous at $x$, it is bounded in a neighborhood $O_x$ of $x$. Then, using (8), and setting $O = O_x - x$, we have

$$\forall v \in O, f^+(x; v) \leq f(x+v) - f(x) \leq \max_{O_x} f - f(x).$$

This shows that $f^+(x; \cdot)$ is bounded in a neighborhood of the origin. Since $f^+(x; \cdot)$ is convex (by sublinearity), we deduce from Corollary 19 that $f^+(x; \cdot)$ is continuous on the interior of $\mathrm{dom}(f^+(x; \cdot)$, i.e. on the whole space since $f^+(x; \cdot)$ takes finite values only. □

## 3.2 Gâteaux and Fréchet differentiability

**Definition 22** (Gâteaux-differentiability)**.** A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ on a topological vector space $\mathcal{X}$ is called *Gâteaux-differentiable* at $x \in \mathrm{dom}\, f$ if and only if the directional derivative $v \in \mathcal{X} \mapsto f^+(x; v)$ is a continuous linear form.

*Remark* 11. The notion of of Gâteaux-differentiability is quite weak. For instance, consider$f : \mathbb{R}^2 \to \mathbb{R}$ defined

$$f(x_1, x_2) = \begin{cases} 1 \text{ if } x_1 \neq 0 \text{ and } x_2 = x_1^2 \\ 0 \text{ if not} \end{cases}$$

Then, $f^+(0; \cdot) \equiv 0$ so that $f$ is Gâteaux-differentiable at the origin. On the other hand, $f$ is not even continuous at $(0, 0)$!

**Corollary 34.** *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be convex and let $x \in \operatorname{cont} f$. Then, $f$ is Gâteaux-differentiable at $x$ if and only if $f^+(x; v) \leq -f^+(x; -v)$ for all $v \in \mathcal{X}$.*
*($\simeq$ iff the left and right derivatives at $x$ coincide for any direction $v$)*

*Remark* 12. The function $v \mapsto f^+(x; v)$ can be linear and discontinuous, e.g. if $f$ is a discontinuous linear form. Note also that the inequality $f^+(x; v) \geq -f^+(x; -v)$ always holds, so that the hypothesis could be replaced by $f^+(x; v) \leq -f^+(x; -v)$.

*Example* 14. Let $f(x) = |x|$ on $\mathbb{R}$, then $f^+(0; 1) = 1$ and $f^+(0; -1) = 1 \neq -f^+(0, 1)$.

Corollary 34 follows directly from Proposition 33 and the following lemmas.

**Lemma 35.** *A sublinear function $g : \mathcal{X} \to \mathbb{R}$ is linear if and only if $g(v) \leq -g(-v)$ for all $v \in \mathcal{X}$.*

*Proof.* The direct implication is obvious, let us prove the converse. By sublinearity, we have $0 \leq g(v) + g(-v)$, i.e. $g(v) \geq -g(-v)$, so that the assumption implies $g(v) = -g(-v)$. Then,

$$\begin{aligned} &g(v + w) \leq g(v) + g(w) \\ &g(-(v + w)) \leq g(-v) + g(-w) = -g(v) - g(w), \end{aligned}$$

where we used the assumption on the last line. Thus,

$$g(v) + g(w) \leq -g(-(v + w)) = g(v + w) \leq g(v) + g(w),$$

and all inequalities must be equalities; in particular $g(v + w) = g(v) + g(w)$. Since in addition we know from Proposition 33 that $g(\lambda v) = \lambda g(v)$, $g$ is linear. ☐

**Definition 23** (Fréchet-differentiability)**.** A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is *Fréchet-différentiable* at $x \in \operatorname{dom}(f)$ if it is Gâteaux-differentiable at $x$ and if

$$\lim_{v \to 0, v \neq 0} \frac{|f(x + v) - f(x) - f^+(x; v)|}{\|v\|} = 0, \tag{10}$$

i.e. $f(x + v) = f(x) + f^+(x; v) + \mathrm{o}(\|v\|)$.

*Remark* 13. Fréchet-differentiability is the usual differentiability (and implies continuity). In general, Fréchet différentiability $\Longrightarrow$ Gâteaux-différentiability $\Longrightarrow$ linearity of $v \mapsto f^+(x; v)$. The converse implications are false in general, but is true when $f$ is a convex function on a finite-dimensional space, see Section 3.6.2.

## 3.3 Definition of the subdifferential and first properties

**Definition 24** (Subgradient and subdifferential)**.** Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ and $x_0 \in \operatorname{dom} f$. A linear form $x^* \in \mathcal{X}^*$ is called a *subgradient* of $f$ at $x_0$ if

$$\forall y \in \mathcal{X},\ \ f(y) \geq f(x_0) + \langle x^* \mid y - x_0 \rangle.$$

The set of subgradients of $f$ at $x_0$ is called the *subdifferential* of $f$ at $x_0$ and is denoted $\partial f(x_0)$. When $x_0 \notin \operatorname{dom} f$, we set $\partial f(x_0) := \emptyset$.

*Remark* 14. When $\mathcal{X}$ is a Hilbert space or $\mathbb{R}^d$, we will often consider the subdifferential of a function at a point as a subset of $\mathcal{X}$, using the isomorphism $\mathcal{X}^* \simeq \mathcal{X}$.

*Example* 15. (i) Let $f(x) = |v| = \max(x, -x)$. Then,

$$\partial f(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ \{1\} & \text{if } x > 0 \\ [-1, 1] & \text{if x} = 0 \end{cases}$$

(ii) If $f$ is defined on $\mathbb{R}^d$ by $f(x) = \sum_{i=1}^{d} g(x_i)$ with $g : \mathbb{R} \to \overline{\mathbb{R}}$ convex,

$$\partial f(x) = \prod_i \partial g(x_i).$$

Thus, if $f(x) = \sum_{1 \leq i \leq N} |x_i|$. Then, denoting $s(x) = \partial |\cdot| (x)$,

$$\partial f(x) = \prod_{1 \leq i \leq d} s(x_i).$$

(iii) Let $\mathcal{X}$ be a normed space and let $f(x) = \|x\|$. Then,

$$x^* \in \partial f(0) \Longleftrightarrow \forall x \in \mathcal{X}, \langle x^* \mid x \rangle \leq \|x\| \Longleftrightarrow \|x^*\|_* \leq 1$$

In other words, $\partial f(0)$ is the dual unit ball.

(iv) Let $f(x) = -\sqrt{x}$ on $[0, +\infty[$. Then, $\partial f(0) = \emptyset$ even though 0 belongs to $\operatorname{dom} f$.

The last example, in the form of a proposition, makes the connection between the notion of subdifferential and the notion of normal cone:

**Proposition 36** (Subdifferential of indicator function)**.** *Let $K \subseteq \mathcal{X}$ be a convex set and let $x \in K$. Then, $\partial \mathrm{i}_K(x) = \mathrm{Nor}_x K$.*

**Proposition 37** (General properties)**.** *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$. Then,*

*(i) (Fermat's rule) $f$ attains its minimum at $x_0 \in \mathcal{X}$ if and only if $0 \in \partial f(x_0)$.*

*(ii) (Convexity and closedness) $\partial f(x)$ is weak*-closed and convex.*

*(iii) (Monotonicity) $\forall x, y \in \mathcal{X}, x^* \in \partial f(x)$ and $y^* \in \partial f(y)$, $\quad \langle x^* - y^* \mid x - y \rangle \geq 0$.*

*(iv) (Upper semi-continuity) Assume that $\mathcal{X}$ is a Banach space and that $f$ is lower semicontinuous. If $(x_n, x_n^*)_{n \in \mathbb{N}}$ is such that $x_n^* \in \partial f(x_n)$, and if*

$$x_n \xrightarrow[n \to +\infty]{\text{strong}} x \text{ and } x_n^* \xrightarrow[n \to +\infty]{\text{weak*}} x^*,$$

*then $x^* \in \partial f(x)$.*

*Proof.* (ii) This follows from the following representation of $\partial f(x)$ as an intersection of weak* closed halfspaces:

$$\partial f(x) = \bigcap_{y \in \mathcal{X}} \{x^* \in \mathcal{X}^* \mid f(y) \geq f(x) + \langle x^* \mid y - x \rangle\}.$$ □

(iii) By definition of $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$ we have

$$f(y) \geq f(x) + \langle x^* \mid y - x \rangle \text{ and } f(x) \geq f(y) + \langle y^* \mid x - y \rangle.$$

We conclude by summing these inequalities.
(iv) Under the assumptions, we have for all $y \in \mathcal{X}$

$$\begin{aligned} f(y) &\geq f(x_n) + \langle x_n^* \mid y - x_n \rangle \\ &= f(x_n) + \langle x_n^* \mid y - x \rangle + \langle x^* \mid x - x_n \rangle + \langle x_n^* - x^* \mid x - x_n \rangle \end{aligned}$$

By lower semi-continuity we have $\liminf_{n \to +\infty} f(x_n) \geq f(x)$ and by strong convergence $\lim_{n \to +\infty} \langle x^* \mid x - x_n \rangle$. Moreover, by Banach-Steinhaus' uniform boundedness principle, we see that the sequence $(x_n^*)$ is bounded in dual norm, so that

$$|\langle x_n^* - x^* \mid x - x_n \rangle| \leq \|x_n^* - x^*\| \, \|x - x_n\| \xrightarrow{n \to +\infty} 0.$$

Taking these limits in the inequality above, we get $f(y) \geq f(x) + \langle x^* \mid y - x \rangle$, thus showing that $x^* \in \partial f(x)$.

*Remark* 15. The upper semi-continuity property is equivalent to the (sequential) closedness of the graph of the subdifferential $\partial f = \{(x, x^*) \in \mathcal{X} \times \mathcal{X}^* \mid x^* \in \partial f(x)\}$ with respect to the strong $\times$ weak-$*$ topology. This property is false in general, see [BFG03] for a counter-example. Thanks to Clément Cosserat for pointing an error in an earlier version of these notes.

**Theorem 38** (Subdifferential of convex functions)**.** *Let $\mathcal{X}$ be a locally convex topological vector space and let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be convex. Then,*

*(i) if $x \in \operatorname{dom}(f)$, $\partial f(x) = \{x^* \in \mathcal{X}^* \mid f^+(x; \cdot) \geq \langle x^* \mid \cdot \rangle\}$.*

*(ii) if $x \in \operatorname{cont} f$ and if $\mathcal{X}$ is a normed space, $\partial f(x)$ is bounded w.r.t $\|\cdot\|_*$.*

*(iii) if $x \in \operatorname{cont} f$, $f^+(x; v) = \sup\{\langle x^* \mid v \rangle \mid x^* \in \partial f(x)\}$.*

*(iv) if $x \in \operatorname{cont} f$, $\partial f(x)$ is non-empty.*

*(v) if $x \in \operatorname{cont} f$, then $f$ is Gâteaux-differentiable at $x$ if and only if $\partial f(x)$ is a singleton $\{x^*\}$, and then $f^+(x; \cdot) = \langle x^* \mid \cdot \rangle$.*

*Remark* 16 (Support function of the subdifferential)*.* Let $f \in \Gamma_0(\mathcal{X})$ and $x \in \operatorname{cont} f$. Note that we endowed $\mathcal{X}^*$ with the weak$*$ topology and that any weak$*$ continuous linear form over $\mathcal{X}^*$ is induced by a vector of $\mathcal{X}$, i.e. $(\mathcal{X}^*)^* \simeq \mathcal{X}$. The support function of $\partial f(x)$ can therefore be defined as

$$\sigma_{\partial f(x)} : y \in \mathcal{X} \mapsto \sup_{x^* \in \partial f(x)} \langle x^* \mid y \rangle \in \mathbb{R}$$

Then Theorem 38 immediately implies that $\sigma_{\partial f(x)} = f^+(x, \cdot)$, i.e. the directional derivative can be thought of as the support function of the subdifferential.

*Proof.* (i) Let $A$ be the second member of the equality. If $x^* \in A$ we have,

$$\forall v \in \mathcal{X}, \quad \langle x^* \mid v \rangle \leq f^+(x; v) = \inf_{t > 0} \frac{f(x + tv) - f(x)}{t} \leq f(x + v) - f(x)$$

Thus, given $y \in \mathcal{X}$ and letting $v = y - x$, we get $\langle x^* \mid y - x\rangle \le f(y) - f(x)$, implying that $x^*$ belongs to $\partial f(x)$. Reciprocally, assume that $x^* \in \partial f(x)$. Then, for all $v \in \mathcal{X}$, $t > 0$ and $y = x + tv$ we have $\langle x^* \mid y - x\rangle \le f(x + tv) - f(x)$. Dividing by $t$ and using $y - x = tv$, we get

$$\langle x^* \mid v\rangle \le \frac{f(x + tv) - f(x)}{t}.$$

Since this is true for all $t > 0$, we have as desired $\langle x^* \mid v\rangle \le f^+(x; v)$, i.e. $x \in A$.

(ii) To show that $\partial f(x)$ is bounded, we use that since $f$ is continuous at $x$, it is $L$-Lipschitz near $x$ for some $L \ge 0$. This directly implies that $f^+(x; v) \le L\,\|v\|$, so that if $x^* \in \partial f(x)$ we have $\langle x^* \mid v\rangle \le f^+(x; v) \le L\,\|v\|$. By definition of the dual norm, we get $\|x^*\| \le L$.

(iii) By Proposition 33, the function $g = f^+(x; \cdot)$ is convex and continuous, so that by Proposition 26, $g$ is equal to the supremum of its affine minorant. Let $(x^*, \alpha) \in x^* \in \mathcal{X}^* \times \mathbb{R}$ be such that $g(x) \ge \langle x^* \mid x\rangle + \alpha$ for all $x \in \mathcal{X}$. Setting $x = ty$ and using the homogeneity of $g$ we get

$$\forall y \in \mathcal{X}, \forall t > 0, \quad t(\langle x^* \mid y\rangle + \alpha/t) \le tg(y).$$

Letting $t \to +\infty$ we get $g \ge \langle x^* \mid \cdot\rangle$. This implies that

$$\begin{aligned} g(x) &= \sup\{\langle x^* \mid x\rangle \mid x^* \in \mathcal{X}^*,\ g \ge \langle x^* \mid \cdot\rangle\} \\ &= \sup\{\langle x^* \mid x\rangle \mid x^* \in \partial f(x)\}, \end{aligned}$$

where we used (i) to get the second equality.

(iv) This follows directly from (iii).

(v) If $f$ is Gâteaux-differentiable at $x$, then $\phi := f^+(x; \cdot)$ is linear continuous, and (i) implies that $\partial f(x) = \{\phi_0\}$. Conversely, if $\partial f(x) = \{x^*\}$ is a singleton, (iv) directly implies that $f^+(x; \cdot) = \langle x^* \mid \cdot\rangle$ is linear continuous. □

### 3.4 Subdifferential calculus

In applications, one often encounters minimization of functions which are defined as a sum, maximum, or composition of other functions with linear operators. It is therefore of prime interest to study the effect of these operations on the subdifferential. We start with a very general and easy inclusion result. We recall the definition of the adjoint operator:

**Definition 25** (Adjoint)**.** Let $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ be the set of continuous linear operators between two topological vector spaces $\mathcal{Y}$ and $\mathcal{X}$. The adjoint of $A \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ is a linear operator between the dual spaces $\mathcal{X}^*$ and $\mathcal{Y}^*$ defined by

$$A^* : x^* \in \mathcal{X}^* \mapsto (y \in \mathcal{Y} \mapsto \langle x^* \mid Ay\rangle) \in \mathcal{Y}^*.$$

It is characterized by the property $\langle A^* x^* \mid y\rangle = \langle x^* \mid Ay\rangle$.

**Proposition 39** (Inclusions). *(i) for all function $f, g : \mathcal{X} \to \overline{\mathbb{R}}$ and all $x \in \mathcal{X}$,*

$$\partial f(x) + \partial g(x) \subseteq \partial(f+g)(x).$$

*(ii) if $(f_i)_{i\in I}$ is a family of functions from $\mathcal{X} \to \overline{\mathbb{R}}$, and $f = \sup_{i\in I} f_i$, then for all $x \in \mathcal{X}$, and $I_x := \arg\max_{i\in I} f_i(x)$,*

$$\overline{\operatorname{conv}} \bigcup_{i\in I_x} \partial f_i(x) \subseteq \partial f(x).$$

*(iii) if $f : \mathcal{X} \to \overline{\mathbb{R}}$, $A \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, and $g = f \circ A$ then for $y \in \mathcal{Y}$,*

$$A^* \partial f(Ay) \subseteq \partial g(y)$$

*Proof.* Exercise. □

*Remark* 17 (A counterexample). We note that these inclusions may sometimes be strict. For instance, consider $f = \mathrm{i}_{[-\infty,0)}$ and $g(x) = -\sqrt{x}$ on for $x \geq 0$ and $g(x) = +\infty$ for $x < 0$. Then, $f + g$ takes value $+\infty$ everywhere expect at $x = 0$, so that $\partial(f+g)(0) = \mathbb{R}$. On the other hand, $\partial g(0) = \emptyset$, so that $\partial f(0) + \partial g(0) = \emptyset$.

In order to avoid this counterexample, we need to assume that the domains of $f$ and $g$ intersect non-trivially. Such assumptions are called *qualification conditions* in the literature. The next theorem gives an example of such a qualification condition, but there exists weaker conditions on Banach spaces (see e.g. Brézis-Attouch [AB86]) or when $\mathcal{X} = \mathbb{R}^d$ (see [Roc70, Chapter 23]).

**Theorem 40** (Moreau-Rockafellar). *Let $\mathcal{X}$ be a locally convex topological vector space, let $f, g \in \Gamma_0(\mathcal{X})$ and assume the* qualification condition *holds:*

$$\operatorname{dom} f \cap \operatorname{cont} g \neq \emptyset.$$

*Then, for all $x \in \mathcal{X}$ one has*

$$\partial f(x) + \partial g(x) = \partial(f+g)(x).$$

*Proof.* Let $x \in \mathcal{X}$ and consider $x^* \in \partial(f+g)(x)$. Thus,

$$\forall y \in \mathcal{X}, f(y) + g(y) \geq f(x) + g(x) + \langle x^* \mid y - x\rangle.$$

Setting $\tilde{f}(y) = f(y) - (f(x) + g(x)) - \langle x^* \mid y - x\rangle$, this is equivalent to

$$\forall y \in \mathcal{X}, \tilde{f}(y) \geq -g(y).$$

Consider the set $K = \operatorname{epi}(\tilde{f})$ and $L = -\operatorname{epi}(g)$ in $\mathcal{X} \times \mathbb{R}$. Since $\operatorname{cont}(g) \neq \emptyset$, $L$ has non-empty interior, ensuring by Exercise 4, that $L$ is the closure of its interior $M := \operatorname{int} L$. As a consequence of the Hahn-Banach theorem (Corollary 5), there exists a continuous affine function on $\mathcal{X} \times \mathbb{R}$ separating the disjoint convex sets $K$ and $M$, with $M$ open. This means that there exists $z^* \in \mathcal{X}^*$ and $\alpha \in \mathbb{R}$ such that

$$\inf_{(y,t)\in K} \langle z^* \mid y\rangle + \alpha t \geq \sup_{(y,s)\in M} \langle z^* \mid y\rangle + \alpha s = \sup_{(y,s)\in L} \langle z^* \mid y\rangle + \alpha s, \tag{11}$$

where the last equality holds because $L$ is the closure of $M$. Taking $y = x_0$, $t > \tilde{f}(x_0)$ and $s < -g(x_0)$ in the previous inequality (so that $t - s > \tilde{f}(x_0) - g(x_0) \geq 0$) gives

$$0 \leq \alpha(t-s),$$

which is only possible if $\alpha \geq 0$. We now rule out the possibility that $\alpha = 0$. By contradiction, assume that $\alpha = 0$. Then, the separation inequality gives

$$\inf_{y \in \operatorname{dom} f} \langle z^* \mid y \rangle \geq \sup_{y \in \operatorname{dom} g} \langle z^* \mid y \rangle,$$

i.e. $\operatorname{dom} f$ and $\operatorname{dom} g$ are linearly separated (note that $z^* \neq 0$ since $(z^*, \alpha) \neq 0$ and $\alpha = 0$). This contradicts the assumption $\operatorname{dom} f \cap \operatorname{cont} g = \emptyset$. Thus, $\alpha > 0$. Replacing $z^*$ by $\frac{1}{\alpha} z^*$ if necessary, we may therefore assume that $\alpha = 1$ in the separation inequality (11). With $(x, \tilde{f}(x)) \in K$ and $(y, -g(y)) \in L$, this inequality gives

$$\forall y \in \mathcal{X}, \quad \langle z^* \mid x \rangle + \tilde{f}(x) \geq \langle z^* \mid y \rangle - g(y),$$

and since $\tilde{f}(x) = -g(x)$,

$$\forall y \in \mathcal{X}, g(y) \geq g(x) + \langle z^* \mid y - x \rangle.$$

In other words, $z^* \in \partial g(x)$. Applying again the separation equality (11) but with $(y, \tilde{f}(y)) \in K$ and $(x, -g(x)) \in L$ we get

$$\forall y \in \mathcal{X}, \quad \langle z^* \mid y \rangle + \tilde{f}(y) \geq \langle z^* \mid x \rangle - g(x),$$

Using $\tilde{f}(y) = f(y) - (f(x) + g(x)) - \langle x^* \mid y - x \rangle$ we get

$$\forall y \in \mathcal{X}, \quad f(y) \geq \langle x^* - z^* \mid y - x \rangle + f(x),$$

so that $x^* - z^* \in \partial f(x)$. Thus, $x^* \in \partial(f+g)(x)$ can be written as the sum of $z^* \in \partial g(x)$ and $x^* - z^* \in \partial f(x)$. $\square$

**Theorem 41** (Moreau-Rockafellar)**.** *Let $f \in \Gamma_0(\mathcal{X})$ and $A : \mathcal{Y} \to \mathcal{X}$ be a continuous linear operator between two topological vector spaces. Assume the following qualification hypothesis holds:*

$$\operatorname{cont} f \cap A(\mathcal{Y}) \neq \emptyset.$$

*Then, for all $y \in \mathcal{Y}$ one has*

$$\partial(f \circ A)(y) = A^* \partial f(Ay).$$

*Proof.* Let $y \in \mathcal{Y}$, and $y^*$ be an element of $\partial(f \circ A)(y)$. This reads

$$\forall z \in \mathcal{Y}, f(Az) \geq \ell(z) := f(Ay) + \langle y^* \mid z - y \rangle.$$

We introduce $K = \operatorname{epi} f$ and $L = \{(Az, \ell(z) \mid z \in \mathcal{Y}\}$ which are two convex subsets of $\mathcal{X} \times \mathbb{R}$. Since the function $f$ is continuous at some point in $\mathcal{X}$, the set $K$ has non-empty interior and is therefore equal to the closure of its interior $M := \operatorname{int} K$. The convex sets $M$ and $L$ are disjoint thanks to the subdifferential inequality above and the set $M$ is open, so that by Corollary 5 there exists a non-zero continuous linear form $(x^*, \alpha) \in \mathcal{X} \times \mathbb{R}$ such that

$$\inf_{(x,t)\in K} \langle x^* \mid z\rangle + \alpha t = \inf_{(x,t)\in M} \langle x^* \mid z\rangle + \alpha t \geq \sup_{(x,s)\in L} \langle x^* \mid x\rangle + \alpha s.$$

We argue similarly as before to prove that $\alpha > 0$. By assumption, there exists a point $y_0 \in \mathcal{Y}$ such that $Ay_0 \in \operatorname{cont} f$. Taking $x = Ay_0$ in the infimum and supremum and choosing $t > f(Ay_0)$ and $s = \ell(y_0)$, so that $t > s$, we obtain that $\alpha \geq 0$. Second, we note that $\alpha = 0$ implies that $\operatorname{dom} f$ and $A\mathcal{Y}$ are linearly separated, a contradiction with $\operatorname{cont} f \cap A\mathcal{Y} \neq \emptyset$. Dividing $z^*$ by $\alpha > 0$ if necessary, we now assume that $\alpha = 1$. The separation inequality can then be rewritten as

$$\forall x \in \mathcal{X}, \forall z \in \mathcal{Y}, \quad \langle x^* \mid x\rangle + f(x) \geq \langle x^* \mid Az\rangle + \ell(z) = \langle x^* \mid Az\rangle + f(Ay) + \langle y^* \mid z - y\rangle.$$

Taking $z = y$ in this inequality inequality, we get

$$\forall x \in \mathcal{X}, \langle x^* \mid x\rangle + f(x) \geq \langle x^* \mid Ay\rangle + f(Ay),$$

i.e. $-x^* \in \partial f(Ay)$. Taking $x = Ay$ on the other hand, we obtain

$$\forall z \in \mathcal{Y}, \quad \langle A^* x^* + y^* \mid y - z\rangle \geq 0,$$

showing that $-A^*x^* = y^*$. Thus, $y^* = A^*(-x^*)$ with $-x^* \in \partial f(Ay)$ as desired. □

**Theorem 42** (Dubovitskii-Milyutin). *Let $\mathcal{X}$ be a topological vector space, let $f_1, \ldots, f_N \in \Gamma_0(\mathcal{X})$ and define $f = \sup_{1\leq i\leq N} f_i$. Then for all $x \in \operatorname{cont}(f_1) \cap \ldots \cap \operatorname{cont}(f_N)$, and $I_x := \arg\max_{i\in I} f_i(x)$,*

$$\partial f(x) = \overline{\operatorname{conv}} \left( \bigcup_{i\in I_x} \partial f_i(x) \right).$$

A more general version of this theorem, where the index set is not finite but compact is proven in [Zal02, Theorem 2.4.18].

*Proof.* We let $K = \overline{\operatorname{conv}} \left(\bigcup_{i\in I_x} \partial f_i(x)\right) \subseteq \mathcal{X}^*$. For proving that $\partial f(x) = K$ we note

$$\begin{aligned} f^+(x; v) &= \lim_{t\to 0, t>0} \frac{f(x+tv) - f(x)}{t} \\ &= \lim_{t\to 0, t>0} \max_{i\in I_x} \frac{f_i(x+tv) - f_i(x)}{t} \\ &= \max_{i\in I_x} \lim_{t\to 0, t>0} \frac{f_i(x+tv) - f_i(x)}{t} \\ &= \max_{i\in I_x} f_i^+(x; v). \end{aligned}$$

Since the $f_i$ are continuous at $x$, the function $f$ is also continous at $x$. Using Remark 16, the previous computation thus yields

$$\sigma_{\partial f(x)} = \max_{i \in I_x} \sigma_{\partial f_i(x)}.$$

By Proposition 13, we deduce that $\sigma_{\partial f(x)} = \sigma_K$, and since both $K$ and $\partial f(x)$ are convex and weak*-closed, we deduce from Proposition 13 that $K = \partial f(x)$. $\square$

## 3.5 Application: optimality conditions

### 3.5.1 Examples

*Example* 16 (Fermat's problem). We consider the following minimization problem

$$\min_{x \in \mathbb{R}^d} \alpha_i \|x - x_1\| + \ldots + \alpha_N \|x - x_N\|,$$

where $\alpha_1, \ldots, \alpha_N \geq 0$ and where $x_1, \ldots, x_N \in \mathbb{R}^d$ are *distinct points*. The minimizer of this problem is called a Fermat point. Putting $f_i = \|\cdot - x_i\|$, we have

$$\partial f_i(x) = \begin{cases} \frac{x - x_i}{\|x - x_i\|} & \text{if } x \neq x_i \\ \mathrm{B}(0,1) & \text{if } x = x_i \end{cases}.$$

Since all the functions $f_i$ are convex and continuous, by Theorem 40

$$\partial(\alpha_1 f_1 + \ldots + \alpha_1 f_N) = \begin{cases} \{\sum_i \alpha_i \frac{x - x_i}{\|x - x_i\|}\} & \text{if } x \notin \{x_1, \ldots, x_N\} \\ B(0, \alpha_j) + \sum_{i \neq j} \alpha_i \frac{x - x_i}{\|x - x_i\|} & \text{if } x = x_j. \end{cases}$$

Thus, $x$ is a Fermat point iff

$$(x = x_j \text{ and } \left\| \sum_{i \neq j} \alpha_i \frac{x_j - x_i}{\|x_j - x_i\|} \right\| \leq \alpha_j)$$

$$\text{or } (x \notin \{x_1, \ldots, x_N\} \text{ and } \sum_i \alpha_i \frac{x - x_i}{\|x - x_i\|} = 0.)$$

*Example* 17 (Lasso problem). Let $A$ a $m$-by-$d$ matrix, $y \in \mathbb{R}^m$ and $\gamma > 0$. We consider the following modification of the least-squared problem

$$\min_{x \in \mathbb{R}^d} \frac{1}{2} \|Ax - y\|_2^2 + \gamma \|x\|_1.$$

Set $f(x) = \frac{1}{2} \|Ax - y\|_2^2$ and $g(x) = \|x\|_1$. Setting $h = |\cdot|$ we have

$$\partial f(x) = \{\nabla f(x)\} = \{A^T(Ax - y)\},$$

$$\partial g(x) = \partial h(x_1) \times \ldots \times \partial h(x_d)$$

Since these functions are continuous and convex, Theorem 40 gives $\partial(f+\gamma g)(x) = \partial f(x) + \gamma \partial g(x)$. Thus, $x$ is a minimizer of the problem if and only if

$$A^T(Ax-y) = -\gamma v \text{ where } v \text{ satisfies } \begin{cases} v_i = \operatorname{sgn}(x_i) & \text{if } x_i \neq 0 \\ v_i \in [-1,1] & \text{if } x_i = 0 \end{cases}$$

or in other words, letting $A_i$ be the $i$th column of $A$, $x$ is a minimizer iff

$$\begin{cases} |\langle A_i \mid Ax-y\rangle| \leq \gamma & \text{if } x_i = 0 \\ \langle A_i \mid Ax-y\rangle = \gamma \operatorname{sgn}(x_i) & \text{otherwise} \end{cases}$$

$$\iff \begin{cases} |\langle A_i \mid Ax-y\rangle| \leq \gamma & \text{if } x_i = 0 \\ \langle A_i \mid Ax-y\rangle = \gamma \operatorname{sgn}(x_i) & \text{otherwise.} \end{cases}$$

One can see from this example that the term $\|x\|_1$ induces sparsity of the solution

### 3.5.2 Karush-Kuhn-Tucker theorem

**Theorem 43** (Variational characterization of optimality)**.** *Let $K \subseteq \mathcal{X}$ be a closed convex subset and $f : \mathcal{X} \to \overline{\mathbb{R}}$ be convex lsc, and assume that*

$$(\operatorname{int} K \cap \operatorname{dom} f \neq \emptyset) \text{ or } (K \cap \operatorname{cont} f \neq \emptyset)).$$

*Then, the following are equivalent :*
*(i) $x$ is a minimizer of $f$ on $K$ ;*
*(ii) $-\partial f(x) \cap \operatorname{Nor}_x K \neq \emptyset$.*
*(iii) $\exists x^* \in \partial f(x)$ s.t.*

$$\forall y \in K, \langle x-y \mid -x^*\rangle \geq 0.$$

*Proof.* The point $x \in \mathcal{X}$ minimizes $f$ on $K$ if and only if it minimizes $f+g$ with $g = \mathrm{i}_K$. This is also equivalent to $0 \in \partial(f+g)(x)$. Since $\operatorname{dom} g = K$ and $\operatorname{cont} g = \operatorname{int} K$, the first hypothesis is equivalent to $\operatorname{cont} g \cap \operatorname{dom} f \neq \emptyset$ and the second one is equivalent to $\operatorname{dom} g \cap \operatorname{cont} f \neq \emptyset$. In either case, one can apply Theorem 40 to get

$$\partial(f+g) = \partial f(x) + \partial \mathrm{i}_K = \partial f(x) + \operatorname{Nor}_x K.$$

We deduce that $x$ is a minimizer if and only if there exists $x^* \in \partial f(x)$ such that $-x^* \in \operatorname{Nor}_x K$ (this is (ii)) and the equivalence with (iii) comes from the definition of the normal cone. □

**Proposition 44** (Normal cone of sublevel set)**.** *Let $g \in \Gamma_0(\mathcal{X})$, let $K = \operatorname{lev}_{\leq 0} g$ and assume Slater's condition:*

$$\begin{cases} K \subseteq \operatorname{cont} g \\ \operatorname{lev}_{<0} g \neq \emptyset \end{cases}$$

*Then,*

$$\operatorname{Nor}_x K = \begin{cases} 0 & \text{if } g(x) < 0 \\ \mathbb{R}^+ \partial g(x) & \text{if } g(x) = 0 \\ \emptyset & \text{if } g(x) > 0 \end{cases}.$$

*Proof.* First we note that if $g(x) < 0$, then by assumption $x \in \operatorname{cont} g$. Thus, $g \leq 0$ on a neighborhood of $x$, which implies that $\mathrm{i}_K$ is locally equal to zero. From this we deduce that $\operatorname{Nor}_x K = \partial \mathrm{i}_K(x) = \{0\}$.

From now on we assume that $g(x) = 0$. We first prove the inclusion $\mathbb{R}^+ \partial g(x) \subseteq \operatorname{Nor}_x K$. Let $x^* \in \partial g(x)$, so that

$$\forall y \in \mathcal{X}, g(y) \geq \langle x^* \mid y - x \rangle + g(x)$$

Multiplying this inequality by $\lambda \geq 0$ we get

$$\forall y \in K, 0 \geq \lambda g(y) \geq \langle \lambda x^* \mid y - x \rangle + \underbrace{g(x)}_{=0},$$

thus showing that $\lambda x^* \in \operatorname{Nor}_x K$.

Let us prove the converse inclusion $\operatorname{Nor}_x K \subseteq \mathbb{R}^+ \partial g(x)$. Let $x^* \in \operatorname{Nor}_x K$. By definition of the normal cone, for all $y \in K$ one has $\langle x^* \mid x \rangle \geq \langle x^* \mid y \rangle$, implying that $K$ is included in the complement of the closed half-space $H = \{y \in X \mid \langle x^* \mid y \rangle \geq \langle x^* \mid x \rangle \geq 0\}$. This in turns implies that $g \geq 0$ on $H$, so that $x$ minimizes $g$ over $H$. Since $x \in \operatorname{cont} g \cap \operatorname{dom}(\mathrm{i}_H)$, we can apply the subdifferential sum rule:

$$0 \in \partial(g + \mathrm{i}_H) = \partial g(x) + \mathbb{R}^+\{-x^*\}.$$

In other words, there exists $y^* \in \partial g(x)$ and $\lambda \geq 0$ such that $\lambda x^* = y^*$. Note that $\lambda$ cannot be zero since $x$ is not a minimizer of $g$. Then, $x^* = \frac{1}{\lambda} y^*$ as desired. □

**Theorem 45** (Karush-Kuhn-Tucker). *Let $f, g_1, \dots, g_N \in \Gamma_0(\mathcal{X})$ let $K = \bigcap_{1 \leq i \leq N} \operatorname{lev}_{\leq 0} g_i$, and assumes that Slater's condition*

$$\begin{cases} \operatorname{lev}_{\leq 0} g_i \subseteq \operatorname{int} \operatorname{dom} g_i & \forall i \in \{1, \dots, N\} \\ \operatorname{dom} f \cap \operatorname{lev}_{<0} g_1 \cap \ldots \cap \operatorname{lev}_{<0} g_N \neq \emptyset \end{cases} \tag{12}$$

*hold. Then the following are equivalent:*

*(i) $x$ is a minimizer of $f$ on $K$ ;*

*(ii) there exists $\lambda_1, \dots, \lambda_N \geq 0$ such that*

$$\begin{cases} 0 \in \partial f(x) + \lambda_1 \partial g_1(x) + \ldots + \lambda_N \partial g_N(x) \\ \lambda_i g_i(x) = 0 & \forall i \in \{1, \dots N\} \end{cases}$$

*Remark* 18. If the functions $g_i$ are continuous, the first hypothesis in Slater's condition automatically holds.

*Remark* 19 (Lagrange multipliers). The scalars $\lambda_1, \dots, \lambda_N$ whose existence is given by the theorem are called *Lagrange multipliers*. If $\lambda_1, \dots, \lambda_N$ are as in the theorem, we directly obtain that $x$ minimizes $f + \lambda_1 g_1 + \ldots + \lambda_N g_N$ over $\mathcal{X}$. Thus, knowing the Lagrange multipliers allows to replace the constrained optimization problem $\min_K f$ by an unconstrained problem.

*Proof.* We consider $K_i = \mathrm{lev}_{\leq 0}\, g_i$. The second assumption's in Slater's condition ensures that there exists a point in $\mathrm{dom}\, f$ in the interior of the sets $K_i$, i.e.

$$\mathrm{dom}\, f \cap \mathrm{cont}(i_{K_1}) \cap \ldots \cap \mathrm{cont}(i_{K_N}) \neq \emptyset.$$

Applying Theorem 40 recursively, we obtain

$$\begin{aligned} \partial(f + \mathrm{i}_{K_1} + \ldots + \mathrm{i}_{K_N})(x) &= \partial f(x) + \partial \mathrm{i}_{K_1}(x) + \ldots + \partial \mathrm{i}_{K_N}(x) \\ &= \partial f(x) + \mathrm{Nor}_x\, K_1 + \ldots + \mathrm{Nor}_x\, K_N, \end{aligned}$$

so that $x \in K$ minimizes $f$ on $K$ iff there exists $x^* \in \partial f(x)$ and $x_i^* \in \mathrm{Nor}_x\, K_i$ such that $-x^* = x_1^* + \ldots + x_i^*$. Since in addition for $x \in K$, we have

$$\mathrm{Nor}_x\, K_i = \begin{cases} \mathbb{R}^+ \partial g_i(x) & \text{if } g_i(x) < 0 \\ \{0\} & \text{if not} \end{cases},$$

we get that $x_i^* = \lambda_i y_i^*$ with $y_i^* \in \partial g_i(x)$ and $\lambda_i = 0$ if $g_i(x) < 0$ and $\lambda_i \geq 0$ if not, as desired. □

### 3.6 Differentiability almost everywhere

**Motivation** Given a compact convex domain $K \subseteq \mathcal{X}$ and $x^* \in \mathcal{X}^*$, we consider the following linear programming problem:

$$\sup_{x \in K} \langle x^* \mid x \rangle \tag{13}$$

We denote $f : x^* \in \mathcal{X}^* \to \mathbb{R}$ the *value function* of this problem, i.e. $f(x^*)$ is the value of the maximum in the previous definition. In other words, $f$ is the support function of $K$. Then, $f$ is convex and if $x$ is a solution to (13), i.e. if $x \in K$ and $f(x^*) = \langle x^* \mid x \rangle$, one has

$$\begin{aligned} f^+(x^*, v^*) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(f^+(x^* + \varepsilon v^*) - f(x^*)) \\ &\geq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\langle x^* + \varepsilon v^* \mid x \rangle - \langle x^* \mid x \rangle) \geq \langle v^* \mid x \rangle \end{aligned}$$

It is often useful to know whether (13) has a unique maximizer. Suppose that there exists $x \neq y \in K$ such that $f(x^*) = \langle x^* \mid x \rangle = \langle x^* \mid y \rangle$. Then, as before,

$$f^+(x^*, v^*) \geq \max(\langle v^* \mid x \rangle, \langle v^* \mid y \rangle).$$

This shows that the application $v^* \in \mathcal{X}^* \mapsto f^+(x^*; v^*)$ is not linear, so that $f$ is not differentiable at $x$. Thus Gâteaux-differentiability of $f$ at $x^*$ implies the uniqueness of the minimizer to the linear programming problem (13). If we prove that $f$ is Gâteaux-differentiable "almost everywhere", we get that the problem (13) has a unique solution for "almost every" linear form $x^* \in \mathcal{X}^*$. (Note that this last property is quite intuitive when $K$ is a convex polytope in $\mathbb{R}^d$.)

*Remark* 20. Characterizing the non-differentiability locus of convex functions also has applications in optimal transport [M+95], in optimal control [CS04], etc. There exists many results on the "size" of the non-differentiability locus of convex functions, both in finite [AAC92] and infinite dimensions [BV+10, §4.6].

On $\mathbb{R}$, convex functions are differentiable almost everywhere thanks to the following proposition.

**Proposition 46.** *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be convex. Then, there are at most a countable number of points in* $\operatorname{dom} f$ *where $f$ is non-differentiable.*

This proposition is false in higher dimension, consider e.g. $f(x_1, x_2) = |x_1|$ on $\mathbb{R}^2$.

*Proof.* Consider $f'_+(x) = f^+(x; 1)$ et $f'_-(x) = -f^+(x, -1)$ the right and left derivatives. These functions are increasing (exercise) and for all $x < x_0$ in $\operatorname{dom}(f)$,

$$f'_+(x) = \inf_{y>x} \frac{f(y) - f(x)}{y - x} \leq \frac{f(x_0) - f(x)}{x_0 - x} \leq f'_-(x_0),$$

thus showing the inequality

$$\lim_{x \to x_0^-} f'_+(x) \leq f'_-(x_0) \leq f'_+(x_0).$$

The function $f$ is differentiable at $x_0$ if and only $f'_-(x_0) = f'_+(x_0)$. Thus, if $f$ is not differentiable at $x_0$, the right derivative $f'_+$ has a jump at $x_0$:

$$\lim_{x \to x_0^-} f'_d(x) < f'_d(x_0).$$

One concludes by using that an increasing function can only have a countable number of jumps. □

### 3.6.1 Gâteaux-différentiability almost everywhere

**Theorem 47** (Mazur)**.** *Let $\mathcal{X}$ be a separable Banach space and let $f : \mathcal{X} \to \mathbb{R}$ be continuous and convex. Then, $f$ is Gâteaux-differentiable on a dense subset of $\mathcal{X}$.*

To prove this theorem, we start by considering a dense sequence $(v_n)_{n\geq 0}$ in $\mathcal{X}$, which exist thanks to the separability assumption. Then, we introduce the sets

$$A_{m,n} = \{x \in \mathcal{X} \mid f^+(x, v_n) + f^+(x, -v_n) \geq 1/m\}, \qquad A = \bigcup_{m,n\geq 1} A_{m,n} \tag{14}$$

The sketch of the proof is as follows:

a. First we prove that $f$ is Gâteaux-differentiable on $\mathcal{X} \setminus A$;
b. Second, that all sets $A_{m,n}$ are closed and have empty interior.

Then, by Baire's theorem, we know that $A$ has empty interior (i.e. $\mathcal{X} \setminus A$ is dense).

**Lemma 48.** *The function $f$ is Gâteaux-differentiable on $\mathcal{X} \setminus A$.*

*Proof.* As $f$ is continuous on $\mathcal{X}$, by Corollary 34,

$$\begin{aligned} &f \text{ is not Gâteaux-differentiable at } x \in \mathcal{X} \\ &\quad \Longrightarrow \exists v \in \mathcal{X}, f^+(x,v) + f^+(x,-v) > 0 \\ &\quad \Longrightarrow \exists v \in \mathcal{X}, \exists m > 1, f^+(x,v) + f^+(x,-v) > 2/m \\ &\quad \Longrightarrow \exists m,n \geq 1, f^+(x,v_n) + f^+(x,-v_n) > 1/m \\ &\quad \Longrightarrow x \in A_{m,n} \subseteq A, \end{aligned}$$

where we used the continuity of the map $v \mapsto f^+(x,v)$. □

**Lemma 49.** *The set $A_{m,n}$ defined in* (14) *is closed.*

This is a consequence of the following proposition, showing that for any $v \in \mathcal{X}$, the directional derivative $f^+(\cdot,v)$ is upper-semicontinuous. Then, $f^+(\cdot,v)+f^+(\cdot,-v)$ is also usc, and $A_{m,n}$ is closed as a superlevel set of a usc function.

**Lemma 50.** *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be convex and let $x \in \operatorname{cont}(f)$. Then, for any sequence $(x_n)_{n\in\mathbb{N}}$ converging to $x$, one has $f^+(x,v) \geq \limsup_{k\to\infty} f^+(x_k,v)$.*

*Proof.* By convexity and continuity, $f$ is $M$-Lipschitz in a neighborhood of $x$. Without loss of generality, we assume that the sequence $(x_n)_{n\in\mathbb{N}}$ remains in this neighborhood. Let $\varepsilon > 0$. Using the Lipschitz property, we get

$$\begin{aligned} \frac{1}{\varepsilon}(f(x+\varepsilon v) - f(x)) &\geq \frac{1}{\varepsilon}(f(x_k+\varepsilon v) - f(x_k) - 2L\,\|x - x_k\|) \\ &\geq f^+(x_k,v) - \frac{2L\,\|x-x_k\|}{\varepsilon} \end{aligned}$$

Taking the infimum on the left-hand side we get

$$f^+(x;v) \geq \limsup_{k\to\infty} f^+(x_k,v).$$ □

**Lemma 51.** *The set $A_{m,n}$ defined in* (14) *has empty interior.*

*Proof.* Assume that the interior of $A_{m,n}$ contains a point $x$, i.e. there exists $r > 0$ such that $\mathrm{B}(x,r) \subseteq A_{m,n}$. Let $x_t := x + tv_n$ and $g : t \in [0,r] \mapsto f(x_t)$. Then,

$$\begin{aligned} &\forall t \in [0,r], \quad -f^+(x_t,-v_n) + 1/m \leq f^+(x_t,v_n) \\ &\qquad \Longrightarrow \forall t \in [0,r], \quad g \text{ is not differentiable at } t \end{aligned}$$

This contradicts Proposition 46, which shows that the non-differentiability set of $g$ is countable. □

### 3.6.2 Fréchet-differentiability almost everywhere on $\mathbb{R}^d$

The behaviour of convex functions on $\mathbb{R}^d$ is much simpler than on infinite-dimensional spaces. Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be a convex function and let $x \in \operatorname{cont} f$. We will prove the following chain of implications:

$$\begin{aligned} &f \text{ admits partial derivatives } \left(\frac{\partial f}{\partial e_i}(x)\right)_{1\leq i\leq d} \\ &\qquad\implies \text{the application } v \mapsto f^+(x;v) \text{ is linear} \\ &\qquad\implies f \text{ is Gâteaux-differentiable at } x \\ &\qquad\implies f \text{ is Fréchet-differentiable at } x \end{aligned}$$

From this, we will deduce that a convex function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is a.e. differentiable on its domain.

**Proposition 52.** *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be convex. If $f$ is Gâteaux-differentiable at $x \in \operatorname{int}(\operatorname{dom}(f))$, then $f$ is also Fréchet-différentiable at $x$.*

This proposition follows from the next lemma and from the fact that $f$ is locally Lipschitz around $x$.

**Lemma 53.** *Let $f : \mathrm{B}(x,r) \subseteq \mathbb{R}^d \to \mathbb{R}$ be convex and Lipschitz. Then, $f$ is Gâteaux-differentiable at $x$ iff it is Fréchet-différentiable at $x$.*

*Proof.* Let $S$ be the unit sphere of $\mathbb{R}^d$. By compactness, for all $\varepsilon > 0$, there exists a finite family of vectors $(v_i)_{1\leq i\leq N}$ of $S$ such that $S \subseteq \cup_i \mathrm{B}(v_i,\varepsilon)$. By Gâteaux-differentiability of $f$ at $x$, for all $i$, there exists $\delta_i$ such that

$$\forall t \in [-\delta_i,\delta_i], \;\; \left\| f(x+tv_i) - (f(x) + tf^+(x;v_i)) \right\| \leq \varepsilon\,|t|$$

Consider $\delta := \min_i \delta_i > 0$. By construction of the $(v_i)$, for all $v$ in $S$, there exists $i \in \{1,\dots,N\}$ such that $\|v_i - v\| \leq \varepsilon$. Then, using that $f$ is Lipschitz (which implies that $f^+(x;\cdot)$ is also Lipschitz), we have

$$\begin{aligned} \|f(x+tv_i) - f(x+tv)\| &\leq M\,|t|\,\varepsilon \\ \left\|f^+(x;v_i) - f^+(x;v)\right\| &\leq M\,|t|\,\varepsilon \end{aligned}$$

Thus, for all $v \in S$ and all $|t| \leq \delta$,

$$\begin{aligned} \left\|f(x+tv) - (f(x) + tf^+(x;v))\right\| &\leq \left\|f(x+tv_i) - (f(x) + tf^+(x;v))\right\| + 2M\varepsilon\,|t| \\ &\leq (2M+1)\varepsilon\,|t| \end{aligned}$$

Equivalently (by homogeneity of $f^+(x;\cdot)$) we have for all $v \in \mathbb{R}^d$, $\|v\| \leq \delta$,

$$\left\|f(x+v) - (f(x) + f^+(x;v))\right\| \leq (2M+1)\varepsilon\,\|v\|\,,$$

thus showing that $f$ is Fréchet-differentiable at $x$. $\square$

**Proposition 54.** *Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be convex. Then, $f$ is Gâteaux-differentiable at $x \in \operatorname{int}(\operatorname{dom}(f))$ iff it admits partial derivatives at $x$.*

**Lemma 55.** *Let $g : \mathbb{R}^d \to \mathbb{R}$ be sublinear. Then, the set*

$$V = \{v \in \mathbb{R}^d \mid f^+(x; v) = -f^+(x; -v)\}$$

*is a linear subspace of $\mathbb{R}^d$.*

*Proof.* The fact that $V$ is stable by multiplication by a scalar follows from the homogeneity of $g$. By sublinearity, $0 = g(u+(-u)) \leq g(u)+g(-u)$, so that $-g(-u) \leq g(u)$. Let $v, w \in V$. We have

$$g(v + w) \leq g(v) + g(w) = -g(-v) + -g(-w) \leq -g(-v - w) \leq g(v + w),$$

where we used sublinearity to get the first and second inequality, the definition of $V$ to get the equality, and the property $-g(u) \leq g(-u)$ to get the third inequality. This shows that $v + w \in V$, proving that $V$ is a linear subspace. □

*Proof of Proposition 54.* The function $f$ is locally Lipschitz near $x$ and $g = f^+(x; \cdot)$ is sublinear. Let $V := \{v \in \mathcal{X} \mid g(v) = -g(-v)\}$. By the previous lemma, $V$ is a linear subspace of $\mathcal{X}$. Since the partial derivatives exist, we have $f^+(x; -e_i) = -f^+(x; e_i)$ for all basis vector $e_i$, thus showing that $e_i \in V$ for all $i$. Therefore, $V = \mathcal{X}$ and $f^+(x; \cdot)$ is linear. □

**Theorem 56.** *Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be a convex function. Then $f$ is Fréchet-différentiable at a.e. point in* $\operatorname{int}(\operatorname{dom}(f))$.

*Proof.* Let $A$ be the non-diffentiability locus of $f$ in $\Omega := \operatorname{int}(\operatorname{dom}(f))$. By Proposition 54, the set $A$ is contained in the union of the sets

$$A_i := \left\{x \in \Omega \mid \frac{\partial f}{\partial e_i} \text{ does not exist } x\right\}.$$

Therefore, to show that $A$ has zero measure, it suffices to prove that all of the sets $A_i$ have zero measure. Without loss of generality, we assume that $i = n$, and we consider $\phi$ the indicator function of $A_n$. By Tonelli's theorem,

$$\lambda(A_n) = \int_{\mathbb{R}^n} \phi(x)\mathrm{d}x = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \phi(y, x_n)\mathrm{d}x_n\mathrm{d}y$$

However, for all $y \in \mathbb{R}^{n-1}$, $t \mapsto \phi(y, t)$ is the non-differentiability locus of the 1D convex function $t \in \mathbb{R} \mapsto f(y, t)$. By Proposition 46, $B_y$ is countable, and therefore has zero Lebesgue measure, thus concluding the proof. □

*Remark* 21. In fact, one can prove that the non-differentiability locus of $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ has Hausdorff dimension $d-1$, and it is in fact a rectifiable set of dimension $\leq d-1$. This means that $S$ can be covered, up to a set with zero $d-1$–Hausdorff measure, by a countable union of Lipschitz maps $(\phi_n)_{n\in\mathbb{N}}$ with $\phi_i : \mathbb{R}^{d-1} \to \mathbb{R}$ [AAC92]. This fact is used, in optimal transport, to give a characterization of pairs of measures for which there exists an optimal transport map, see e.g. [M+95].

# 4 Proximal operator

In this short chapter, we assume that $\mathcal{X}$ is a Hilbert space, which we identify with its dual. We introduce the proximal operator, which is a basic building block of many first-order methods for minimizing non-smooth convex functions. We recall that a function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is coercive if $\lim_{\|x\|\to\infty} f(x) = +\infty$.

## 4.1 Definition and properties

**Definition 26** (Proximal operator)**.** The *proximal operator* associated to a function $f \in \Gamma_0(\mathcal{X})$ is defined by

$$\mathrm{Prox}_{f(x)} = \arg\min_{y\in\mathcal{X}} \frac{1}{2}\|x-y\|^2 + f(y).$$

*Example* 18 (Projection)*.* If $K$ is a closed convex subset of $\mathcal{X}$ and $f = \mathrm{i}_C$, then for all $\gamma > 0$ one has $\mathrm{Prox}_{\gamma \mathrm{i}_K} x = p_K(x)$. The proximal operator generalizes the projection on a convex set, and shares some of its properties.

**Proposition 57.** *Let $f \in \Gamma_0(\mathcal{X})$ and $\gamma > 0$. Then,*

*(i) The minimization problem*

$$\min_{y\in\mathcal{X}} \frac{1}{2\gamma}\|x-y\|^2 + f(y),$$

*has a unique solution, implying that $\mathrm{Prox}_{\gamma f}$ is well defined.*

*(ii) The point $p = \mathrm{Prox}_\gamma f(x)$ is characterized by the relation $x \in (\mathrm{id} + \gamma\partial f)(p)$.*

*(iii) The point $x$ minimizes $f$ on $\mathcal{X}$ iff $x = \mathrm{Prox}_\gamma f(x)$;*

*Proof.* Let $g = \frac{1}{2\gamma}\|x-\cdot\|^2$ and $h = f+g$. (i) The function $h$ belongs to $\Gamma_0(\mathcal{X})$. Since $f \in \Gamma_0(\mathcal{X})$, $f$ is equal to the supremum of its affine minorants (Proposition 26), so that in particular it admits an affine minorant: there exists $x^* \in \mathcal{X}$ and $\alpha \in \mathbb{R}$ such that $f \geq \langle x^* \mid \cdot\rangle + \alpha$. Thus,

$$h = f + g \geq \langle x^* \mid \cdot\rangle + \alpha + \frac{1}{2\gamma}\|x-\cdot\|^2,$$

from which we deduce that $h$ is coercive. In addition, $h$ is strictly convex as the sum of a convex and a strictly convex function. By Proposition 30, we deduce that $h$ admits a unique minimizer.
(ii) Applying Theorem 40 on the subdifferential of a sum of two functions, which we can use since $\mathrm{cont}(g) = \mathcal{X}$ and $\mathrm{dom}\, f \neq \emptyset$, we see that $p = \mathrm{Prox}_{\gamma f}(x)$ if and only if

$$0 \in \partial(f+g)(p) = \partial f(p) + \frac{1}{\gamma}\{p - x\} \iff x \in (\mathrm{id} + \gamma\partial f)(p).$$

(iii) The point $x$ minimizes $f$ if and only if 0 belongs to $\partial[\gamma f](x) = \gamma\partial f(x)$, or equivalently if $x$ belongs to $(\mathrm{id} + \gamma\partial f)(x)$. □

*Example* 19 (Proximal of $\ell^1$ norm). Let $h : \mathbb{R} \to \mathbb{R}, x \mapsto |x|$. Then,

$$\partial h(y) = \begin{cases} -1 & \text{if } y < 0 \\ [-1, 1] & \text{if } y = 0 \\ 1 & \text{if } y > 0 \end{cases} \implies (\mathrm{id} + \gamma \partial h)(y) = \begin{cases} y - \gamma & \text{if } y < 0 \\ [-\gamma, \gamma] & \text{if } y = 0 \\ y + \gamma & \text{if } y > 0 \end{cases}.$$

By the characterization given in the previous proposition, we get

$$\mathrm{Prox}_{\gamma h}(x) = \begin{cases} x - \gamma & \text{if } x \geq \gamma \\ 0 & \text{if } -\gamma \leq x \leq \gamma \\ x + \gamma & \text{if } x \leq -\gamma. \end{cases}$$

Thus, $\mathrm{Prox}_{\gamma h}$ is exactly the so-called soft thresholding operator

$$s_\gamma(r) = \begin{cases} r - \gamma \,\mathrm{sgn}(r) & \text{if } |r| \geq \gamma \\ 0 & \text{otherwise} \end{cases}.$$

already introduced in Example 17. If $\mathcal{X} = \mathbb{R}^n$ and if $f(x) = \|x\|_1 = \sum_i |x_i|$, then

$$\min_{y \in \mathbb{R}^n} \frac{1}{2\gamma} \|x - y\|_2^2 + f(y) = \sum_{1 \leq i \leq n} \min_{y_i \in \mathbb{R}} \frac{1}{2\gamma} (x_i - y_i)^2 + |y_i|,$$

so that $\mathrm{Prox}_\gamma f(x) = (s_\gamma(x_1), \ldots, s_\gamma(x_n))$.

## 4.2 Proximal point algorithm

By Proposition 57, minimizing $f$ is equivalent to finding a fixed point of the proximal operator. This suggests the following minimization algorithm:

$$\begin{cases} x_0 \in \mathcal{X} \\ x_{n+1} = \mathrm{Prox}_\gamma f(x_n) \end{cases} \tag{PPA}$$

This algorithm is called the *proximal point algorithm*. It has been originally introduced by Martinet [Mar70, Mar72] in the 1970s, and was later generalized by Rockafellar [Roc76]. Before proving the convergence of this algorithm, we list a very useful property of the proximal operator.

**Definition 27** (Firm non-expansiveness)**.** A map $T : \mathcal{X} \to \mathcal{X}$ is *firmly non-expansive* it it satisfies one of the following equivalent conditions

(i) $\forall x, y \in \mathcal{X},\ \ \|T(x) - T(y)\|^2 \leq \langle T(x) - T(y) \mid x - y \rangle$

(ii) $\forall x, y \in \mathcal{X},\ \ \|T(x) - T(y)\|^2 \leq \|x - y\|^2 - \|(x - T(x)) - (y - T(y))\|^2$

To see that these two conditions are equivalent, it suffices to remark that

$$\|(x - T(x)) - (y - T(y))\|^2 = \|x - y\|^2 + \|T(x) - T(y)\|^2 - 2\langle x - y \mid T(x) - T(y) \rangle.$$

Note that a firmly non-expansive operator is 1-Lipschitz – but firm non-expansiveness is a stronger property.

**Proposition 58.** *Let $f \in \Gamma_0(\mathcal{X})$ and let $\gamma > 0$. Then*

*(i) The point $p = \mathrm{Prox}_\gamma f(x)$ is characterized by the inequality*

$$\forall q \in \mathcal{X},\ \frac{1}{\gamma}\langle x - p \mid q - p\rangle \leq f(q) - f(p)$$

*(ii) The operator $T : x \mapsto \mathrm{Prox}_\gamma f(x)$ is firmly non-expansive.*

Note that (i) is a generalization of a well-known characterization of the projection of a point on a convex set.

*Proof.* (i) The point $p = \mathrm{Prox}_\lambda f(x)$ is characterized by $x \in (\mathrm{id} + \gamma\partial f)(p)$ or equivalently by $\frac{1}{\gamma}(x - p) \in \partial f(p)$. This is equivalent to

$$\forall q \in \mathcal{X},\ f(q) \geq f(p) + \frac{1}{\gamma}\langle x - p \mid q - p\rangle.$$

(ii) Let $x_1, x_2 \in \mathcal{X}$ and let $p_i = \mathrm{Prox}_\gamma f(x_i)$. We apply the inequality form (i) using first $x = x_1$, $p = p_1$ et $q = p_2$ and then switching the roles:

$$f(p_2) - f(p_1) \geq \frac{1}{\gamma}\langle x_1 - p_1 \mid p_2 - p_1\rangle$$
$$f(p_1) - f(p_2) \geq \frac{1}{\gamma}\langle x_2 - p_2 \mid p_1 - p_2\rangle$$

Summing these inequalities and multipliying by $\gamma > 0$, we obtain

$$\|p_2 - p_1\|^2 \leq \langle p_1 - p_2 \mid x_1 - x_2\rangle.$$ □

We note that since $T = \mathrm{Prox}_\gamma f$ is merely 1-Lipschitz, we cannot deduce the convergence of the proximal point algorithm from Picard's fixed point theorem (one would need $T$ to be $k$-Lipschitz for some $k < 1$). However, the property of non-expansiveness allows prove convergence of the PPA.

**Theorem 59** (Martinet)**.** *Let $f \in \Gamma_0(\mathcal{X})$ and assume that $f$ is coercive, meaning that $\lim_{\|x\|\to+\infty} f(x) = +\infty$. Then, the sequence of points generated by the proximal point algorithm* (PPA) *weakly converges to a global minimizer of $f$.*

This theorem is a direct consequence of the next theorem about firmly non-expansive operators, where where we have set $T = \mathrm{Prox}_{\gamma f}$. Note that by assumption, $f$ has a minimizer, implying that the set of fixed-points of $T$ is non-empty

$$\mathrm{Fix}(T) = \{x \in \mathcal{X} \mid T(x) = x\} \neq \emptyset.$$

**Theorem 60** (Martinet)**.** *Let $T$ be firmly non-expansive and such that* $\mathrm{Fix}(T) \neq \emptyset$*. Then the sequence defined by $x_{n+1} = T(x_n)$ converges weakly to a fixed point of $T$.*

A modern proof of this theorem placing it in the general setting of fixed point iterations for $\alpha$-averaged operators can be found in the book of Bauschke and Combettes [BC$^+$11]. We nonetheless present the original proof of Martinet, which is self-contained, for the completeness of these notes.

*Proof. Step 1.* We first prove that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded and that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Let $c \in \mathrm{Fix}(T)$. Firm non-expansiveness gives:

$$\|Tx_n - Tc\|^2 \leq \|x_n - c\|^2 - \|(x_n - T(x_n)) - (c - T(c))\|^2 .$$

Using $Tx_n = x_{n+1}$ and $T(c) = c$, we deduce that

$$\|x_{n+1} - c\|^2 \leq \|x_n - c\|^2 - \|x_n - x_{n+1}\|^2 .$$

The sequence $\|x_n - c\|_{n\geq 1}$ is therefore decreasing and bounded from below, and thus admits a limit. This gives $\|x_n - x_{n+1}\|^2 \leq \|x_n - c\|^2 - \|x_{n+1} - c\|^2 \underset{n\to+\infty}{\longrightarrow} 0$.

*Step 2.* We prove that every weak cluster point $\bar{x}$ of $(x_n)_{n\in\mathbb{N}}$ is a fixed point of $T$. First, we note that

$$\begin{aligned}\|x_n - T(\bar{x})\|^2 &= \|x_n - \bar{x} + \bar{x} - T(\bar{x})\|^2 \\ &= \|x_n - \bar{x}\|^2 + \|\bar{x} - T(\bar{x})\|^2 - 2\langle x_n - \bar{x} \mid \bar{x} - T(\bar{x})\rangle,\end{aligned}$$

Second, using that $x_{n+1} = T(x_n)$ and that $T$ is 1-Lipschitz we get

$$\begin{aligned}\|x_n - T(\bar{x})\| &= \|x_n - x_{n+1} + x_{n+1} - T(\bar{x})\| \\ &\leq \|x_n - x_{n+1}\| + \|T(x_n) - T(\bar{x})\| \\ &\leq \|x_n - x_{n+1}\| + \|x_n - \bar{x}\| ,\end{aligned}$$

Combining these computations, we therefore get

$$\begin{aligned}\|\bar{x} - T(\bar{x})\|^2 &= \|x_n - T(\bar{x})\|^2 - \|x_n - \bar{x}\|^2 + 2\langle x_n - \bar{x} \mid \bar{x} - T(\bar{x})\rangle \\ &\leq (\|x_n - x_{n+1}\| + \|x_n - \bar{x}\|)^2 - \|x_n - \bar{x}\|^2 + 2\langle x_n - \bar{x} \mid \bar{x} - T(\bar{x})\rangle \\ &= \|x_n - x_{n+1}\|^2 + 2\|x_n - x_{n+1}\| \|x_n - \bar{x}\| + 2\langle x_n - \bar{x} \mid \bar{x} - T(\bar{x})\rangle \xrightarrow{n\to+\infty} 0.\end{aligned}$$

Note that we have used Step 1. to prove that the first two terms converge to zero and the hypothesis that $\bar{x}$ is a weak cluster point to control the last term.

*Step 3.* Let $c_1$ and $c_2$ be two weak cluster points of the sequence, which by Step 2 belong to $\mathrm{Fix}(T)$. By Step 1, we know that $\|x_{n+1} - c_i\|^2 \leq \|x_n - c_i\|^2$, implying that the sequence

$$\|x_n\|^2 - 2\langle x_n \mid c_i\rangle = \|x_n - c_i\|^2 - \|c_i\|^2 ,$$

is decreasing and therefore converging. Substrating these sequences for $i = 1$ and 2, we see that

$$(\|x_n\|^2 - 2\langle x_n \mid c_1\rangle) - (\|x_n\|^2 - 2\langle x_n \mid c_2\rangle) = 2\langle x_n \mid c_1 - c_2\rangle$$

is converging. Since $c_1$ and $c_2$ are weak cluster points of $x_n$, we obtain by taking limit over the corresponding subsequences

$$\langle c_1 \mid c_1 - c_2\rangle = \langle c_2 \mid c_1 - c_2\rangle,$$

implying $\|c_1 - c_2\|^2 = 0$, i.e. $c_1 = c_2$. The sequence $(x_n)_{n\geq 1}$ is bounded and has a unique weak cluster point, and is therefore weakly converging. □

### 4.3 Proximal of the convex conjugate

**Theorem 61** (Subdifferential and conjugation)**.** *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be proper. Then, for any $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$ the following two conditions are equivalent:*
*(i)* $x^* \in \partial f(x)$ *;*
*(ii)* $f(x) + f^*(x^*) = \langle x^* \mid x\rangle$
*If in addition $f$ belongs to $\Gamma_0(\mathcal{X})$, then (i) and (ii) are equivalent with*
*(iii)* $x \in \partial f^*(x^*)$ *;*

*Proof.* (i)$\iff$ (ii) A linear form $x^*$ belongs to $\partial f(x)$ if and only if

$$\begin{aligned}\forall y \in \mathcal{X}, f(y) \geq f(x) + \langle x^* \mid y - x\rangle &\iff \forall y \in \mathcal{X}, \langle x^* \mid y\rangle - f(y) \leq \langle x^* \mid x\rangle - f(x)\\ &\iff f^*(x^*) = \sup_{y\in Y}\langle x^* \mid y\rangle - f(y) \leq \langle x^* \mid x\rangle - f(x)\\ &\iff f^*(x^*) + f(x) \leq \langle x^* \mid x\rangle\end{aligned}$$

Since Fenchel-Young's inequality asserts that $f^*(x^*) + f(x) \geq \langle x^* \mid x\rangle$ always holds, we get the equivalence between (i) and (iii).
(i)$\iff$ (iii) Since $f \in \Gamma_0(\mathcal{X})$, we know from Fenchel-Moreau that $f^{**} = f$. Therefore,

$$\begin{aligned}x^* \in \partial f(x) &\iff f(x) + f^*(x^*) = \langle x^* \mid x\rangle\\ &\iff f^{**}(x) + f^*(x^*) = \langle x^* \mid x\rangle\\ &\iff x \in \partial f^*(x^*),\end{aligned}$$

where we applied the equivalence between (i) and (iii) to the function $f^*$. ☐

*Example* 20 (A sufficient condition for Gâteaux-differentiability). Let $f \in \Gamma_0(\mathcal{X})$. If $f^*$ is strictly convex, then $f$ is Gâteaux-differentiable in $\mathrm{cont}(f)$.

*Proof.* Let $x \in \mathrm{cont}(f)$. To show that $f$ is Gâteaux-differentiable at $x$ it suffices to establish that $\mathrm{Card}\,\partial f(x) \leq 1$. Assume by contradiction that there exists $x_0^* \neq x_1^*$ such that $x_0^*, x_1^* \in \partial f(x)$. By convexity of $\partial f(x)$ one has $x_t^* = (1-t)x_0^* + tx_1^* \in \partial f(x)$ for all $t \in [0, 1]$. Thus, using the previous proposition,

$$\forall t \in [0,1],\ f(x) = \langle x_t^* \mid x\rangle - f^*(x_t^*) = \langle x_0^* \mid x\rangle - f^*(x_0^*) = \langle x_1^* \mid x\rangle - f^*(x_1^*).$$

Therefore, $f^*$ is not strictly convex on $[x_0^*, x_1^*]$. ☐

**Corollary 62** (Prox of conjugate)**.** *Let $\mathcal{X}$ be a Hilbert space and $f \in \Gamma_0(\mathcal{X})$. Then,*

$$\forall x \in \mathcal{X}, \mathrm{Prox}_f(x) + \mathrm{Prox}_{f^*}(x) = x. \tag{15}$$

*Proof.* Let $p = \mathrm{Prox}_f(x)$. Then, by definition,

$$\begin{aligned}x \in (\mathrm{id} + \partial f)(p) &\iff x - p \in \partial f(p)\\ &\iff p \in \partial f^*(x - p)\\ &\iff x \in x - p + \partial f^*(x - p)\\ &\iff x - p = \mathrm{Prox} f^*(x).\end{aligned}$$

☐

*Remark* 22. Eq. (15) generalizes the formula $x = \mathrm{proj}_V x + \mathrm{proj}_{V^\perp} x$, where $V^\perp$ is the orthogonal of the subspace $V \subseteq \mathcal{X}$. See Exercise 27 for a generalization of this formula to convex cones.

# 5 Convex duality

## 5.1 Perturbations of convex problems

This section is inspired by the presentation of Ekeland and Temam [ET99] and by a forthcoming book by Guillaume Carlier. We consider the problem of minimizing a convex function $f \in \Gamma_0(\mathcal{X})$ on a space $\mathcal{X}$

$$P = \inf_{x \in \mathcal{X}} f(x),$$

and we assume that the function $f$ can be written as $f(x) = \Phi(x, 0)$ where $\Phi \in \Gamma_0(\mathcal{X} \times \mathcal{Y})$ is also convex, and where $\mathcal{Y}$ is another space. In other words, we assume that the original problem is a special instance of the following problem, which is parameterized by a vector $y \in \mathcal{Y}$ :

$$P_y := \inf_{x \in \mathcal{X}} \Phi(x, y).$$

In the following, we regard $\Phi^*$ as a function on $\mathcal{X}^* \times \mathcal{Y}^*$, through the identification $(\mathcal{X} \times \mathcal{Y})^* \simeq \mathcal{X}^* \times \mathcal{Y}^*$. The dual problem to the minimization problem $P$ is then:

$$D = \sup_{y^* \in \mathcal{Y}^*} -\Phi^*(0, y^*).$$

The construction of this dual problem is be found in the proof of the following proposition. Note that there is no uniqueness of the dual problem: there is one dual problem to $P$ associated to each perturbation $\Phi$ of $f$.

**Proposition 63** (Weak duality)**.** *The weak duality inequality $P \geq D$ always hold. Moreover, for $(x, y^*) \in \mathcal{X} \times \mathcal{Y}^*$, the following statements are equivalent:*

*(i)* $\Phi(x, 0) = -\Phi^*(0, y^*)$,

*(ii)* *$x$ is a minimizer of $P$ and $y^*$ a maximizer of $D$ and $P = D$,*

*(iii)* $(0, y^*) \in \partial\Phi(x, 0)$,

*(iv)* $(x, 0) \in \partial\Phi^*(0, y^*)$.

*Proof.* Fenchel-Young's formula asserts that

$$\forall (x, y^*) \in \mathcal{X} \times \mathcal{Y}^*, \quad \Phi(x, 0) + \Phi^*(0, y^*) \geq \langle (0, y^*) \mid (x, 0) \rangle = 0,$$

thus implying $\inf_x \Phi(x, 0) \geq \sup_{y^*} -\Phi^*(0, y^*)$. We deduce at once the equivalence between the statements (i) and (ii). To see the equivalence between (i) and (iii), we use the equality case in Fenchel-Young's inequality (Theorem 61): equality holds if $(0, y^*) \in \partial\Phi(x, 0)$. The equivalence between (i) and (iii) uses the same equality case but applied to $\Phi^*$ and $\Phi^{**} = \Phi$. □

**Theorem 64** (Strong duality)**.** *Let $\mathcal{X}, \mathcal{Y}$ be two spaces, let $\Phi \in \Gamma_0(X, Y)$ and consider the following minimisation problem: $P = \inf_{\mathcal{X}} \Phi(\cdot, 0)$. Assume:*

- *$P$ is finite ;*
- *the following qualification hypothesis is verified: $\exists x_0 \in \mathcal{X}, \quad 0 \in \mathrm{cont}(\Phi(x_0, \cdot))$.*

*Then, the maximum in the the dual problem* $D = \max_{y^* \in \mathcal{Y}^*} -\Phi^*(0, y^*)$ *is attained and strong duality holds*

$$P = D.$$

*In addition, the set of the maximizers of D is the subdifferential of the* value function

$$v : y \in \mathcal{Y} \mapsto \inf_{x \in \mathcal{X}} \Phi(x, y).$$

The following lemma is central:

**Lemma 65.** *Under the assumptions of Theorem 64, the value function v satisfies:*

*(i)* $v$ *is convex;*

*(ii)* $v^*(y^*) = \Phi^*(0, y^*)$ *;*

*(in particular,* $v(0) = P$ *and* $v^{**}(0) = \sup_{y^* \in \mathcal{Y}^*} -\Phi^*(0, y^*) = D$ *;*

*(iii)* $v$ *is continuous at 0 ;*

*(iv)* $\partial v(0) \neq \emptyset$.

Since $P = v(0)$ and $D = v^{**}(0)$, Moreau-Rockafellar's theorem (Theorem 28) would directly imply $P = D$ if $v$ was lower-semicontinuous on $\mathcal{Y}$. This is not a priori the case, but the last claim will nonetheless allow us to prove that $v(0) = v^{**}(0)$.

*Proof.* (i) Let $y_0, y_1 \in \mathcal{Y}$ and let $y_t = (1-t)y_0 + ty_1$. Then, for any $t \in [0, 1]$,

$$\begin{aligned} v(y_t) &\leq \inf_{x_0, x_1} \Phi((1-t)x_0 + tx_1, y_t) \\ &\leq \inf_{x_0, x_1} (1-t)\Phi(x_0, y_0) + t\Phi(x_1, y_1). \\ &= (1-t)v(y_0) + tv(y_1) \end{aligned}$$

(ii) Given $y^* \in \mathcal{Y}^*$ one has

$$\begin{aligned} v^*(y^*) &= \sup_{y \in \mathcal{Y}} \langle y^* \mid y \rangle - v(y) \\ &= \sup_{y \in \mathcal{Y}} \langle y^* \mid y \rangle - \inf_{x \in \mathcal{X}} \Phi(x, y) \\ &= \sup_{(x,y) \in \mathcal{Y} \times \mathcal{X}} \langle (0, y^*) \mid (0, y) \rangle - \Phi(x, y) = \Phi^*(0, y^*). \end{aligned}$$

(iii) By the qualification hypothesis, there exists $x_0 \in \mathcal{X}$ such that $\Phi(x_0, \cdot)$ is continuous near 0. Then,

$$v(y) = \inf_{x \in \mathcal{X}} \Phi(x, y) \leq \Phi(x_0, y),$$

is bounded near $y = 0$. Since the function $v$ is convex, finite at and bounded near the origin, $v$ is continuous there (see Proposition 17).
(iv) From Theorem 38, the continuity of $v$ at 0 implies the non-emptiness of the subdifferential. $\square$

*Proof of Theorem 64.* By the previous lemma, the subdifferential $\partial v(0)$ of the value function at $y = 0$ contains a linear form $y^* \in \mathcal{Y}$. Then, by the equality case in Fenchel-Young's inequality,

$$\begin{cases} v(0) + v^*(y^*) = 0 \\ v(0) + v^*(z^*) \geq 0 \quad \forall z^* \in \mathcal{Y} \end{cases}.$$

which can be rewritten as $\forall z^* \in \mathcal{Y}^*, \ -v^*(z^*) \leq v^*(y)$. Thus,

$$v^{**}(0) = \sup_{z^* \in \mathcal{Y}^*} -v^*(z^*) = -v^*(y^*) = v(0).$$

This shows that $P = D$, and that $y^*$ is a maximizer of the dual problem. □

*Example* 21 (Application: linear programming). As a first simple application, we consider the case where $\mathcal{X} = \mathbb{R}^n$, $\mathcal{Y} = \mathbb{R}^m$, which we identify with their dual spaces through the Euclidean scalar product. Given a matrix with $n$ columns and $m$ rows and $b \in \mathcal{Y}$ and $c \in \mathcal{X}^*$, we consider

$$\Phi(x, y) = \langle c \mid x \rangle + \mathrm{i}_{Ax-b \leq y}.$$

Thus,

$$P = \inf_{x \in \mathcal{X}} \Phi(x, 0) = \min_{Ax \leq b} \langle c \mid x \rangle,$$

where $Ax \leq b$ means that for all $j$, $(Ax)_j \leq b_j$. The primal problem consists in minimizing the linear form $\langle c \mid x \rangle$ over the polyhedron $K = \{x \in \mathcal{X} \mid Ax \leq b\}$. For simplicity, we assume that $K$ has non-empty interior, thus ensuring the existence of $x_0 \in K$ such that $\Phi(x_0, \cdot)$ is continuous at 0. By strong duality (Theorem 64), we therefore obtain $P = D$ where the dual problem is given by

$$D = \max_{y^* \in \mathcal{Y}^*} -\Phi^*(0, y^*).$$

Let us now compute the convex conjugate appearing in $D$:

$$\begin{aligned} \Phi^*(0, y^*) &= \sup_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \langle (0, y^*) \mid (x, y) \rangle - \Phi(x, y) \\ &= \sup_{(x,y) \in \mathcal{X} \times \mathcal{Y} \mid Ax-b \leq y} \langle y^* \mid y \rangle - \langle c \mid x \rangle. \end{aligned}$$

At this point, we note that if $y^*_i > 0$ for some $i$, then for all $\lambda \geq 0$, setting $y = \lambda e_i$ and $x = x_0$ in the supremum, we obtain

$$\Phi^*(0, y^*) \geq \lambda y^*_i - \langle c \mid x_0 \rangle \xrightarrow{\lambda \to +\infty} +\infty,$$

thus showing that $\Phi^*(0, y^*) = +\infty$. On the other hand, if $y^*_i \leq 0$ for all $i$, then the supremum in $y$ is attained for $y = Ax - b$, i.e.

$$\Phi^*(0, y^*) = \sup_{x \in \mathcal{X}} \langle y^* \mid Ax - b \rangle - \langle c \mid x \rangle = \begin{cases} -\langle y^* \mid b \rangle & \text{if } A^T y^* = c \\ +\infty & \text{if not} \end{cases}$$

Thus, the dual problem can be written as

$$D = \max_{y^*} -\Phi^*(0, y^*) = \max_{y^* \leq 0 \text{ and } A^T y^* = c} \langle y^* \mid b \rangle.$$

Note that the maximum is attained in $D$ as a consequence of the strong duality theorem.

## 5.2 Application: Lagrangian duality

In this section, we briefly see how the method of perturbation can be used to recover Lagrangian duality for constrained optimization problems where the constraint set is defined by a family of inequalities. Let $f, g_1, \ldots, g_N \in \Gamma_0(\mathcal{X})$, and consider

$$P = \inf_K f, K = \{x \in \mathcal{X} \mid g_1(x) \leq 0, \ldots, g_N(x) \leq 0\}.$$

We construct the perturbed problems for $y \in \mathbb{R}^d$ by

$$P_y = \inf\{f(x) \mid g_1(x) \leq -y_1, \ldots, g_N(x) \leq -y_N\}.$$

This amounts to introducing the following perturbation function

$$\begin{aligned} \Phi : \mathcal{X} \times \mathbb{R}^N &\to \overline{\mathbb{R}} \\ (x, y) &\mapsto \begin{cases} f(x) & \text{if } g_i(x) \leq -y_i \text{ for } 1 \leq i \leq N \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

The function $\Phi$ can also be expressed as

$$\Phi(x, y) = f(x) + \sum_{1 \leq i \leq N} \mathrm{i}_{\mathrm{epi}(g_i)}(x, -y_i),$$

and this expression clearly shows that $\Phi$ is convex and lower-semicontinuous. We assume the following condition, similar to (12):

$$\exists x_0 \in \operatorname{dom} f \text{ s.t. } g_i(x_0) < 0 \text{ for } i \in \{1, \ldots, N\}. \tag{16}$$

Thus, $\Phi(x_0, y) = f(x_0)$ as soon as $\|y\|_\infty \leq \min_{1 \leq i \leq N} |g_i(x_0)|$, implying that $\Phi(x_0, y)$ is continuous near $y = 0$. We can therefore apply the strong duality theorem (Theorem 64), which gives us $P = D = \sup_{y^* \in \mathbb{R}^N} -\Phi^*(0, y^*)$. The conjugate $\Phi^*(0, y^*)$ can be computed explicitely in terms of the *Lagrangian* of the problem, i.e.

$$\begin{aligned} L : \mathcal{X} \times \mathbb{R}^N &\to \overline{\mathbb{R}} \\ (x, \lambda) &\mapsto f(x) + \sum_{1 \leq i \leq N} \lambda_i g_i(x). \end{aligned} \tag{17}$$

Note that we replaced the dual variable $y^* \in (\mathbb{R}^N)^*$ by $\lambda \in \mathbb{R}^N$ to follow the standard notation for Lagrange multipliers.

**Lemma 66.** $\Phi^*(0,\lambda) = \begin{cases} -\inf_{x\in\mathcal{X}} L(x,y) & \textit{if } \forall i, \lambda_i \geq 0 \\ +\infty & \textit{if not} \end{cases}.$

*Proof.* By definition,

$$\begin{aligned}\Phi^*(0,\lambda) &= \sup_{(x,y)\in\mathcal{X}\times\mathbb{R}^d} \langle\lambda \mid y\rangle - \Phi(x,y) \\ &= \sup\{\langle\lambda \mid y\rangle - f(x) \mid (x,y)\in\mathcal{X}\times\mathbb{R}^d \text{ s.t. } \forall i, g_i(x) \leq -y_i\}.\end{aligned}$$

Note that if $\lambda_i < 0$ for some $i$, one can take $x = x_0$ and $y = -re_i$, where $e_i$ is a coordinate vector and $r$ a large number, to show that $\Phi^*(0,\lambda) = +\infty$. On the other hand, if $\lambda_i \geq 0$ for all $i$, the scalar product $\langle y^* \mid y\rangle$ is maximized when $-y_i = g_i(x)$, so that in this case

$$\sup_{x\in\mathcal{X}} -\lambda_i \sum_i g_i(x) - f(x) = -\inf_{x\in\mathcal{X}} L(x,\lambda)$$ □

This leads to the following expression for the dual problem:

$$D = \sup_{\lambda\in\mathbb{R}^N_+} -\Phi^*(0,y^*) = \sup_{\lambda\in\mathbb{R}^N_+} \inf_{x\in\mathcal{X}} L(x,\lambda).$$

**Theorem 67** (Lagrangian duality)**.** *Let $f, g_1, \dots, g_N \in \Gamma_0(\mathcal{X})$ satisfying the qualification condition* (16)*, and define the Lagrangian $L$ as in* (17)*. Then,*

$$\inf_{x\in\mathcal{X}} \sup_{\lambda\in\mathbb{R}^N_+} L(x,\lambda) = \max_{\lambda\in\mathbb{R}^N_+} \inf_{x\in\mathcal{X}} L(x,\lambda),$$

*if one assumes that the left-hand-side of this equation is finite.*

*If $\lambda \in \mathbb{R}^N_+$ is a solution of the dual problem, then $x$ is a solution to the primal problem if and only if it satisfies the three Karush-Kuhn-Tucker conditions*

- *admissibility: $g_i(x) \leq 0$ for $i \in \{1,\dots,N\}$*
- *Lagrangian minimization: $x \in \arg\min_{\mathcal{X}} L(\cdot,\lambda)$,*
- *complementary slackness: $\lambda_i g_i(x) = 0$ for $i \in \{1,\dots,N\}$.*

### 5.3 Application: Fenchel-Rockafellar' theorem

Let $\mathcal{X}, \mathcal{Y}$ be two spaces, $g \in \Gamma_0(\mathcal{X})$, $h \in \Gamma_0(\mathcal{Y})$ and $A : \mathcal{X} \to \mathcal{Y}$ be a continuous linear operator. Consider the primal minimization problem

$$P = \inf_{x\in\mathcal{X}} g(x) + h(Ax),$$

and the dual maximization problem

$$D = \sup_{y^*\in\mathcal{Y}^*} -g^*(A^*y^*) - h^*(-y^*).$$

**Proposition 68** (Weak duality)**.** *$P \geq D$.*
*Moreover, if $P = D$ then the following statements are equivalent:*

*(i) $x$ is a minimizer in $P$ and $y^*$ is a maximizer of $D$ ;*
*(ii) $A^*y^* \in \partial g(x)$ and $-y^* \in \partial h(Ax)$ ;*
*(iii) $x \in \partial g^*(A^*y)$ and $Ax \in \partial h^*(-y^*)$.*

*Proof.* We first prove the weak duality inequality $P \geq D$: by Fenchel-Young,

$$\forall (y, y^*) \in \mathcal{X} \times \mathcal{X}^*, g(y) + g^*(A^*y^*) \geq \langle y \mid A^*y^* \rangle$$

$$\forall (y, y^*) \in \mathcal{X} \times \mathcal{X}^*, h(Ay) + h^*(-y^*) \geq \langle Ay \mid -y^* \rangle,$$

thus giving

$$P = \inf_{x \in \mathcal{X}} g(x) + h(Ax) \geq \sup_{y^* \in \mathcal{X}^*} -g(A^*y^*) - h(-y^*).$$

Now assume that $x$ is a solution of $P$ and $y^*$ a solution of $D$ and that $P = D$. Then,

$$g(x) + h(Ax) = -g(A^*y^*) - h(-y^*).$$

Using Fenchel-Young's inequality, we may rewrite this as

$$\underbrace{g(y) + g^*(A^*y^*) - \langle y \mid A^*y^* \rangle}_{\geq 0} + \underbrace{h(Ay) + h^*(-y^*) - \langle Ay \mid -y^* \rangle}_{\geq 0} = 0.$$

By Theorem 61, these two equalities hold if and only if $A^*y \in \partial g(x)$ and $-y^* \in \partial h(Ax)$ iff $x \in \partial g^*(A^*y)$ and $Ax \in \partial h(-y^*)$. □

**Theorem 69** (Fenchel-Rockafellar)**.** *If $P$ is finite and if the following qualification holds*

$$\exists x_0 \in \operatorname{dom} g \text{ s. t. } Ax_0 \in \operatorname{cont} h.$$

*Then, $P = D$.*

*Proof of Theorem 69.* We introduce $f(x) = g(x) + h(Ax)$ and the perturbation

$$\Phi(x, y) = g(x) + h(Ax - y).$$

The dual problem associated to this perturbation is $D = \sup_{y^*} -\Phi^*(0, y^*)$. We now compute $\Phi^*(0, y^*)$:

$$\begin{aligned}
\Phi^*(0, y^*) &= \sup_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \langle (0, y^*) \mid (x, y) \rangle - \Phi(x, y) \\
&= \sup_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \langle (0, y^*) \mid (x, y) \rangle - (g(x) + h(\underbrace{Ax - y}_{:=z})) \\
&= \sup_{(x,z) \in \mathcal{X} \times \mathcal{Y}} \langle y^* \mid Ax - z \rangle - (g(x) + h(z)) \\
&= \sup_{x \in \mathcal{X}} \langle A^*y^* \mid x \rangle - g(x) + \sup_{z \in \mathcal{Y}} \langle y^* \mid -z \rangle - h(z) \\
&= g^*(A^*y) + h^*(-y^*).
\end{aligned}$$

Therefore, the dual problem is

$$D = \sup_{y^*} -\Phi^*(0, y^*) = \sup_{y^*} -g^*(A^*y) - h^*(-y^*).$$

By assumption, there exists $x_0 \in \operatorname{dom} g$ such that $Ax_0 \in \operatorname{cont} h$. Thus, $\Phi(x_0, \cdot) = g(x_0) + h(Ax_0 - y)$ is continuous at $y = 0$. By Theorem 64, we get $P = D$ and the existence of a maximizer for $D$. $\square$

*Remark* 23 (Proof by subdifferential calculus). If the minimum in the primal problem $P$ is attained, one can prove Theorem 69 using subdifferential calculus. Assume for simplicity that $\mathcal{Y} = \mathcal{X}$ and $A = \text{Id}$. To prove the $D \geq P$, we assume that $x \in \arg\min g + h$. Thanks to the qualification hypothesis we can apply Theorem 40 on the subdifferential of the sum, giving

$$0 \in \partial(g + h)(x) = \partial g(x) + \partial h(x).$$

Thus, there exists $x^* \in \partial g(x)$ such that $-x^* \in \partial h(x)$. Therefore, using the equality case in Fenchel-Young (Theorem 61) we get

$$g(x) + g^*(x^*) = \langle x^* \mid x \rangle, h(x) + h^*(-x^*) = \langle -x^* \mid x \rangle.$$

Summing these inequalities, we obtain $g(x) + h(x) = -g^*(x^*) - h(-x^*)$, implying the strong duality $P \leq D$.

*Example* 22 (Von Neumann minimax theorem). Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert spaces, and let $C \subseteq \mathcal{X}$, $D \subseteq \mathcal{Y}$ be two closed and bounded convex sets. Finally, let $L : \mathcal{X} \to \mathcal{Y}$ be continuous. We consider the problem

$$P = \inf_{x \in C} \sup_{y \in D} \langle Lx \mid y \rangle.$$

We note that, introducing $\sigma_D$ the support function of $D$, one has

$$\sup_{y \in D} \langle Lx \mid y \rangle = \sigma_D(Lx),$$

so that

$$P = \inf_{x \in C} \sigma_D(Lx) = \inf_{x \in \mathcal{X}} i_C(x) + \sigma_D(Lx).$$

We can therefore apply Fenchel-Rockafellar's theorem with $g = \sigma_C$ and $h = \sigma_D$. We note that since $D$ is bounded, one has $\sigma_D(y) \leq M \|y\|$ with $M = \max_{z \in D} \|z\|$. Thus, $\sigma_D$ is continuous on $\mathcal{Y}$, so that the qualification hypothesis $\exists x_0 \in \operatorname{dom}(g)$ s.t. $Lx_0 \in \operatorname{cont} h$ trivially holds. By Fenchel-Rockafellar's theorem, we deduce $P = D$ with

$$D = \sup_{y \in \mathcal{Y}} -g^*(L^*y) - h^*(-y).$$

Recalling that $g^* = (i_C)^* = \sigma_C$ and $h^* = (\sigma_D)^* = i_D$, we see that

$$\begin{aligned} D &= \sup_{y \in \mathcal{Y}} -\sigma_C(L^* y) - i_D^*(-y) \\ &= \sup_{y \in \mathcal{Y}} -\sigma_C(-L^* y) - i_D^*(y) \\ &= \sup_{y \in D} - \sup_{z \in C} \langle -L^* y \mid z \rangle \\ &= \sup_{y \in D} \inf_{z \in C} \langle y \mid Lz \rangle. \end{aligned}$$

*Example* 23 (LASSO). Consider again the LASSO problem

$$\min_{x \in \mathbb{R}^d} \frac{1}{2} \|Ax - b\|_2^2 + \gamma \|x\|_1 ,$$

which is under the desired form by setting $h(y) = \frac{1}{2} \|y - b\|_2^2$ and $g(x) = \gamma \|x\|_1$. To compute the dual problem we need to compute the conjugate functions to $f$ and $g$:

$$\begin{aligned} g^*(x^*) &= \sup_{x \in \mathbb{R}^d} \langle x^* \mid x \rangle - \gamma \|x\|_1 \\ &= \gamma \sup_{x \in \mathbb{R}^d} \langle \frac{x^*}{\gamma} \mid x \rangle - \gamma \|x\|_1 \\ &= \gamma \mathrm{i}_{[-1,1]^d} \left( \frac{x^*}{\gamma} \right) \\ &= \gamma \mathrm{i}_{[-\gamma,\gamma]^d} = \mathrm{i}_{[-\gamma,\gamma]^d}, \end{aligned}$$

where we used that the convex conjugate of the norm is the unit ball of the dual norm. Similarly, one can compute $h^*$:

$$h^*(y^*) = \sup_{y \in \mathbb{R}^d} \langle y^* \mid y \rangle - \frac{1}{2} \|y - b\|_2^2 = \frac{1}{2} \|y^* + b\|^2 - \|b\|^2 .$$

Thus, the dual problem is

$$D = \max_{x^* \in \mathbb{R}^d} -g^*(A^* y^*) - h^*(-y^*) = \max \left\{ -\frac{1}{2} \|A^* x^* + b\|^2 + \|b\|^2 \mid A^* x^* \in [-\gamma, \gamma]^d \right\}.$$

The unconstrained non-smooth optimization problem $P$ is tranformed into the constrained smooth optimization problem $D$.

*Example* 24 (Rudin-Osher-Fatemi). As a second example, we consider an abstract version of the Rudin-Osher-Fatemi model for image denoising: given some $x_0 \in \mathbb{R}^d$ and a linear operator $D : \mathbb{R}^d \to \mathbb{R}^n$ we consider

$$P = \min_{x \in \mathbb{R}^d} \frac{1}{2} \|x - x_0\|_2^2 + \gamma \|Dx\|_1 .$$

Fenchel-Rockafellar's theorem can be applied by setting $g(x) = \frac{1}{2} \|x - x_0\|^2$, $h(x) = \gamma \|\cdot\|_1$ and $A = D$. The dual problem is then

$$D = \max_{y^* \in \mathbb{R}^n} -g^*(D^* y^*) - h^*(-y^*) = \max_{y^* \in [-\gamma,\gamma]^d} -\frac{1}{2} \left\| D^T y^* + x_0 \right\|_2^2 + \|x_0\|_2^2 .$$

This problem is much easier than the primal problem, because the optimized function is quadratic and the constraint set is separable (i.e. a product of segments). This problem is, for instance, directly amenable to projected gradient descent. Once a solution $y^*$ to $D$ is found, we know by Theorem 69 that if $x$ is a solution of $P$, then

$$x \in \partial g^*(D^T y^*) = \{\nabla g^*(D^T y^*)\} = \{D^T y^* + x_0\}.$$

This gives us the explicit expression $x = x_0 + D^T y^*$, allowing to recover the primal solution.

*Example* 25 (Minimization over a polyhedron). Let $A$ be a $n \times d$ matrix, let $b \in \mathbb{R}^n$ and $g : \mathbb{R}^d \to \mathbb{R}$ be a convex function. We consider the minimization problem

$$P = \inf\{g(x) \mid Ax \leq b\},$$

where the inequality $Ax \leq b$ should be understood coordinatewise, i.e. $(Ax)_i \leq b_i$ for all $i$. This problem can be recast under the form

$$P = \inf_{x \in \mathbb{R}^d} g(x) + h(Ax)$$

by setting $h = i_L$ where $L = \{y \in \mathbb{R}^n \mid \forall i, y_i \leq b_i\}$. Assume that:

- there exists $x_0 \in \mathbb{R}^d$ such that $Ax_0 \in \operatorname{int} L$. Note that this implies that the polyhedron $K = \{x \in \mathbb{R}^d \mid Ax \leq b\}$ has non-empty interior,
- $P$ is finite.

Then, the assumptions in Fenchel-Rockafellar's theorem are satisfied and we thus get

$$P = D = \max_{y^* \in \mathcal{Y}^*} -g^*(A^* y^*) - h^*(-y^*).$$

Let us compute $h^*$:

$$h^*(y^*) = \sup_{y \in \mathbb{R}^n} \langle y^* \mid y \rangle - i_L(y) = \sup_{y \leq b} \langle y^* \mid y \rangle = \begin{cases} \langle y^* \mid b \rangle & \text{if } y \leq 0, \\ +\infty & \text{otherwise.} \end{cases}.$$

Thus,

$$D = \max\{-g^*(A^* y^*) - \langle y^* \mid b \rangle \mid y \in \mathbb{R}^n, y \leq 0\}.$$

If $g(x) = \langle c^* \mid x \rangle$, $P$ is a linear programming problem, and we recover linear programming duality (under the unnecessary assumption $\operatorname{int} K \neq \emptyset$) as in Example 21. Indeed, by Example 9, $g^*(x^*) = i_{c^*}$, so that

$$D = \max\{-\langle y^* \mid b \rangle \mid y \in \mathbb{R}^n, y \leq 0 \text{ and } A^* y^* = c^*\}.$$

### 5.4 Application: duality in optimal transport

Given a compact subset $\Omega$ of $\mathbb{R}^d$, we denote:

- $\mathcal{C}^0(\Omega)$ the space of continuous functions over $\Omega$,

- $\mathcal{M}(\Omega) = (\mathcal{C}^0(\Omega))^*$ the space of finite measures over $\Omega$ with the weak$*$ topology: a sequence of measures $(\mu_n)_{n\geq 0}$ converges to $\mu$ weakly if

$$\forall \phi \in \mathcal{C}^0(\Omega), \lim_{n\to+\infty} \langle \mu_n \mid \phi\rangle = \langle \mu \mid \phi\rangle.$$

- $\mathcal{M}^+(\Omega)$ the space of non-negative measures, i.e. measures $\mu$ such that for all non-negative test function $\phi \in \mathcal{C}^0(\Omega)$ one has $\langle \mu \mid \phi\rangle \geq 0$.
- $\mathcal{P}(\Omega)$ the space of probability measures, i.e. non-negative measures $\mu$ such that $\langle \mu \mid 1\rangle = 1$.

Let $\Omega, \Omega'$ be two compact subsets of $\mathbb{R}^d$. Given $(\phi,\psi) \in \mathcal{X} := \mathcal{C}^0(\Omega)\times\mathcal{C}^0(\Omega')$, we denote

$$\phi\oplus\psi : (x,y) \in \Omega\times\Omega' \mapsto \phi(x)+\psi(y).$$

Given a cost function $c \in \mathcal{Y} := \mathcal{C}^0(\Omega\times\Omega')$ and two probability measures $\mu,\nu \in \mathcal{P}(\Omega)$, we consider the following optimization problem:

$$\sup_{(\phi,\psi)\in\mathcal{X} \mid \phi\oplus\psi\leq c} \langle \mu \mid \phi\rangle + \langle \nu \mid \psi\rangle,$$

which can be written as $P = -\inf_{(\phi,\psi)\in\mathcal{X}} \Phi((\phi,\psi),0)$, where

$$\begin{aligned} &\Phi : \mathcal{X}\times\mathcal{Y} \to \mathbb{R}^d \\ &((\phi,\psi),p) \mapsto \begin{cases} -(\langle \mu \mid \phi\rangle + \langle \nu \mid \psi\rangle) & \text{if } \phi\oplus\psi \leq c-p \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

The associated dual problem is $D = \sup_{\gamma\in\mathcal{Y}^*} -\Phi^*((0,0),\gamma)$ with

$$\begin{aligned} \Phi^*((0,0),\gamma) &= \sup_{((\phi,\psi),p)} \langle (\phi,\psi) \mid (0,0)\rangle + \langle p \mid \gamma\rangle - \Phi((\phi,\psi),p) \\ &= \sup_{((\phi,\psi),p) \mid \phi\oplus\psi\leq c-p} \langle p \mid \gamma\rangle + \langle \mu \mid \phi\rangle + \langle \nu \mid \psi\rangle \\ &= \begin{cases} \sup_{((\phi,\psi),p) \mid \phi\oplus\psi\leq c-p} \langle c-\phi\oplus\psi \mid \gamma\rangle + \langle \mu \mid \phi\rangle + \langle \nu \mid \psi\rangle & \text{if } \gamma \in \mathcal{M}^+(\Omega\times\Omega') \\ +\infty & \text{otherwise} \end{cases} \\ &= \begin{cases} \langle c \mid \gamma\rangle \text{ if } \gamma \in \Gamma(\mu,\nu) \\ +\infty & \text{otherwise,} \end{cases} \end{aligned}$$

where we have denoted $\Gamma(\mu,\nu)$ the space of *transport plan* between $\mu$ and $\nu$, i.e. non-negative measures $\gamma \in \mathcal{M}^+(\Omega\times\Omega)$ such that for all test function $(\phi,\psi)$ one has $\langle \gamma \mid \phi\oplus\psi\rangle = \langle \mu \mid \phi\rangle + \langle \nu \mid \psi\rangle$. We have thus proven Kantorovich's duality:

$$\begin{aligned} \sup_{(\phi,\psi)\in\mathcal{X} \mid \phi\oplus\psi\leq c} \langle \mu \mid \phi\rangle + \langle \nu \mid \psi\rangle &= -\inf_{(\phi,\psi)\in\mathcal{X}} \Phi((\phi,\psi),0) \\ &= -\max_{\gamma\in\mathcal{Y}^*} -\Phi^*((0,0),\gamma) \\ &= \min_{\gamma\in\Gamma(\mu,\nu)} \langle \gamma \mid c\rangle. \end{aligned}$$

# A Topology and functional analysis

## A.1 Point-set topology

We briefly recall some elementary notions from topology, even if we assume some familiarity with them.

**Definition 28** (Topology)**.** A *topology* on a set $X$ is a family of subsets $\tau$ of $X$, called *open sets*, which satisfy the following axioms:
(i) the empty set is open, i.e. $\emptyset \in \tau$;
(ii) the intersection of a finite family of open subsets is open: if $\omega_1, \ldots, \omega_k \in \tau$, then $\omega_1 \cap \ldots \cap \omega_k \in \tau$.
(iii) any union of open subsets if open: if $(\omega_i)_{i \in I}$ belongs to $\tau$, then so does $\cup_i \omega_i$.
The space $(X, \tau)$ is then called a topological space. The complement of an open set is called a *closed set*.

*Example* 26 (Metric topology). Let $(X, d)$ be a metric space, i.e. $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ satisfies the assumptions
(i) $d(x, y) \geq 0$ with equality if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
We denote $\mathrm{B}(x, r) = \{y \in \mathcal{X} \mid d(x, y) \leq r\}$ the closed ball. We call a set $\omega$ open if for any $x \in \omega$, there exists $r > 0$ such that $B(x, r) \subseteq \omega$. Then, the family $\tau_d$ of such open sets satisfies the axioms of a topology.

**Definition 29** (Neighborhood)**.** Let $(X, \tau)$ be a topological space. A neighborhood of $x \in \mathcal{X}$ is any set that contains an open set, which itself contains $x$. A *neighborhood basis* at $x$ is a family $\mathcal{B}_x$ of subsets of $\mathcal{X}$ sucht that (i) every set $B \in \mathcal{B}_x$ is a neighborhood of $x$ and (ii) for any neighborhood $N$ of $x$, there exists $B \in \mathcal{B}_x$ such that $B \subseteq N$.

*Example* 27. In a metric space, the sets $\mathcal{B}_x = \{B(x, \delta) \mid \delta > 0\}$ and $\mathcal{B}'_x = \{B(x, \frac{1}{n}) \mid n \in \mathbb{N}\}$ form two neighborhood bases at $x$.

**Definition 30** (Continuity at a point)**.** A function $f : (\mathcal{X}, \sigma) \to (\mathcal{Y}, \tau)$ between topological spaces is *continuous* at $x$ if for any neighborhood $B$ of $f(x)$, the inverse image $f^{-1}(B)$ is a neighborhood of $x$. Equivalently, $f$ is continuous at $x \in \mathcal{X}$ iff there are neighborhood bases $\mathcal{B}_x$ at $x$ and $\mathcal{B}_{f(x)}$ at $f(x)$ such that

$$\forall C \in \mathcal{B}_{f(x)}, \exists B \in \mathcal{B}_x \text{ s.t. } f^{-1}(C) \subseteq B.$$

In the case of metric spaces, one recovers the definition of continuity using $\varepsilon$ and $\delta$.

The notion of continuity depends on the choice of the topologies $\sigma$ and $\tau$. This is important to keep in mind, because we will sometimes consider two topologies over the same space.

**Definition 31** (Continuity)**.** The function $f : (\mathcal{X}, \sigma) \to (\mathcal{Y}, \tau)$ is *continuous* on $\mathcal{X}$ if it is continuous at every $x \in \mathcal{X}$. Equivalently, $f$ is continuous on $\mathcal{X}$ if the inverse image of any open set under $f$ is open, i.e. $\forall \omega \in \tau,\ f^{-1}(\omega) \in \sigma$.

**Definition 32.** A topological space $(\mathcal{X}, \tau)$ is called

- *separated* or *Hausdorff* if for any distinct points $x, y \in \mathcal{X}$ have neighborhoods $O_x, O_y$ such that $O_x \cap O_y = \emptyset$.
- *metrizable* if the topology $\tau$ is induced by a metric;

*Remark* 24. Metric spaces are automatically separated. In addition, for a metric space, and hence for a metrizable space, all topological notions may be recovered through sequences. For instance,

- a set $C \subseteq \mathcal{X}$ is closed if and only if $C$ contains the limit of every converging sequence in $C$ ;
- a function $f$ is continuous at $x$ if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ converging to $x$ one has $\lim_{n\to+\infty} f(x_n) = f(x)$.

## A.2 Topological vector spaces

Most of the statements of this course hold when $\mathcal{X}$ is a normed space i.e. a vector space endowed with a norm $\|\|$ which endows $\mathcal{X}$ with a topology. The closed ball of radius around a point $x$ is then denoted $\mathrm{B}(x, r) = \{y \in \mathcal{X} \mid \|x - y\| \leq r\}$, and the topology is induced by the metric $d(x, y) = \|x - y\|$. In particular, the results of this course can be applied to Hilbert spaces, Banach spaces, and even $\mathbb{R}^d$.

However, it is often the case that our statement hold in the more setting of (locally convex) topological vector spaces, which is particularly adequate when one wants to consider weak/weak$*$ topologies, which are important in some applications.

**Definition 33** (Topological vector space)**.** A *topological vector space* is a vector space $\mathcal{X}$ endowed with a topology $\tau$ which makes the addition of vectors $(x, y) \in \mathcal{X}^2 \mapsto x + y \in \mathcal{X}$ and the multiplication by a scalar $(\lambda, x) \in \mathbb{R} \times \mathcal{X} \mapsto \lambda x \in \mathcal{X}$ continuous.

In order to define the continuity of the maps $\mathcal{X}^2 \to \mathcal{X}$ and $\mathbb{R} \times \mathcal{X} \to \mathcal{X}$, we need to define a topology on these product spaces, called the product topology.

**Definition 34** (Product topology)**.** Let $(X_1, \tau_1)$ and $(X_2, \tau_2)$ be topological spaces. A set $O \subseteq X_1 \times X_2$ is open for the *product topology* $\tau_1 \otimes \tau_2$ if for any $(x_1, x_2) \in O$, there exists neighborhoods $N_1$ and $N_2$ of $x_1$ and $x_2$ such that $N_1 \times N_2 \subseteq O$.

One can check that if $O$ is an open subset of a topological vector space, then for all $x \in X$, $O + x = \{x + y \mid y \in O\}$ is also open. This is because the map $T : y \mapsto y - x$, and $O + x = T^{-1}(O)$. In particular, if $\mathcal{B}$ is a basis of neighborhood of the origin, then $\{B + x \mid B \in \mathcal{B}\}$ is a basis of neighborhood at $x$.

**Definition 35** (Locally convex topological vector space)**.** A topological vector space is called *locally convex* if the origin admits a neighborhood basis made of convex sets.

## A.3 Continuous linear forms

**Definition 36** (Dual space)**.** The *topological dual* of $\mathcal{X}$ is the space of all continuous linear functions on $\mathcal{X}$, and is denoted $\mathcal{X}^*$.

The elements of $\mathcal{X}^*$ are denoted with a star, e.g. $x^* \in \mathcal{X}^*$, and we denote the pairing between $\mathcal{X}^*$ and $X$ using the notation for scalar product, i.e. $\langle x^* \mid x\rangle := x^*(x)$, to underline the similarity with the case of Hilbert spaces.

*Remark* 25. When $(\mathcal{X}, \|\cdot\|)$ is a normed vector space, the topological dual $\mathcal{X}^*$ may be endowed with the dual norm

$$\|x^*\| = \sup_{x\in\mathcal{X}} \langle x^* \mid x\rangle.$$

One could construct a "strong" topology on $\mathcal{X}^*$ even when $\mathcal{X}$ is not a normed space, but this will not be necessary in this course.

We recall the following characterization of continuous linear forms on a topological vector space, which generalizes a similar criterion for normed spaces.

**Lemma 70.** *(Continuous linear forms) Let $\phi : \mathcal{X} \to \mathbb{R}$ be a linear form on a topological vector space $(X, \sigma)$. Then, the following are equivalent*

*(i) $\phi$ is continuous,*

*(ii) $\phi$ is continuous near the origin (i.e. for all $\varepsilon > 0$, there exists an open set $O$ containing the origin such that $|\phi| < \varepsilon$ on $O$),*

*(iii) $\phi$ is bounded (from above or from below) in a neighborhood of the origin,*

*(iv) $\phi$ is bounded (from above or from below) on a non-empty open subset.*

*Proof.* The equivalence between (i) and (ii) (resp. (iii) and (iv)) is obvious, and so is the implication (ii) $\Longrightarrow$ (iii). We therefore only prove the implication (iii) $\Longrightarrow$ (ii). Let $\omega$ be a neighborhood of the origin such that $|\phi| \leq M$ on $\omega$. For any $\varepsilon > 0$, define $\omega_\varepsilon = (\varepsilon/M)O$. Then, $\omega_\varepsilon$ is open and $|\phi| \leq \varepsilon$ on $\omega_\varepsilon$. This proves the continuity of $\phi$ at zero. ☐

**Lemma 71.** *(Closed halfspaces) Let $\phi : \mathcal{X} \to \mathbb{R}$ be a linear form over a topological vector space $\mathcal{X}$. Then, the following are equivalent:*

*(i) $\phi$ is continuous,*

*(ii) the set $H = \{x \in \mathcal{X} \mid \phi(x) \leq \alpha\}$ is closed,*

*(iii) the set $H' = \{x \in \mathcal{X} \mid \phi(x) = \alpha\}$ is closed,*

*Proof.* If $\phi$ is continuous, then the halfspace $H$ (resp. hyperplane $H'$) is closed as the inverse image of $(-\infty, \alpha]$ (resp. $\{\alpha\}$) under $\phi$. Conversely, if $H$ is closed (resp. $H'$ is closed), then $\phi > \alpha$ on the non-empty open set $\mathcal{X} \setminus H$ (resp. $\mathcal{X} \setminus H'$), so that $\phi$ is continuous by the previous lemma. ☐

## A.4 Weak and weak∗ topologies

In many cases, the topology associated with the norm is too fine, in the sense that it has too few compact sets, making it difficult to to prove existence to optimization problems. This is why we will try to define coarser topologies, i.e. topologies with less open sets and thus more compact sets. One standard way to do this is to start with a family of maps, and to construct the *coarsest* (i.e. smallest with respect to inclusion) topology that makes all these maps continuous.

We will explain the construction of the coarsest topology making a family of linear forms $\mathcal{Y}$ over a vector space $\mathcal{X}$ continuous.

**Lemma 72.** *Let $\mathcal{X}$ be a vector space and let $\mathcal{Y}$ be a vector space of linear forms over $\mathcal{X}$. Assume that $\sigma$ is a topology over $\mathcal{X}$ and that every linear form in $\mathcal{Y}$ is continuous with respect to $\mathcal{X}$. Then, all sets of the following form are open with respect to $\sigma$:*

$$B_{x,y_1,\ldots,y_N,\varepsilon} := \{z \in \mathcal{X} \mid \forall i \in \{1,\ldots,N\}, |\langle y_i \mid z - x\rangle| < \varepsilon\},$$

*with $x \in \mathcal{X}$, $y_1,\ldots,y_N \in \mathcal{Y}$ and $\varepsilon > 0$.*

*Proof.* Let $y_1,\ldots,y_N \in \mathcal{Y}$. Then,

$$B_{x,y_1,\ldots,y_N,\varepsilon} = \bigcap_{i=1}^{N} y_i^{-1}((\langle y_i \mid x\rangle - \varepsilon, \langle y_i \mid x\rangle + \varepsilon)).$$

Thus, $B_{x,y_1,\ldots,y_N,\varepsilon}$ is open with respect to $\sigma$ as a finite intersection of inverse images of the open sets $(\langle y_i \mid x\rangle - \varepsilon, \langle y_i \mid x\rangle + \varepsilon)$ under the maps $y_1,\ldots,y_N$ which we have assumed continuous with respect to $\sigma$. □

This lemma suggests the following definition.

**Definition 37** (Topology generated by linear forms)**.** Let $\mathcal{X}$ be a vector space and let $\mathcal{Y}$ be a vector space of linear forms over $\mathcal{X}$. A set $O$ is open with respect to the topology generated by $\mathcal{Y}$, denoted $\sigma(\mathcal{X},\mathcal{Y})$ if for any $x \in O$, there exists $y_1,\ldots,y_N \in \mathcal{Y}$ and $\varepsilon > 0$ such that $B_{x,y_1,\ldots,y_N,\varepsilon} \subseteq O$.

The sets $B_{x,y_1,\ldots,y_N,\varepsilon}$ are open with respect to $\sigma(\mathcal{X},\mathcal{Y})$ (exercise!) and a neighborhood basis of the point $x \in \mathcal{X}$ for $\sigma(\mathcal{X},\mathcal{Y})$ is given by

$$\mathcal{B}_x = \{B_{x,y_1,\ldots,y_N,\varepsilon} \mid N \in \mathbb{N}, y_1,\ldots,y_N \in \mathcal{Y}, \text{ and } \varepsilon > 0\}.$$

In some sense, the sets $B_{x,y_1,\ldots,y_N,\varepsilon}$ play the role of balls for the weak topology. However, this topology has a rather surprising feature: if $\mathcal{X}$ is infinite dimensional, then any set with non-empty $\sigma(X,Y)$-interior must contain a set of the above form and must therefore be unbounded. In particular, in infinite dimension the unit ball $\mathrm{B}(0,1)$ has empty $\sigma(X,Y)$-interior.

**Proposition 73.** *Let $\mathcal{X}$ be a vector space and let $\mathcal{Y}$ be a vector space of linear forms over $\mathcal{X}$. Then,*

*(i) $(\mathcal{X},\sigma(\mathcal{X},\mathcal{Y}))$ is a locally convex topological vector space;*
*(ii) A linear form $y : \mathcal{X} \to \mathbb{R}$ is continuous with respect to the topology $\sigma(\mathcal{X},\mathcal{Y})$ if and only if it belongs to $\mathcal{Y}$;*

**Corollary 74.** *Given a linear form $y : \mathcal{X} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, the hyperplane $y^{-1}(\alpha)$ (resp. the halfspace $y^{-1}((-\infty, \alpha])$) is closed w.r.t. $\sigma(\mathcal{X}, \mathcal{Y})$ if and only if $y \in \mathcal{Y}$.*

*Remark* 26 (Coarsest topology). Lemma 72 shows that if $\sigma$ is a topology on $\mathcal{X}$ such that all the linear forms in $\mathcal{Y}$ are continuous, then it contains all the sets of the form $B_{x,y_1,\dots,y_N,\varepsilon}$ with $y_1, \dots, y_N \in \mathcal{Y}$. Thus, any such topology $\sigma$ must contain $\sigma(\mathcal{X}, \mathcal{Y})$, implying that $\sigma(\mathcal{X}, \mathcal{Y})$ is the coarsest topology on $\mathcal{Y}$ making all the linear forms in $\mathcal{Y}$ are continuous. (This also means that it has the most compact subsets, or the most converging sequences...)

*Remark* 27 (Convergence of sequences). One can show that a sequence $(x_n)_{n\geq 1}$ of points of $\mathcal{X}$ converges to $x \in \mathcal{X}$ with respect to the topology of $\sigma(\mathcal{X}, \mathcal{Y})$ — meaning that for all neighborhood $O$ of $x$ one has $x_n \in O$ for $n$ large enough — if and only if

$$\forall y \in \mathcal{Y}, \lim_{n\to+\infty} \langle y \mid x_n \rangle = \langle y \mid x \rangle.$$

*Proof.* (i) The family $\sigma(\mathcal{X}, \mathcal{Y})$ contains $\emptyset$, is (easily) stable under arbitrary unions, and is finite under finite intersections because

$$B_{x,y_1,\dots,y_N,y_1',\dots,y_M',\min(\varepsilon,\varepsilon')} \subseteq B_{x,y_1,\dots,y_N,\varepsilon} \cap B_{x,y_1',\dots,y_M',\varepsilon'}.$$

This proves that $\sigma(\mathcal{X}, \mathcal{Y})$ is a topology. We will prove that the map $S : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ defined by $S(x, y) = x + y$ is continuous with respect to $\sigma(\mathcal{X}, \mathcal{Y})$. To see this, take $x, y \in \mathcal{X}$ and consider a neighborhood $N_z$ of $z = S(x, y) = x + y$. We want to prove that there exists neighborhoods $N_x$, $N_y$ of $x$ and $y$ so that $S(N_x, N_y) \subseteq N_z$. By definition, the neighborhood $N_z$ contains a set of the form $B_z := B_{z,y_1,\dots,y_N,\varepsilon}$. Define $B_x := B_{x,y_1,\dots,y_N,\frac{1}{2}\varepsilon}$ and $B_y = B_{y,y_1,\dots,y_N,\frac{1}{2}\varepsilon}$. For any $x' \in B_x$ and $y' \in B_y$, let $z' = S(x', y') = x' + y'$). We have

$$\left|\langle y_i \mid z' - z \rangle\right| = \frac{1}{2} \left|\langle y_i \mid x' + y' - (x + y) \rangle\right| \leq \frac{1}{2}(\left|\langle y_i \mid x' - x \rangle\right| + \left|\langle y_i \mid x' - x \rangle\right|) < \varepsilon,$$

thus ensuring that $z' \in B_z$. In other words, $S(B_x, B_y) \subseteq B_z$, and $B_x$ and $B_y$ are neighborhoods of $x$ and $y$ respectively: this shows the continuity of $S$ at $(x, y)$. We would prove similarly that the map $P : (\lambda, x) \in \mathbb{R} \times \mathcal{X} \mapsto \lambda x \in \mathcal{X}$, thus proving that $(\mathcal{X}, \sigma(\mathcal{X}, \mathcal{Y}))$ is a topological vector space. Finally, the space is locally convex because all the sets $B_{x,y_1,\dots,y_N,\varepsilon}$ are convex.

(ii) Let $y$ be a linear form over $\mathcal{X}$ and assume that it is continuous with respect to the topology $\sigma(\mathcal{X}, \mathcal{Y})$. Thus, there exists a neighborhood $N$ of the origin on which the linear form $y$ is bounded by 1. By construction of the topology, there exists $y_1, \dots, y_N$ and $\varepsilon > 0$ such that $B_{0,y_1,\dots,y_N,\varepsilon} \subseteq N$. Thus, we have

$$(\forall i \in \{1, \dots, N\}, \quad \langle y_i \mid x \rangle < \varepsilon) \Longrightarrow |\langle y \mid x \rangle| < 1,$$

which implies in particular that if $\mathrm{Ker}(y_1) \cap \dots \mathrm{Ker}(y_n) \subseteq \mathrm{Ker}(y)$. By a the linear algebra lemma below, this implies that $y \in \mathrm{vect}(y_1, \dots, y_N) \subseteq \mathcal{Y}$. □

**Lemma 75.** *Let $\mathcal{X}$ be a vector space, and let $y_1, \dots, y_N, y$ be linear forms on $\mathcal{X}$ such that* $\mathrm{Ker}(y_1) \cap \dots \mathrm{Ker}(y_n) \subseteq \mathrm{Ker}(y)$. *Then* $y \in \mathrm{vect}(y_1, \dots, y_N)$.

*Proof.* Let $F : \mathcal{X} \to \mathbb{R}^{N+1}$ be defined by $F(x) = (y(x), y_1(x), \dots, y_N(x))$. The assumption implies that the subspace $L = F(\mathcal{X})$ and the point $z = (1, 0, \dots, 0)$ are disjoint. Denoting $p \in L$ the orthogonal projection of $z$ on $L$ and $p - z = (\lambda, \lambda_1, \dots, \lambda_N) \neq 0$, the characterization of the orthogonal projection gives

$$\forall x \in \mathcal{X}, \langle F(x) \mid \lambda \rangle = 0,$$

i.e. $\lambda y + \sum_i \lambda_i y_i = 0$. Moreover, $\langle p \mid z - p \rangle > 0$, giving $\lambda \neq 0$. □

**Definition 38** (Weak-topology)**.** Let $\mathcal{X}$ be a topological vector space, and let $\mathcal{X}^*$ be the space of continuous linear forms over $\mathcal{X}$. The topology over $\mathcal{X}$ generated by the linear forms $\mathcal{X}^*$, namely $\sigma(\mathcal{X}, \mathcal{X}^*)$, is called the weak topology.

*Remark* 28 (Relation to the strong topology). We will call the original topology on $\mathcal{X}$ the *strong* topology to distinguish it from the weak topology $\sigma(\mathcal{X}, \mathcal{X}^*)$. The sets $B_{x, x_1^*, \dots, x_N^*, \varepsilon}$, which form a basis of the weak topology, are open with respect to the original topology. This directly implies that weakly open sets are strongly open, and similarly that weakly closed sets are strongly closed. Maybe unintuitively, this also implies that if a function $f : \mathcal{X} \to \mathbb{R}$ is weakly continuous, then it is strongly continuous (Proof: assume that $f$ is weakly continuous. Then, for any open subset $O$ of $\mathbb{R}$, the set $f^{-1}(O)$ is weakly open, therefore strongly open. Thus, $f$ is strongly continuous). The converse implications are false in general, but we will show that they are true when convexity hypotheses are added (see e.g Proposition 12).

**Definition 39** (Weak∗-topology)**.** Let $\mathcal{X}$ be a topological vector space, and consider the canonical injection

$$i : \mathcal{X} \to \mathcal{X}^{**}, x \mapsto (x^* \mapsto \langle x^* \mid x \rangle).$$

Thanks to this injection, $\mathcal{X}$ can be identified with the set $i(\mathcal{X})$ of linear forms over $\mathcal{X}^*$. The topology over $\mathcal{X}^*$ generated by the linear forms $i(\mathcal{X})$ is called the weak topology and often denoted $\sigma(\mathcal{X}^*, \mathcal{X}) := \sigma(\mathcal{X}, i(\mathcal{X}))$.

We refer to [Bre10]) for a more thorough treatment of the weak/weak* topology.

*Remark* 29 (Weak∗ and pointwise convergence). A good exercise is to show that a sequence $(x_n^*)$ of elements of $\mathcal{X}^*$ converges to $x^*$ with respect to $\sigma(\mathcal{X}^*, \mathcal{X})$ (for short we will say that $(x_n^*)$ *weak-∗ converges to* $x^*$) if and only if

$$\forall x \in \mathcal{X}, \quad \lim_{n \to +\infty} \langle x_n^* \mid x \rangle = \langle x^* \mid x \rangle.$$

In plain words, $(x_n^*)$ weak-∗ converges to $x^*$ if only if $(x_n^*)$ converges *pointwise* to $x^*$, when these linear forms are seen as functions on $\mathcal{X}$.

*Remark* 30 (Separation). The space $(\mathcal{X}^*, \sigma(\mathcal{X}^*, \mathcal{X}))$ is separated, i.e. Hausdorff. Indeed, let $x^*, y^* \in \mathcal{X}^*$ be two distinct linear forms. Then, there exists $x \in \mathcal{X}$ such that $\langle x^* \mid x \rangle \neq \langle y^* \mid x \rangle$, for instance $\langle x^* \mid x \rangle < \langle y^* \mid x \rangle$. Thus, denoting $r = \langle \frac{1}{2}(x^* + y^*) \mid x \rangle$, the two weak∗ open sets $O_- = \{\langle \cdot \mid x \rangle < r\}$ and $O_+ = \{\langle \cdot \mid x \rangle > r\}$ separate the points $x^*$ and $y^*$: they are disjoint and $x^* \in O_-$ and $y^* \in O_+$.

*Example* 28 (Measures). Let $K \subseteq \mathbb{R}^d$ be compact and let $\mathcal{X} = \mathcal{C}^0(K)$ be the space of continuous functions over $K$. What are examples of linear functionals over $\mathcal{X}$ ? Let $\mu$ be a finite (Radon) measure over $\mathcal{X}$: then, the functional $f \in \mathcal{C}^0(K) \mapsto \int f \mathrm{d}\mu$ is linear. Conversely, Riesz-Markov's theorem asserts that any linear functional over $\mathcal{X}$ is induced by a finite (Radon) measure. Thus, in this course, we will define the space of Radon measures as

$$\mathcal{M}(K) := \mathcal{C}^0(K)^*.$$

An example of Radon measure is the Dirac mass $\delta_x \in \mathcal{M}(K)$, defined by

$$\forall \phi \in \mathcal{C}^0(K), \langle \delta_x \mid \phi \rangle = \delta_x(\phi) := \phi(x).$$

The dual norm on $\mathcal{M}(K)$ is called the *total variation* of the measure $\mu$. It is defined by $\|\nu\|_{TV} = \sup\{\langle \nu \mid \phi \rangle \mid \|\phi\|_\infty \leq 1\}$. Note that if $x, y$ are distinct points in $K$, it is possible to construct a function $\phi \in \mathcal{C}^0(X)$ so that $\phi(x) = 1$ and $\phi(y) = -1$ and $\|\phi\|_\infty \leq 1$. Thus,

$$\|\delta_x - \delta_y\|_{TV} \geq \int \phi \mathrm{d}(\delta_x - \delta_y) = \phi(x) - \phi(y) = 2.$$

In other words, even if $(x_n)$ converges to $x \in K$, the Dirac mass $\delta_{x_n}$ does *not* converge to $\delta_x$ with respect to the total variation $\|\cdot\|_{TV}$. On the other hand,

$$\forall \phi \in \mathcal{C}^0(X), \langle \phi \mid \delta_{x_n} \rangle = \phi(x_n) \longrightarrow_{n \to +\infty} \langle \phi \mid \delta_x \rangle,$$

so that $(\delta_{x_n})$ weak$*$-converges to $\delta_x$, i.e. with respect to the topology $\sigma(\mathcal{M}(X), \mathcal{C}^0(X))$. This is an illustration of the fact that the weak-$*$ topology $\sigma(\mathcal{M}(X), \mathcal{C}^0(X))$ has more converging sequences, and thus more compact sets as well.

*Remark* 31 (Continuous linear forms over $(\mathcal{X}^*, \sigma(\mathcal{X}^*, \mathcal{X}))$). By Proposition 73, we know that a linear form over $\mathcal{X}^*$ is continuous if and only if is induced by an element to $\mathcal{X}$, i.e. $(\mathcal{X}^*, \sigma(\mathcal{X}^*, \mathcal{X}))$ can be identified with $\mathcal{X}$. This implies, for instance, that the dual of the space of measures endowed with its weak$*$ topology is $(\mathcal{M}(K), \sigma(\mathcal{M}(K), \mathcal{C}^0(K)))^* \simeq \mathcal{C}^0(K)$, while the dual $(\mathcal{M}(K), \|\cdot\|_*)^*$ is a much larger space and much less understood (there is a book devoted to this topic [Kap11]).

**Definition 40** (Separability)**.** A topological space is *separable* if $\mathcal{X}$ contains a countable dense subset.

*Example* 29. Examples of separable spaces include

- the space $(\mathcal{C}^0(X), \|\cdot\|_\infty)$ of continuous functions over a compact subset $X$ of $\mathbb{R}^d$ endowed with the norm of uniform convergence. This is a consequence of the Stone-Weierstrass theorem, which guarantees that continuous functions over a compact subset of $\mathbb{R}^d$ can be uniformly approximated by polynomials ;
- the spaces $(\ell^p, \|\cdot\|_p)$ of $p$-summable sequences and the spaces $(\mathrm{L}^p(\Omega), \|\cdot\|_p)$ of $p$-integrable functions over a bounded domain of $\mathbb{R}^d$, for $p \in [1, +\infty)$. Note that $\ell^\infty$ and $\mathrm{L}^\infty(\Omega)$ are *not* separable.

**Theorem 76** (Metrizability of the dual ball). *Let $\mathcal{X}$ be a separable normed vector space, and let $(x_n)_{n\geq 1}$ be a dense sequence in $\mathcal{X}$. Then the weak$*$ topology on a bounded subset $S$ of $\mathcal{X}^*$ is induced by the distance*

$$\mathrm{d}(x^*, y^*) = \sum_{n\geq 1} 2^{-n} \left| \langle x^* - y^* \mid \frac{x_n}{\|x_n\|} \rangle \right|.$$

*Remark* 32. The open sets of the weak$*$ topology on $S$ are of the form $O' = O \cap S$, where $O$ is a weak$*$ open sets of $\mathcal{X}^*$.

*Proof.* The series defining $\mathrm{d}(x^*, y^*)$ is converging. Indeed, by definition of $\|\cdot\|_*$,

$$\left| \langle x^* - y^* \mid \frac{x_n}{\|x_n\|} \rangle \right| \leq \|x^* - y^*\|_* \leq \mathrm{diam}(S).$$

The symmetry of $d$, and the triangle inequality are clear. If $\mathrm{d}(x^*, y^*) = 0$, then for all $x_n$ we have $\langle x^* \mid x \rangle = \langle y^* \mid x \rangle$. Thus, the continuous linear forms $x^*$ and $y^*$ agree on the dense subset $(x_n)_{n\geq 1}$, and therefore they agree on $\mathcal{X}$, i.e. $x^* = y^*$.

Let $S$ be a bounded set in $\mathcal{X}^*$; we assume that $S \subseteq \mathrm{B}(0^*, R)$ for some $R > 0$. Let $O' = O \cap S$ be an open set with respect to the distance $d$. Our goal is to prove that $O \cap S$ is also weak$*$ open. Let $x^* \in O \cap S$: by openness of $O$ with respect to the distance $d$, there exists $\varepsilon > 0$ such that $\mathrm{B}(x^*, \varepsilon) \subseteq O$. For any $y^*, z^* \in S$, we have $\|y^* - z^*\| \leq 2R$, so that there exists some $N \in \mathbb{N}$ such that

$$\forall y^*, z^* \in S, \quad \sum_{n\geq N} 2^{-n} \left| \langle x^* - y^* \mid \frac{x_n}{\|x_n\|} \rangle \right| \leq \frac{\varepsilon}{2}.$$

If we consider some point $z^* \in S \cap B_{x^*, y_1, \dots, y_N, \frac{\varepsilon}{4}}$, then

$$\forall i \in \{1, \dots, N\} \, |\langle x^* - z^* \mid y_i \rangle| < \frac{\varepsilon}{4}.$$

This implies that such a point belongs to the ball $\mathrm{B}(x^*, \varepsilon)$ because

$$d(x^*, z^*) = \sum_{1\leq n<N} 2^{-n} \left| \langle x^* - y^* \mid \frac{x_n}{\|x_n\|} \rangle \right| + \sum_{n\geq N} 2^{-n} \left| \langle x^* - y^* \mid \frac{x_n}{\|x_n\|} \rangle \right| \leq \varepsilon.$$

To summarize, the set $S \cap B_{x^*, y_1, \dots, y_N, \frac{\varepsilon}{4}}$ is included in $S \cap \mathrm{B}(x^*, \varepsilon)$, itself included in $O'$. By definition, this implies that $O'$ is weak$*$ open in $S$.

To conclude that the weak$*$ and the metric topology agree on $S$, we need to prove that conversely, any weak$*$ open set of $S$ is open with respect to the distance $d$. The proof of this fact is similar, and is left as an exercise. □

**Theorem 77** (Banach-Alaoglu). *Let $\mathcal{X}$ be a separable normed space. Then, any bounded sequence in $\mathcal{X}^*$ admit a weak*–converging subsequence.*

*Proof.* Let $A$ be a dense countable subset of $\mathcal{X}$. Let $(x_n^*)$ be a bounded sequence in $\mathcal{X}^*$. Then, for all $x \in A$, the sequence $(\langle x_n^* \mid x\rangle)_{n\in\mathbb{N}}$ is bounded and admits a converging subsequence. By a diagonal argument, and taking subsequences where necessary, we can assume that for all $x \in A$ there exists $f_a \in \mathbb{R}$ such that

$$\lim_{n\to+\infty} \langle x_n^* \mid x\rangle = \lim_{n\to+\infty} x_n^*(x) = f_a.$$

Thus, the sequence of functions $x_n^*|_A$ converges pointwise to the function $f : A \to \mathbb{R}$ defined by $f(x) = f_a$. By boundedness of the sequence $(x_n^*)$, there exists $R > 0$ such that $\|x_n^*\|_* \leq R$ for all $n \in \mathbb{N}$. Then,

$$\forall x, y \in \mathcal{X}, |\langle x_n^* \mid x\rangle - \langle x_n^* \mid y\rangle| \leq R\,\|x - y\|\,.$$

Passing to the limit as $n \to +\infty$, this proves that the function $f$ is $R$-Lipschitz on the dense set $A \subseteq \mathcal{X}$ and can therefore be extended uniquely into a $R$-Lipschitz function function $\hat{f} : \mathcal{X} \to \mathbb{R}$. We will now show that $x_n^*$ converges pointwise to $\hat{f}$ on $\mathcal{X}$. Let $x$ be an arbitrary point $\mathcal{X}$, and let $\varepsilon > 0$. By density of $A$, there exists $y \in A$ such that $\|x - y\| \leq \frac{1}{3R}\varepsilon$; by convergence of $x_n^*(y)$ to $\hat{f}(y)$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $\left|\hat{f}(y) - x_n^*(y)\right| \leq \frac{1}{3}\varepsilon$. Then,

$$\left|\hat{f}(x) - x_n^*(x)\right| \leq \left|\hat{f}(x) - \hat{f}(y)\right| + \left|\hat{f}(y) - x_n^*(y)\right| + |x_n^*(y) - x_n^*(x)| \leq \varepsilon.$$

This shows that $\lim_{n\to+\infty} x_n^*(x) = \hat{f}(x)$, and that weak$*$ converges to $\hat{f}$. Finally, we note that a pointwise limit of linear functions is linear to conclude that $\hat{f} \in \mathcal{X}^*$. □

**Theorem 78** (Banach-Alaoglu)**.** *Let $\mathcal{X}$ be a separable normed space. Then, the unit ball of $\mathcal{X}^*$ is weakly*–closed.*

*Proof.* The unit ball $B$ of $\mathcal{X}^*$ is bounded, so that its weak$*$ topology is metrizable by Theorem 76. Thus, $B$ is weak$*$ compact if and only if it is sequentially weak$*$ compact, i.e. iff any sequence of elements of $B$ admits a weak$*$ converging subsequence. One concludes by invoking Theorem 77. □

*Example* 30 (Probability measures). Let $K$ be a compact subset of $\mathbb{R}^d$, and let $\mathcal{X} = \mathcal{C}^0(K)$. Then, the dual space $\mathcal{X}^*$ is the space of Radon measures over $K$, i.e. $\mathcal{X}^* = \mathcal{M}(K)$, and its unit ball is

$$B = \{\mu \in \mathcal{M}(K) \mid \|\mu\|_{TV} \leq 1\},$$

and by Banach-Alaoglu's theorem, $B$ is weak$*$ compact. Now let $\mathcal{M}^+(K)$ be the set of positive measures on $K$, i.e.

$$\mathcal{M}^+(K) = \{\mu \in \mathcal{M}(K) \mid \forall\phi \in \mathcal{C}^0(K), \phi \geq 0 \Longrightarrow \langle\mu \mid \phi\rangle \geq 0\}.$$

Then, $\mathcal{M}^+(K)$ is a weak$*$ closed set (which is also convex). Thus, the set of probability measures

$$\mathcal{P}(X) = \{\mu \in \mathcal{M}^+(K) \mid \langle\mu \mid \mathbf{1}_K\rangle = 1\} = \mathcal{M}^+(K) \cap B,$$

is a weak$*$-compact set.

We finish by an application of the Banach-Alaoglu theorem to reflexive spaces, a class of spaces that include Hilbet spaces.

**Definition 41** (Reflexive space)**.** A normed space $\mathcal{X}$ is *reflexive* if and only if $\mathcal{X}^{**}$, the dual space to $\mathcal{X}^*$, can be identified with $\mathcal{X}$, meaning that the *canonical injection*

$$\begin{aligned} i : \mathcal{X} &\to \mathcal{X}^{**}, \\ x &\mapsto (\phi_x : x^* \in \mathcal{X}^* \mapsto \langle x^* \mid x \rangle) \end{aligned}$$

is a bijection. In other words, a space $\mathcal{X}$ is reflexive if every continuous linear form on its dual $\mathcal{X}^*$, i.e. $\phi : \mathcal{X}^* \to \mathbb{R}$, can be written under the form $\phi(x^*) = \langle x^* \mid x \rangle$ for some $x \in \mathcal{X}$.

Note that the canonical injection is an isometry, i.e. $\|i(x)\| = \|x\|$. In this course, it will be sufficient to know a few facts about reflexive spaces, such as

- finite-dimensional spaces are reflexive;
- Hilbert spaces are reflexive;
- $L^p$ spaces and Sobolev spaces $W^{1,p}$ are separable and reflexive for $1 < p < +\infty$.

**Corollary 79.** *In a separable and reflexive space, the unit ball is weakly compact.*

*Proof.* We only prove sequential weak compactness.Let $K$ be a bounded closed convex set and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{X}$ with $\|x_n\| \leq 1$. We regard these points as elements of $\mathcal{X}^{**}$ through the canonical injection, i.e. $x_n^{**} := i(x_n^*)$. The sequence $(x_n^{**})_{n\in\mathbb{N}}$ is also bounded (since $\|i(x)\| = \|x\|$), ensuring by Banach-Alaoglu's theorem the existence of a weak*–converging subsequence. Relabeling if necessary, we may therefore assume that there exists $x^{**} \in \mathcal{X}^{**}$ s.t.

$$\forall x^* \in \mathcal{X}^*, \langle x_n^{**} \mid x^* \rangle \xrightarrow{n\to+\infty} \langle x^{**} \mid x^* \rangle.$$

Since the canonical injection is surjective (by reflexivity of $\mathcal{X}$), there exists some $x$ such that $x^{**} = i(x)$. The previous convergence then yields

$$\forall x^* \in \mathcal{X}^*, \langle x^* \mid x_n \rangle \xrightarrow{n\to+\infty} \langle x^* \mid x \rangle,$$

i.e. $(x_n)_{n\in\mathbb{N}}$ weakly converges to $x$. Since $K$ is closed and convex, it is sequentially weakly closed, from which we deduce that $x \in K$. □

# B Exercises

## B.1 Chapter 2

**Exercise 1.** *Convex hull of union.* **1**. Given two convex sets $K$, $L$, prove that: $\operatorname{conv}(K \cup L) = \{(1-\alpha)x + \alpha y \mid (x, y, \alpha) \in K \times L \times [0, 1]\}$.

**2**. Given convex sets $K_1, \ldots, K_n$, prove that

$$\operatorname{conv}(K_1 \cup \ldots \cup K_n) = \left\{ \sum_{1\leq i\leq n} \alpha_i x_i \mid \alpha \in \Delta_n \text{ and } \forall i,\ x_i \in K_i \right\},$$

where $\Delta_n$ is the unit simplex in $\mathbb{R}^n$.

**Exercise 2.** Prove Lemma 4 when $X$ is a normed space and $K$ contains a ball $\mathrm{B}(0,r)$. What is the continuity modulus of $p_K$?

**Exercise 3.** Prove that the closure of a convex set is convex. Deduce that the closed convex hull of $A$ is equal to the closure of the convex hull of $A$.

**Exercise 4.** *Interior of a convex set.* **1**. Prove that if $B, C \subseteq K$ are subsets of a convex set $K$, then $(1-t)B + tC \subseteq K$.

**2**. Deduce that if $x$ belongs to the interior of $K$ and $y \in K$, then $[x, y)$ lies in the interior of $K$.

**3**. Prove that the interior of a convex set is convex.

**4**. Prove that if $K$ is closed and has non-empty interior, then $K$ is the closure of its interior.

**Exercise 5.** Let $\mathcal{X} = \mathrm{L}^2([0,1])$ and let $K = \{f \geq 0 \text{ a.e.} \mid f \in \mathcal{X}\}$. Prove that $K$ is convex and closed and therefore sequentially weakly closed. Write explicitely $K$ as an intersection of closed half-spaces.

**Exercise 6.** *Discontinuous linear form.* Consider $\mathcal{X} = \mathbb{R}[X]$ the space of polynomials, endowed with the sup-norm (if $P = a_n X^n + \ldots + a_0$, then $\|P\| = \max_i |a_i|$), then the linear form $\phi(P) = P'(1)$ is discontinuous everywhere.

**Exercise 7.** *Mazur's lemma.* Let $(x_n)_{n\in\mathbb{N}}$ be a weakly converging sequence in a normed space $\mathcal{X}$, with weak limit $x$. Considering the set $K = \mathrm{conv}(\{x_n \mid n \in \mathbb{N}\})$, prove that there exists a sequence $y_n$ of convex combinations of the $x_n$ (i.e. $y_n \in K$) such that $y_n$ converges strongly to $x$.

**Definition 42** (Extreme point)**.** Let $K \subseteq \mathcal{X}$ convex. An *extreme point* is a point $x \in K$ that cannot be obtained by taking a nontrivial convex combination of points in $K$, i.e. there is no $y, z \in K \setminus \{x\}$ and $\alpha \in (0,1)$ such that $x = (1-\alpha)y + \alpha z$. The set of extremal points of $K$ is denoted $\mathrm{ext}(K)$.

**Exercise 8.** Let $A \subseteq \mathcal{X}$ and $K = \mathrm{conv}(A)$. Prove that $\mathrm{ext}(K) \subseteq A$.

**Exercise 9.** *Extreme points and strict convexity of the ball.* We recall that a convex set $K$ is *strictly convex* if

$$\forall x \neq y \in K, \forall \alpha \in (0,1), \quad (1-\alpha)x + \alpha y \in \mathrm{int}(K).$$

**1**. Prove that the unit ball in a Hilbert space is strictly convex and that all points are extreme.

**2**. Given $x \in \mathbb{R}^d$, we define $\|x\|_1 = \sum_{1\leq i\leq d} |x_i|$ and $\|u\|_\infty = \max_{1\leq i\leq d} |x_i|$. Are the unit balls of $(\mathbb{R}^d, \|\cdot\|_1)$ and $(\mathbb{R}^d, \|\cdot\|_\infty)$ strictly convex ? What are their extreme points?

**3**. Same question for $\mathcal{X} = \mathcal{C}^0([0,1])$ endowed with the sup-norm $\|\cdot\|_\infty$ and with $\mathcal{X} = \mathrm{L}^1([0,1])$.

**Exercise 10.** *Support functions.* Compute the support functions of the following objects:

1. a segment $[a,b]$ in $\mathbb{R}^d$,
2. a square $[0,1]^2$ in $\mathbb{R}^2$,
3. the *unit simplex* $\Delta = \{x \in \mathbb{R}^d_+ \mid \sum_i x_i = 1\}$ in $\mathcal{X} = \mathbb{R}^d$,

**Exercise 11.** Let $(\mathcal{X}, \|\|)$ be a normed space and let $B$ be the unit ball of $\mathcal{X}$. Prove that $\sigma_B$ coincides with the dual norm $\|\cdot\|_*$.

**Exercise 12.** *Hausdorff distance.* Let $K, L$ be two compact convex bodies in $\mathbb{R}^d$, and consider the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$. We denote $\mathrm{d}_H(K,L)$ the Hausdorff distance between $K$ and $L$, i.e.

$$\mathrm{d}_H(K,L) = \left( \max_{x\in K} \min_{y \in L} \|x-y\|, \max_{y\in L} \min_{x\in K} \|x-y\| \right),$$

1. Prove that $\mathrm{d}_H(K,L) = \min\{\varepsilon \geq 0 \mid K \subseteq L + \mathrm{B}(0,\varepsilon) \text{ and } L \subseteq K + \mathrm{B}(0,\varepsilon)\}$,
2. Prove that $K \subseteq L + \mathrm{B}(0,\varepsilon)$ iff $\sigma_K \leq \sigma_L + \varepsilon$ on $\mathbb{S}^{d-1}$,
3. Conclude that $\mathrm{d}_H(K,L) = \|\sigma_K - \sigma_L\|_{\infty,\mathbb{S}^{d-1}}$.

## B.2 Chapter 3

**Exercise 13.** *Distance functions.* Let $K \subseteq \mathcal{X}$, and define $\mathrm{d}_K(x) = \inf_{p\in K} \|x-p\|$.
1. Prove that if $K$ is convex, then $\mathrm{d}_K$ is convex.
2. Prove that if $\mathcal{X}$ is a Hilbert space, then $\|\cdot\|^2 - \mathrm{d}_K^2$ is convex[2] and lsc.
   (nb one does not need to assume that $K$ is convex)

**Exercise 14.** Suppose that $f : \mathcal{X} \to \mathbb{R}$ is convex and satisfies $f(x) \leq L\|x\|$ for all $x \in \mathcal{X}$. Prove that $f$ is $L$-Lipschitz.

We recall that a Banach space is a complete normed space. Baire's theorem asserts that if $(F_n)_{n\in\mathbb{N}}$ is a countable family of closed subsets of a Banach space $\mathcal{X}$ (or more generally of a complete metric space), each with empty interior, then $\cup_{n\in\mathbb{N}} F_n$ has empty interior. We use it to deduce a characterization of the continuity set of lsc convex functions.

**Exercise 15.** Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a lower semicontinuous convex function on a Banach space, and assume that $\mathrm{int}\,\mathrm{dom} f$ contains a point $x$.
1. Assume that $x = 0$ and that $\mathrm{B}(0,r) \subseteq \mathrm{dom} f$. Letting

$$F_n = \{y \in \mathrm{B}(0,r) \mid \max(f(y), f(-y)) \leq n\},$$

   prove that $f$ is bounded on a set with non-empty interior.
2. Conclude that $\mathrm{cont} f = \mathrm{int}\,\mathrm{dom} f$.

**Exercise 16.** *Banach-Steinhaus.* Let $(T_\alpha)_{\alpha\in A}$ be a family of continuous linear operators, $T_\alpha : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, and define $f(x) := \sup_{\alpha\in A} \|T_\alpha(x)\|$.
1. Prove that $f$ is convex and lower-semicontinuous.

[2] one says that $\mathrm{d}_K^2$ is *semi-concave*

2. We now assume that for all $x \in \mathcal{X}$, $\exists M_x \geq 0$ s.t. $\sup_\alpha \|T_\alpha(x)\| < M_x$. Prove that $f$ is continuous on $\mathcal{X}$, and locally Lipschitz near the origin.
3. Deduce the existence of $M \geq 0$ such that $\sup_\alpha \|T_\alpha\| \leq M$.

**Exercise 17.** *Characterization of support functions*. Let $\sigma : \mathcal{Y} \to \overline{\mathbb{R}}$ be a lower-semicontinuous sublinear (hence convex) function.

1. Let $g(x) = \langle x^* \mid x\rangle + t$ be an affine minorant of $\sigma$. Prove that $g(0) = 0$.
2. Deduce that $\sigma(x) = \sup\{\langle x^* \mid x\rangle \mid \sigma \geq \langle x^* \mid \cdot\rangle\}$.
3. Prove that if $\mathcal{Y} = \mathcal{X}^*$ and if $\mathcal{X}^*$ is reflexive, then there exists a convex set $K \subseteq \mathcal{X}$ such that $\sigma = \sigma_K$

## B.3 Chapter 4

**Exercise 18.** *Envelope theorem*. Let $f_i \in \mathcal{C}^1(\mathbb{R}^d)$ be convex functions satisfying

$$\forall i_0 \neq i_1 \in I, \forall x \in \mathbb{R}^n, \nabla f_{i_0}(x) \neq \nabla f_{i_1}(x) \tag{18}$$

We define $f = \max_{i\in I} f_i$ the pointwise maximum of these functions, and we assume that the maximum is attaine at any $x \in \mathbb{R}^d$.

1. Prove that if $I$ is finite, then $f$ is differentiable almost everywhere. *(Hint: invoke the implicit function theorem.)*
2. In the general case, consider the set

$$A = \{x \in \mathbb{R}^n \mid \exists i_0 \neq i_1 \in I,\ f(x) = f_{i_0}(x) = f_{i_1}(x)\}.$$

   Prove that if $x$ belongs to $A$, then $f$ is not Gâteaux-differentiable at $x$.
3. Deduce that $f$ that for almost every $x \in \mathbb{R}^d$, $f$ is differentiable at $x$ and there exists a unique $i_x \in I$ s.t. $f(x) = f_{i_x}(x)$ and $\nabla f(x) = \nabla f_{i_x}(x)$.

**Exercise 19.** *Simplex*. Let $f : \mathbb{R}^d \to \mathbb{R}$ be defined by $f(x_1, \ldots, x_d) = \max(x_1, \ldots, x_d)$. Prove that $\partial f(0) = \{x \in \mathbb{R}^d \mid x_1 + \ldots + x_d = 1 \text{ and } \forall i, x_i \geq 0\}$.

**Exercise 20.** *Failure of subdifferential sum rule*. Let $A = B((0,0),1)$, $B = B((0,2),1)$ bet closed balls in $\mathbb{R}^2$, and $f = \mathrm{i}_A$, $g = \mathrm{i}_B$. Compute the subdifferentials $\partial f(x), \partial g(x)$ and $\partial(f+g)(x)$ at $x = (0,1)$.

**Exercise 21.** *Characterization of convexity*. Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a proper function so that for all $x \in \operatorname{dom} f$, the subdifferential $\partial f(x)$ is non-empty.

1. Prove that $f$ is equal to the supremum of its affine minorants.
2. Deduce that $f$ is convex lsc.
3. Conversely, prove that if $f : \mathbb{R}^d \to \mathbb{R}$ is convex then $\partial f(x) \neq \emptyset$ for all $x \in \mathbb{R}^d$

**Exercise 22.** *Subdifferential sum rule*. Prove using support functions (see Remark 16) that $\partial(f+g)(x) = \partial f(x) + \partial g(x)$ when $x \in \operatorname{cont} f \cap \operatorname{cont} g$.

**Exercise 23.** *Exact penalization using distance*. Let $K$ be a closed non empty convex subset of a Hilbert space $\mathcal{X}$. We will consider subdifferentials, normal cones, etc. as subsets of $\mathcal{X} \simeq \mathcal{X}^*$.

1. Show that $\partial i_K(p) = \{v \in \mathcal{X} \mid \forall q \in K,\ \langle v \mid p\rangle \geq \langle v \mid q\rangle\}$. Using the characterization of the orthogonal projection on $K$, prove that

$$\partial i_K(p) = \{v \in E;\ \ \mathrm{p}_K(p+v) = p\}. \tag{19}$$

2. *Application:* Let $f : \mathcal{X} \to \mathbb{R}$ be convex and continuous. Prove that $\bar{p} \in K$ if a minimizer of

$$\min_{p \in K} f(p) \tag{20}$$

iff there exists $w \in \partial f(\bar{p})$ such that $\mathrm{p}_K(\bar{p} - w) = \bar{p}$. (*Indication: use the subdifferential sum rule.*)

3. We now prove that if $p \in K$ and $v \in \mathrm{Nor}_p K = \partial i_K(p)$ then $v/\|v\| \in \partial \mathrm{d}_K(p)$, or equivalently that

$$\forall x \in \mathcal{X},\ \mathrm{d}_K(x) \geq \mathrm{d}_K(p) + \langle \frac{v}{\|v\|} \mid x - p\rangle. \tag{21}$$

   a. Prove that (21) holds if $\langle v \mid x\rangle \leq \langle v \mid p\rangle$.
   b. Let $x \notin H := \{x \in \mathcal{X} \mid \langle v \mid p\rangle \geq \langle v \mid x\rangle\}$, and define $x_H = \mathrm{proj}_H(x)$. Prove that $x_H = x - \langle \frac{v}{\|v\|} \mid x - p\rangle \frac{v}{\|v\|}$.
   c. Using $K \subseteq H$, prove that $\mathrm{d}_K(x) \geq \mathrm{d}_H(x) = \|x - x_H\| = \langle \frac{v}{\|v\|} \mid x - p\rangle$, and deduce that $v/\|v\| \in \partial \mathrm{d}_K$.

4. *Application:* Let $\bar{p} \in K$ be a minimizer of (20).
   a. Prove that there exists $w \in \partial f(\bar{p})$ such that $-w \in \partial i_K(\bar{p})$, so that

$$0 \in \partial(f + \|w\|\, \mathrm{d}_K)(\bar{p}).$$

   b. Deduce the existence of $\lambda \geq 0$ such that $\bar{p}$ minimizes the penalized problem

$$\min_{p \in \mathcal{X}} f(p) + \lambda \mathrm{d}_K(p).$$

**Exercise 24.** *Limiting subdifferential, Exam 2020.* Let $f \in \mathcal{C}^0(\mathbb{R}^n)$ be convex.

1. Fix a point $x \in \mathbb{R}^n$, and define $K = \overline{\mathrm{conv}} S$ where

$$S = \left\{ s \in \mathbb{R}^n \mid \exists x_n \to x,\ \text{s.t. } f \text{ is differentiable at } x_n \text{ and } \lim_{n \to \infty} \nabla f(x_n) = s \right\}.$$

   a. Prove that $S \subseteq \partial f(x)$ and $K \subseteq \partial f(x)$.
   *The goal of the next questions is to prove the converse inclusion.*
   b. Fix some vector $v \in \mathbb{R}^n$. Prove that for all $t_n = 1/n$, there exists $v_n \in \mathbb{R}^n$ such that $\|v_n - v\| \leq t_n$ and such that $f$ is differentiable at $x_n = x + t_n v_n$.
   c. Prove that, taking a subsequence if necessary, one can assume that $\nabla f(x_n)$ converges to a vector $s \in S$. Show that $f^+(x, v) \leq \langle s \mid v\rangle$.
   d. Deduce that $f^+(x, v) \leq \sigma_K(v)$ where $\sigma_K$ is the support function of $K$. Conclude that $\partial f(x) \subseteq K$.

2. *Application.* Assume there exists $G \in \mathcal{C}^0(\mathbb{R}^d, \mathbb{R}^d)$ such that $\forall x \in \mathbb{R}^n, G(x) \in \partial f(x)$. Prove that $f$ belongs to $\mathcal{C}^1(\mathbb{R}^n)$ and that $\nabla f = G$.

subdifferential of TV norm

## B.4 Chapter 5

**Exercise 25.** Let $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, where $\mathcal{X}_1, \mathcal{X}_2$ are two closed subspaces, let $f_i \in \Gamma_0(\mathcal{X}_i)$ and $f = f_1 \oplus f_2$, i.e. $f : \mathcal{X} \to \overline{\mathbb{R}}$ is defined by

$$\forall (x_1, x_2) \in \mathcal{X}_1 \oplus \mathcal{X}_2 \mapsto f_1(x_1) + f_2(x_2).$$

Prove that $\mathrm{Prox}_{\gamma f}(x_1 + x_2) = \mathrm{Prox}_{\gamma f_1}(x_1) + \mathrm{Prox}_{\gamma f_2}(x_1)$.

**Exercise 26.** Let $f \in \Gamma_0(\mathcal{X})$ be coercive, let $T = \mathrm{Prox}_f$ and define $x_{n+1} = T(x_n)$.

1. Prove that $\forall x \in \mathcal{X},\ f(x) \geq f(x_{n+1}) + \frac{1}{\gamma}\langle x - x_{n+1} \mid x_n - x_{n+1}\rangle$.
2. Using this inequality,
   (i) prove that $(f(x_n))_{n\in\mathbb{N}}$ is decreasing and that $(x_n)_{n\in\mathbb{N}}$ is bounded;
   (ii) prove that any weak cluster point of $(x_n)_{n\in\mathbb{N}}$ minimizes $f$ globally;
   *(Hint: use that* $\lim_{n\to+\infty} \|x_{n+1} - x_n\| = 0$*, as in Theorem 60.)*
3. Deduce that $(x_n)_{n\in\mathbb{N}}$ is a minimizing sequence.
4. Conclude that if $f$ is strictly convex, the sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to the unique minimizer of $f$.

## B.5 Chapter 6

**Exercise 27.** *Convex cones*. Let $K$ be a non-empty convex cone (i.e. for all $x \in K$ and $\lambda \geq 0$, $\lambda x \in K$) and let $K^* \subseteq \mathcal{X}^*$ be its polar of $K$

$$K^* = \{x^* \in \mathcal{X}^* \mid \forall x \in K, \langle x^* \mid x\rangle \leq 0\}.$$

1. Prove that if $f = \mathrm{i}_K$, then $f^* = \mathrm{i}_{K^*}$ and that $\partial f(0) = K^*$.
2. Prove that $K^{**} = \{x \in \mathcal{X} \mid \forall x^* \in K^*, \langle x^* \mid x\rangle \leq 0\}$ is closed, convex, and contains $K$.
3. Prove that if $K$ is a closed convex cone, then $K^{**} = K$.
4. Assume that $\mathcal{X}$ is a Hilbert space, let $K$ be a closed convex cone, $K^* \subseteq \mathcal{X} \simeq \mathcal{X}^*$ be its polar. Prove that

$$\forall x \in \mathcal{X}, \quad x = \mathrm{proj}_K(x) + \mathrm{proj}_{K^*}(x).$$

**Exercise 28.** *Strong convexity and subdifferential*. Let $\mathcal{X}$ be a Hilbert space. A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is called $\alpha$-strongly convex if it satisfies one the following equivalent conditions:

$$\forall x, y \in \mathrm{dom}(f), \forall \lambda \in [0,1],$$
$$\frac{\alpha}{2}\lambda(1-\lambda)\|x-y\|^2 + f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \tag{22}$$
$$\forall x_0 \in \mathcal{X},\ f_{x_0} : x \in \mathcal{X} \mapsto f - \frac{\alpha}{2}\|x - x_0\|^2 \text{ is convex} \tag{23}$$

1. Assume that $f \in \Gamma_0(\mathcal{X})$ is $\alpha$-strongly convex and let $v \in \partial f(x_0)$. Prove that

$$0 \in \partial\left(f - \frac{\alpha}{2}\|\cdot - x_0\|^2 - \langle v \mid \cdot\rangle\right)(x_0),$$

   and deduce that $\forall x \in \mathcal{X},\quad f(x) \geq f(x_0) + \langle v \mid x - x_0\rangle + \frac{\alpha}{2}\|x - x_0\|^2$.

2. Prove that for all $x_0, y_0 \in \mathcal{X}$, $v \in \partial f(x_0)$ and $\forall w \in \partial f(y_0)$ one has

$$\alpha \|x_0 - y_0\| \leq \|v - w\| .$$

3. Deduce that the same inequality holds if $x_0 \in \partial f^*(v)$ and $y_0 \in \partial f^*(w)$.
4. Let $D = \{v \in \mathcal{X} \mid \partial f^*(v) \neq \emptyset\}$. Prove that $f^*$ is Gâteaux-differentiable on $D$, and that the application $v \in D \mapsto \nabla f^*(v)$ is $(1/\alpha)$-Lipschitz.

**Exercise 29.** *Moreau-Yosida regularization*. Let $\mathcal{X}$ be a Hilbert space and $f \in \Gamma_0(\mathcal{X})$. Given $\tau > 0$, define the Moreau-Yosida regularization of $f$ as

$$f_\tau(x) = \inf_{z \in \mathcal{X}} f(z) + \frac{1}{2\tau} \|x - z\|^2$$

1. Prove that $f_\tau$ is convex, finite everywhere, and bounded on every bounded subset of $\mathcal{X}$.
2. Prove that $f_\tau$ is continuous and that $\partial f_\tau(x) \neq \emptyset$ for all $x \in \mathcal{X}$.
3. Prove that $f_\tau = (f^* + g^*)^*$ where $g(x) = \frac{1}{2\tau} \|x\|^2$.
4. Using the previous exercise, deduce that $f_\tau$ is Gâteaux-differentiable at all $x \in \mathcal{X}$ and that $\nabla f_\tau$ is $\tau$-Lipschitz.
5. Prove that $f_\tau$ converges pointwise to $f$ as $\tau \to 0$.
6. Prove that $\mathrm{Prox}_{\tau f}(x) = x - \tau \nabla f_\tau(x)$.
*(Hint: Let $p = \mathrm{Prox}_{\tau f}(x)$, and prove that $\frac{x-p}{\tau} \in \partial f_\tau(x)$ using the equality case in Fenchel-Young's inequality.)*

This last question shows that the proximal point algorithm can be interpreted as "usual" gradient descent for the Moreau-Yosida regularization.

For the next exercise, recall that a *measure* on a compact set $X$ is a linear form $\mu : \mathcal{C}^0(X) \to \mathbb{R}$ which is continous for the topology induced by the sup-norm $\|\cdot\|_\infty$. The space of measures is denoted

$$\mathcal{M}(X) = (\mathcal{C}^0(X))^*.$$

A measure $\mu$ is *non-negative* if $\forall f \in \mathcal{C}^0(X), (f \geq 0 \implies \mu(f) \geq 0)$. Finally, a probability measure on $X$ is a non-negative measure $\mu$ such that $\|\mu\|_{TV} = 1$. The space of non-negative measures is denoted $\mathcal{M}_+(X)$, and the space of probability measures is $\mathcal{P}(X)$.

**Exercise 30.** *Kantorovich Duality*. Let $X, Y$ be two compact sets, let $\mu$ be a probability measure on $X$, let $\nu$ be a probability measure on $Y$ and finally let $c \in \mathcal{C}^0(X \times Y)$. Define a linear form $\Lambda : \mathcal{C}^0(X) \times \mathcal{C}^0(Y) \to \mathcal{C}^0(X \times Y)$ by $\Lambda(\phi, \psi) = \phi \oplus \psi$ where

$$\phi \oplus \psi : (x, y) \in X \times Y \mapsto \phi(x) + \psi(y).$$

We consider the following optimization problem

$$P := \inf\{-(\langle \mu \mid \phi \rangle + \langle \nu \mid \psi \rangle) \mid (\phi, \psi) \in \mathcal{C}^0(X) \times \mathcal{C}^0(Y), \phi \oplus \psi \leq c\}.$$

1. Prove that $P = \inf_{(\phi,\psi)\in\mathcal{C}^0(X)\times\mathcal{C}^0(Y)} f(\Lambda(\phi,\psi)) + g(\phi,\psi)$, where

$$f : \sigma \in \mathcal{C}^0(X \times Y) \mapsto= \begin{cases} 0 & \text{if } \sigma \leq c \\ +\infty & \text{otherwise,} \end{cases}$$
$$g : (\phi,\psi) \in E \times F \mapsto -(\langle \mu \mid \phi\rangle + \langle \nu \mid \psi\rangle).$$

2. Prove that $f^*$ and $g^*$ are given by:

$$f^* : \gamma \in \mathcal{M}(X \times Y) : \gamma \mapsto = \begin{cases} \langle \gamma \mid c\rangle & \text{if } \gamma \in \mathcal{M}_+(X \times Y) \\ +\infty & \text{otherwise} \end{cases}$$

$$g^* : (\kappa,\lambda) \in \mathcal{M}(X) \times \mathcal{M}(Y) \mapsto \begin{cases} 0 & \text{if } (\kappa,\lambda) = -(\mu,\nu) \\ +\infty & \text{otherwise} \end{cases}$$

3. Let $\gamma \in \mathcal{M}(X \times Y)$. The *marginal of* $\gamma$ *on* $X$ is the measure $\Pi_X\gamma \in \mathcal{M}(X)$ defined by $\Pi_X\gamma : \phi \in \mathcal{C}^0(X) = \gamma(\phi\oplus 0)$. The marginal on $Y$ is defined similarly. Prove that the adjoint $\Lambda^*$ of $\gamma$ is given by

$$\Lambda^*(\gamma) = (\Pi_X\gamma, \Pi_Y\gamma) \in \mathcal{M}(X) \times \mathcal{M}(Y).$$

4. Deduce from Fenchel-Rockafellar that $P = D$ where

$$\begin{aligned} D &:= \max_{\gamma\in\mathcal{C}^0(X\times Y)^*} -g^*(-\Lambda^*\gamma) - f^*(\gamma) \\ &= \max\{-\langle\gamma \mid c\rangle \mid \gamma \in \mathcal{M}_+(X \times Y) \text{ s.t. } \Pi_X\gamma = \nu \text{ and } \Pi_Y\gamma = \nu\}. \end{aligned}$$

This duality formula is due to Leonid Kantorovich, and is one of the most important results in the theory of optimal transport.

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